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Discussion on: “An Algorithm for Solving a Perturbed Algebraic Riccati Equation”

G. Freiling

Institut für Mathematik, Universität Duisburg-Essen, D-47048 Duisburg, Germany

The solution of stochastic linear quadratic optimal control problems with infinite time horizon leads (see [4,14,15]) to generalized algebraic matrix equations of the form

$$\begin{aligned}\mathcal{R}(X) &:= A^*X + XA + Q + \Pi_1(X),\\ &-P(X)[R+\Pi_2(X)]^{-1}P(X)^* = 0,\end{aligned} \tag{1}$$

with $P(X) = L + XB + \Pi_{12}(X)$ and where A, B, Q, R and L are given matrices of sizes $n\times n$, $n\times m$, $n\times n$, $m\times m$ and $n\times m$, respectively, such that

$$T := \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix}$$

is hermitian.

The operator $\Pi : \mathcal{H}^n \to \mathcal{H}^{n+m}$ with

$$\Pi(X) := \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{bmatrix}$$

is positive, that is, $X \geq 0$ implies $\Pi(X) \geq 0$. Here, $\mathcal{H}^n$ stands for the real vector space of hermitian matrices of size n, and by $X \geq 0$ (or $X > 0$) it is denoted that $X = X^*$ is positive semi-definite (or positive definite); we mention that in most applications Π is a linear function.

In the finite time horizon, one has to solve (in the continuous-time case) the corresponding differential equation

$$\dot{X} = -\mathcal{R}(X), \quad X(t_f) = X_f. \tag{2}$$

Notice that $\mathcal{R}(X)$ is the Schur complement of $R + \Pi_2(X)$ in the matrix

$$\mathcal{D}_{\mathcal{R}}(X) := \begin{bmatrix} A^*X + XA + Q + \Pi_1(X) & P(X) \\ P(X)^* & R + \Pi_2(X) \end{bmatrix}.$$

Hence on

$$\begin{aligned} &D(\mathcal{R}) := \{X \in \mathcal{H}^n \mid R + \Pi_2(X) > 0\},\\ &\mathcal{D}_{\mathcal{R}}(X) \geq 0 \iff \mathcal{R}(X) \geq 0, \end{aligned} \tag{3}$$

and

$$\dot{X} + \mathcal{R}(X) \geq 0 \iff \begin{bmatrix} \dot{X} & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{D}_{\mathcal{R}}(X) \geq 0.$$

For $\Pi_2 \equiv 0, \Pi_{12} \equiv 0$ and $L \equiv 0(1)$ coincide with the linearly perturbed continuous-time algebraic Riccati equation

$$A^*X + XA + Q + \Pi_1(X) - XSX = 0,$$

where $S := BR^{-1}B^*$, which appears in control problems with stochastically jumping parameters (see [14]); the corresponding algebraic equations and inequalities also play an important role in the application of the Lyapunov–Krasovskii method to linear time-delay systems.

The paper under discussion by E.F. Costa and J.B.R. do Val presents an algorithm for solving the perturbed algebraic Riccati equation (4) that generates under minimal assumptions (namely

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E-mail: freiling@math.uni-duisburg.de

monotonicity of the perturbation term Π_1 in [4]) a monotonically increasing sequence (X_k) with $X_0 = 0$; this sequence converges to the minimal positive semidefinite solution of (4), provided it exists.

The same algorithm (and also some variants) were studied earlier in [2] under an additional assumption that guaranteed the convergence of the algorithm; subsequently, several variants of this algorithm have been proposed in the literature – see Section 6.7 of our textbook [1] for a more detailed discussion of questions related to (4) and for variants of the algorithm presented by Costa and do Val.

It is worthwhile to mention that several numerical algorithms for solving (4) (e.g. Lyapunov and Riccati iterations and variants of Newton's method) were already used successfully (but without having sufficient conditions for convergence) by engineers working in this area, before [2] was written. Therefore, the main purpose of [2] was to provide sufficient conditions for the existence of a positive semidefinite (or stabilizing) solution of (4) as well as for the convergence of the proposed algorithms, because at this time the theoretical background for (1) and even (4) was not yet well understood.

Since the fundamental work of Wonham [14] (see also [13, Chapter 2, Section 9]), it took around three decades to understand why generalized matrix Riccati equations of the form (1) (see, e.g. [3,5,6,9,10]) or equations of the more special form (4) (see, e.g. [8]) and also the corresponding differential equations share some of the nice properties of unperturbed matrix Riccati equations although they cannot be related via Radon's lemma to a corresponding linear equation.

For a summary of the state of the art of generalized and perturbed matrix Riccati equations we refer the reader to Sections 6.7–9 of [1] (see also [4]); for convenience of the reader we mention below some of the key facts from the theory of generalized Riccati equations:

- The solutions of (1) and (2) depend monotonically on T; moreover, the solutions of (2) depend monotonically on X_f – this results from the general comparison theorem ([1, Theorem 6.7.33]).
- Necessary and sufficient conditions for the existence of solutions of perturbed matrix Riccati equations can be formulated via generalized algebraic Riccati inequalities. For example, Eq. (4) has at least one positive semidefinite solution if and only if (see [11, Theorem 3.1]) there exist matrices $X_1, X_2 \in \mathcal{H}^n$ such that

$$X_1 \leq X_2 \quad \text{and} \quad \mathcal{R}(X_2) \leq 0 \leq \mathcal{R}(X_1);$$

  this solution is not necessarily stabilizing. Notice that on account of (3) the solvability of $\pm\mathcal{R}(X) \leq 0$ is (in case of a linear function Π) equivalent to the solvability of a linear matrix inequality (LMI).
- Each solution of $\pm\mathcal{R}(X_f) \leq 0$ generates a solution of (2) that is monotonic (for a precise formulation see [10, Lemma 6.1] or [1, Lemma 6.7.49]) – this fact indicates the strong connection between the solutions of generalized Riccati inequalities and/or LMIs and monotonic solutions of (2).
- In order to ensure the existence of a unique stabilizing solution of (4) or (1) it is not sufficient to assume (like in the case of classical linear quadratical control problems) that (A, B) is stabilizable and that $(\sqrt{Q}, A)$ is detectable. Instead one has to introduce more general concepts of (stochastical) stabilizability and (stochastical) detectability (see [1, Section 6.7, 4, 8]) for a discussion of this topic and for further references.
- Discrete-time versions of the above-mentioned results can be found in [1,9, Section 6.8]. Detailed results on the solutions of (2) in the case of time-variant coefficients have been reported recently in [6]. For further investigations on the iterative solution of generalized algebraic and differential Riccati equations we refer to [5–7,12] and the references given therein.

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Discussion on: "An Algorithm for Solving a Perturbed Algebraic Riccati Equation"

Hiroaki Mukaidani

Graduate School of Education, Hiroshima University, Japan

This paper presents the valuable contribution of the algorithm for solving a perturbed algebraic Riccati equation. The proposed method, which consists of Riccati iterations, guarantees monotone convergence to the minimal solution. As another important feature, the convergence is unconditionally ensured without stability or any other condition. However, the convergence rate is unclear. In fact, the convergence speed seems to be slow when the proposed algorithm is carried out. Moreover, the required iterations depend most strongly on the design parameter which is included in the Riccati iterations.

This discussion focuses on two other issues. First, instead of the Lyapunov and the Riccati iterations recent advances in the theory of the Newton–Kantorovich theorem [2] will allow a revisiting of Newton's method. Second, the possibility of implementing the computation of a Hessenberg–Schur method [3] into Newton's method is considered. This addresses a more practical aspect because the reduced order computation will avoid the calculation with the large dimension for the Kronecker product method [4].

Let us consider the following algebraic Riccati-type equation (ARTE) of the specific form [1,6]

$$\mathcal{F}(X) := A^{\mathrm{T}}X + XA - XSX + Q + \sigma C^{\mathrm{T}}XC = 0, \tag{1}$$

where $A \in \boldsymbol{R}^{n\times n}$ and $C \in \boldsymbol{R}^{n\times n}$ are appropriate constant matrices; $S \in \boldsymbol{R}^{n\times n}$ and $Q \in \boldsymbol{R}^{n\times n}$ are the symmetric positive semidefinite matrices; σ is a positive scalar constant; $X \in \boldsymbol{R}^{n\times n}$ is an unknown matrix.

The following algorithm which is given in terms of generalized Lyapunov iterations was considered in [5,6].

$$\begin{aligned}&(A - SX^{(k)})^{\mathrm{T}}X^{(k+1)} + X^{(k+1)}(A - SX^{(k)})\\ &\quad + X^{(k)}SX^{(k)} + Q + \sigma C^{\mathrm{T}}X^{(k+1)}C = 0,\\ &\quad k = 0, 1, \ldots,\end{aligned} \tag{2}$$

with a stabilizing initial guess $X^{(0)}$. Although it was shown from [5] that $X^{(k)} \to X (k \to \infty)$, the convergence rate has not been proved. As a matter of fact, however, a quadratic convergence has been shown in [7]. Moreover, it will be shown in this discussion that the above algorithm (2) is equivalent to Newton's

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E-mail: mukaida@hiroshima-u.ac.jp

method. Taking the partial derivative of the function $\mathcal{F}(X)$ with respect to X yields

$$\begin{aligned}\nabla\mathcal{F}(X) &:= \frac{\partial \mathrm{vec}\mathcal{F}(X)}{\partial(\mathrm{vec}X)^{T}}\\ &= (A-SX)^{\mathrm{T}}\otimes I_n + I_n\otimes(A-SX)^{\mathrm{T}}\\ &\quad + \sigma C^{\mathrm{T}}\otimes C^{\mathrm{T}}.\end{aligned}\tag{3}$$

Taking the vec-operator transformation on both sides of (1) and (2) results in

$$\begin{aligned}\mathrm{vec}\,\mathcal{F}(X^{(k)}) = [&(A-SX^{(k)})^{\mathrm{T}}\otimes I_n\\ &+ I_n\otimes(A-SX^{(k)})^{\mathrm{T}}\\ &+ \sigma C^{\mathrm{T}}\otimes C^{\mathrm{T}}]\mathrm{vec}\,X^{(k)}\\ &+ \mathrm{vec}[X^{(k)}SX^{(k)}+Q],\end{aligned}\tag{4a}$$

$$\begin{aligned}&[(A-SX^{(k)})^{\mathrm{T}}\otimes I_n + I_n\otimes(A-SX^{(k)})^{\mathrm{T}}\\ &\quad + \sigma C^{\mathrm{T}}\otimes C^{\mathrm{T}}]\mathrm{vec}\ X^{(k+1)}\\ &\quad + \mathrm{vec}[X^{(k)}SX^{(k)}+Q] = 0,\end{aligned}\tag{4b}$$

where vec and $\otimes$ denote an ordered stack of the columns and the Kronecker product, respectively [9]. Subtracting (4b) from (4a) and noting that

$$\nabla\mathcal{F}(X^{(k)})\cdot(\mathrm{vec}\,X^{(k+1)} - \mathrm{vec}\,X^{(k)}) = -\mathrm{vec}\,\mathcal{F}(X^{(k)}),\tag{5}$$

the following equation holds

$$\mathrm{vec}X^{(k+1)} = \mathrm{vec}X^{(k)} - [\nabla\mathcal{F}(X^{(k)})]^{-1}\cdot\mathrm{vec}\mathcal{F}(X^{(k)}),\tag{6}$$

which is the desired result.

Newton’s method (2) can be constructed by setting $X^{(k+1)} = X^{(k)} + \Delta X^{(k)}$ and neglecting $O(\Delta X^{(k)\mathrm{T}}\Delta X^{(k)})$ term. Newton’s method is well known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood. However, the choice of the initial guess needs special attention.

Since the algorithm (2) is based on Newton’s method, it will be potentially faster than the widely used Lyapunov iterations. However, there exist drawbacks that cannot be ignored. In view of the numerical example, whenever Newton’s method is computed via the Kronecker product method [4], since the usual Gaussian elimination technique requires $O(n^6)$ flops, it may be formidable because of quite large dimension n. This operation count is obviously unacceptable.

If $C = I_n$, then the Hessenberg–Schur method [3] can be applied. Such a method is based upon the equivalence of the problem. Let us consider the generalized algebraic Lyapunov-type equation (ALTE) which is equivalent to the ALTE (2).

$$\begin{aligned}\mathcal{G}(P) &:= L^{\mathrm{T}}P + PL + \sigma P + M = 0\\ &\Leftrightarrow (U^{-1}L^{\mathrm{T}}U)(U^{-1}PV) + (U^{-1}PV)(V^{-1}LV)\\ &\qquad + \sigma(U^{-1}PV) + (U^{-1}MV) = 0,\end{aligned}\tag{7}$$

where $E := U^{-1}L^{\mathrm{T}}U = U^{\mathrm{T}}L^{\mathrm{T}}U = (e_{ij})$ and $F := V^{-1}L^{\mathrm{T}}V = V^{\mathrm{T}}L^{\mathrm{T}}V = (f_{ij})$ are upper Hessenberg and upper quasi-triangular, respectively, and U and V are orthogonal. A matrix $E = (e_{ij})$ is upper Hessenberg if $e_{ij} = 0$ for all $i > j+1$. The orthogonal reduction of L^{T} to the upper Hessenberg form can be accomplished with Householder matrices in $O(n^3)$ flops. On the other hand, the calculation of the orthogonal reduction of L to the upper quasi-triangular form needs $O(n^3)$ flops.

Using the above reductions lead to the transformed system

$$\begin{aligned}&EY + YF^{\mathrm{T}} + \sigma Y + Z = 0,\quad Z = U^{\mathrm{T}}MV,\\ &\quad Y = U^{\mathrm{T}}PV.\end{aligned}\tag{8}$$

Assuming $f_{l(l-1)}$, $F = (f_{ij})$ is zero, it follows that

$$[E + (f_{ll}+\sigma)I_n]y_l = -z_l - \sum_{j=l+1}^{n} f_{lj}y_j,\tag{9}$$

where $Y = [y_1|y_2|\cdots|y_n]$ and $Z = [z_1|z_2|\cdots|z_n]$. Thus, y_l can be found from $y_{l+1},\ldots,y_n$ by solving (9). In the case where the Gaussian elimination with partial pivoting is used for the above calculation, $O(n^3)$ flops are needed. Finally, this system can be solved in not $O(n^6)$ but $O(n^3)$. That is, the use of the Hessenberg–Schur method [3] results in the reduction of operation count.

In order to demonstrate the efficiency of Newton’s method, an illustrative example is given. Consider the following ARTE with

$$\begin{aligned}&A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},\quad S = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\quad Q = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\\ &\quad C = I_2,\quad \sigma = 1.\end{aligned}$$

It should be noted that algorithm (2) converges to the exact solution with an accuracy of $\|\mathcal{F}(X^{(k)})\| < 1.0\mathrm{e}{-}10$ after seven iterations. The convergence solution is given below:

$$X = \begin{bmatrix}4.9524\mathrm{e}{+}01 & 1.2230\mathrm{e}{+}01\\ 1.2230\mathrm{e}{+}01 & 8.1311\end{bmatrix}.$$

It may be noted that the initial guess $X^{(0)}$ satisfies the following algebraic Riccati equation:

$$A^{\mathrm{T}}X^{(0)} + X^{(0)}A - X^{(0)}SX^{(0)} + Q = 0,$$

$$X^{(0)} = \begin{bmatrix}1.8487\mathrm{e}{+}01 & 6.1623\\ 6.1623\mathrm{e} & 6.1623\end{bmatrix}.$$

In order to verify the exactitude of the solution, the remainder per iteration is computed by substituting $X^{(k)}$ into the ARTE (1). In Table 1, the results for the error $\|\mathcal{F}(X^{(k)})\|$ per iterations are given. It can be seen that the algorithm (2) has the quadratic convergence.

The required iterations of the proposed algorithm (2) versus the Riccati iterations [1] are presented in Table 2.

It can be seen from Table 2 that the proposed algorithm (2) succeeds in reducing the iterations compared with the Riccati iterations for different values of ρ, where

$$\begin{aligned}\left(A-\frac{\rho}{2}I_n\right)^{\mathrm{T}}X^{(k+1)}+X^{(k+1)}\left(A-\frac{\rho}{2}I_n\right)\\ -X^{(k+1)}SX^{(k+1)}+Q+\sigma C^{\mathrm{T}}X^{(k)}C\\ +\rho X^{(k)}=0,\quad k=1,2,\dots\,.\end{aligned}\tag{10}$$

Hence, Newton's method seems to be very reliable because the required iteration is small.

The conclusion of this discussion is as follows. Since the Riccati iterations [1] guarantee the convergence without stability or any other condition, the algorithm will be carried out without any constraint condition.

Table 1. The results for the error $\|\mathcal{F}(X^{(k)})\|$ per iterations

k	$\|\mathcal{F}(X^{(k)})\|$
0	2.1039e+01
1	4.2645e+02
2	9.2980e+01
3	1.4086e+01
4	6.8886e−01
5	2.1267e−03
6	2.1705e−08
7	1.6065e−11

Table 2. Number of iterations

ρ	Newton's method	Riccati iterations
0.5	7	30
1.0e−01	7	25
1.0e−02	7	24
1.0e−03	7	23
1.0e−04	7	23

This algorithm, however, requires the solution of the nonlinear algebraic equations at each iteration and its convergence speed is slow. The Lyapunov iterations consist of the linear equations and require much less computational efforts than previously used techniques, while the convergence rate is unclear. On the other hand, the proposed algorithm where Newton's method and the Hessenberg–Schur method are combined guarantees the quadratic convergence. Moreover, it attains the small dimension for computation. However the choice of the initial guess which satisfies the minimal solution is very difficult. Finally, the choice of these algorithms have to be taken into consideration by the reader's purpose adequately.

It is expected that Riccati iterations [1] can also be applied to the discrete-time systems. This problem will be addressed in future investigations. Fortunately, it should be noted that a numerical algorithm for solving the stochastic discrete algebraic Riccati equation has been studied in [8].

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Discussion on: "An Algorithm for Solving a Perturbed Algebraic Riccati Equation"

Zoran Gajic

Department of Electrical and Computer Engineering, Rutgers University, 94 Brett Road, Piscataway, NJ 08854-8058, USA

The paper [1] presents an algebraic Riccati equation (ARE) iterative scheme whose solutions monotonically converge from below to the minimal solution of the linearly perturbed algebraic Riccati equation (PARE), assuming that such a solution exists. The PARE (also known as generalized algebraic Riccati equation [2]) is defined by

$$A^{\mathrm{T}}X + XA - XSX + Q + F(X) = 0,$$

with $A, S, Q \in R^{n\times n}$ being real matrices and $F(X)$ representing a monotonically increasing operator in the positive semi-definite sense, $U \geq V \;\Rightarrow\; F(U) \geq F(V)$. Riccati equations of this type appear in several control theory areas such as optimal control of linear systems with state dependent noise [3], mean square exponential stability of linear stochastic systems [4], minimax control of switched systems under sampling [5], and optimal control of jump parameter linear stochastic systems [6]. In [5,6], the linearly coupled systems of algebraic Riccati equations (CARE) were derived. It was shown in [1] that the CARE can be treated as a special case of the PARE.

The algorithm of [1] for solving the above algebraic equation is based on ARE iterations of the form

$$\begin{aligned}&\left(A-\frac{\rho}{2}I\right)^{\mathrm{T}}X^{(i+1)} + X^{(i+1)}\left(A-\frac{\rho}{2}I\right)\\ &\quad - X^{(i+1)}SX^{(i+1)} + Q + F(X^{(i)}) + \rho X^{(i)} = 0,\\ &\quad X^{(0)} = 0\end{aligned}$$

where $\rho \geq 0$ is a parameter conveniently chosen to assure the solution's existence and uniqueness at each iteration. The most important feature of the presented algorithm is that the convergence to the sought solution is assured by a proper choice of the parameter p, which provides the initial step stabilizability–detectability condition, respectively, for the pairs $(A-0.5\rho I, S)$ and $(A-0.5\rho I, Q)$. This parameter can be always chosen since in the extreme case choosing ρ large enough can make the matrix $A-0.5\rho I$ asymptotically stable. The paper establishes that the initial stabilizability–detectability condition is preserved in each iteration step producing a monotonically non-decreasing sequence of the solutions of ARE iterations (Lemma 1). Assuming that the required solution of PARE exists, Lemma 2 shows that the monotonically non-decreasing sequence is bounded from above by that solution, which in fact represents the sequence converging point, that is $X^{(i)} \to X$, $i = 1, 2, 3, \ldots$. If the solution of the PARE does not exist, the monotonically non-decreasing sequence diverges, $X^{(\infty)} \to \infty$.

Note that stabilizability–detectability condition alone does not provide the existence condition of a positive semi-definite stabilizing solution of PARE [2,3,7]. In addition to stabilizability–observability of the pairs (A,S) and (A,Q) [7], a difficult to test, a matrix function norm condition is required to provide a sufficient condition for the existence of the unique positive definite stabilizing solution of the PARE (see [7, Theorem 4.1]). For the CARE, matrix function norm conditions and stabilizability conditions, imposed for every system mode, can be replaced by stochastic stabilizability condition, given in terms of coupled algebraic Lyapunov equations [8]. Another existence condition for the CARE [10], requires that Riccati inequalities have positive semidefinite solutions. None of these existence conditions is easy for testing.

It should be pointed out that ARE iterations were proposed for the first time in [9] to solve the considered PARE. Those ARE iterations are different from ARE iterations presented in [1]. The ARE iterations of [9] have the form

$$\begin{aligned}&A^{\mathrm{T}}X^{(i+1)} + X^{(i+1)}A - X^{(i+1)}SX^{(i+1)}\\ &\quad + Q + F(X^{(i)}) = 0, \quad X^{(0)} = 0.\end{aligned}$$

The existence condition here requires stabilizability–detectability, respectively, of the pairs (A,S) and (A,Q). In the extreme case, when $\rho \to 0$, the algorithms of [1] and [9] are equivalent. Even more, it was

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E-mail: gajic@ece.rutgers.edu

shown in [1] that the smaller ρ the faster convergence. Similarly, in [10], decoupled systems of ARE iterations were used to solve the CARE of jump parameter linear systems. As indicated in [1], the algorithm proposed in [1] can be used to solve the CARE, and for $\rho \to 0$, the algorithms of [1] and [10] are equivalent. Note that in [11], algebraic Lyapunov iterations (ALE) were used to solve the PARE, and in [12,13], ALE iterations were used to solve the CARE.

It should be emphasized that ARE iterations of [1] are more general than ARE iterations of [9], [10], and include those for $\rho \to 0$. In the case of ARE iterations of [1], an important degree of freedom is achieved by introducing the parameter ρ, which can be appropriately chosen to make the pair $(A - 0.5\rho I, S)$ stabilizable and the pair $(A - 0.5\rho I, Q)$ detectable.

In conclusion, the results of [1] represent an important contribution to the theory of generalized and linearly coupled algebraic Riccati equations.

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