THE WEYL-TYPE MATRIX AND THE METHOD OF SPECTRAL MAPPINGS IN THE INVERSE PROBLEM THEORY

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Abstract. A short review on inverse problems of spectral analysis for higher-order differential equations and systems on the half-line and on a finite interval is given. As the main spectral characteristics for the formulation and solution of inverse problems we use the so-called Weyl matrix, which is one of the possible generalizations of Weyl's classical m - function. Using the concept of the Weyl matrix and the method of spectral mappings we provide the solution of the inverse problem for non-selfadjoint differential equations and systems. We also obtain necessary and sufficient conditions for the solvability of the inverse problem.

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1. Introduction. This short review deals with the inverse spectral problem for the following two main objects:

(i) The higher-order differential equation

$$\ell y := y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y, \quad n > 2, \tag{1}$$

here $p_k(x)$ are complex-valued integrable functions, and

(ii) the system of differential equations

$$lY(x) := Q_0 Y'(x) + Q(x)Y(x) = \rho Y(x), \tag{2}$$

here $Y = [y_k]^T_{k=\overline{1,n}}$ is a column-vector (T is the sign for the transposition), $Q_0 = \mathrm{diag}[q_k]_{k=\overline{1,n}}$, $Q(x) = [q_{kj}(x)]_{k,j=\overline{1,n}}$, where q_k are complex numbers, $q_{kj}(x)$ are complex-valued integrable functions, and $q_{kk}(x) \equiv 0$, $k = \overline{1,n}$.

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences and engineering.

We will consider (1) and (2) on the half-line $x > 0$ or on a finite interval $x \in (0, b)$. As the main spectral characteristics for formulation and solution of inverse problems for (1) and (2) we introduce and study the so-called Weyl matrices which are generalizations of the classical Weyl function for the Sturm-Liouville operator. These matrices express most completely the spectral properties of the non-selfadjoint operators ℓ and l . The concept of the Weyl matrix and the use of the method of spectral mappings (see [1]) enable us to construct a general theory of the inverse problem for non-selfadjoint operators (1) and (2) when the behaviour of the spectrum is arbitrary.

We note that there is another generalization of the Weyl function of the Sturm-Liouville operator (see, for example, [2] and Remark 3 below), which is convenient for studying *direct* spectral problems for (1) and (2) in the selfadjoint case, namely for proving completeness and expansion theorems. However, for solving inverse problems this generalization turned out to be unsuitable, and the concept of the Weyl matrix is more natural and convenient to construct the inverse problem theory. We also note that the Weyl matrix was first introduced in [3]-[4] for differential operators (1) (for more details see [5]).

2. Inverse problems for higher-order differential equations. The greatest success in the inverse problem theory for equation (1) was achieved for $n = 2$ (i.e. for the Sturm-Liouville equation). The central role there was played by the transformation operator method (see [6], [7]) which allowed one to give a constructive procedure for the solution of the inverse problem as well as necessary and sufficient conditions for its solvability. Inverse Sturm-Liouville problems were studied in many works (see the monographs [6]-[11] and the references therein). We note that in [11, Ch.1] one can find a historical review on the inverse problems of spectral analysis for ordinary differential operators.

Unlike Sturm-Liouville operators, inverse problems for $n > 2$ turned out to be much more difficult for studying. For $n > 2$ transformation operators have a more complicated structure than for the Sturm-Liouville operator, and they do not work in the general case. However, in the case of analytic coefficients the transformation operators have the same "triangular" form as for Sturm-Liouville operators. Sakhnovich [12] and Khachatryan [13] used a "triangular" transformation operator to investigate the inverse problem of recovering selfadjoint differential operators on the half-line with *analytic* coefficients from the spectral function, as well as the inverse scattering problem.

A more effective and universal method in the inverse spectral theory is the method of spectral mappings (see [1]) connected with ideas of the contour integral method. This method enables us to construct a general theory of inverse problems for non-selfadjoint differential operators (1) and for many other classes of operators. In particular, in [5] the inverse problem of recovering operator (1) from its Weyl matrix was studied on the half-line and on a finite interval. The inverse scattering problem on the line for operator (1) has been treated in various settings in [14]-[17] and other works. We note that the use of the Riemann problem in the inverse scattering theory can be considered as a particular case of the method of spectral mappings (see, for example, [15]).

Let us briefly formulate the main results on the inverse problem for equation (1) on the half-line $x > 0$ (for more details see [5]).

The Weyl matrix. Consider the linear forms $U = [U_\xi]_{\xi=\overline{1,n}}$:

$$U_\xi(y) = y^{(\sigma_\xi)}(0) + \sum_{\nu=0}^{\sigma_\xi - 1} u_{\xi\nu} y^{(\nu)}(0), \quad \xi = \overline{1,n},$$

where $0 \le \sigma_\xi \le n-1$, $\sigma_\xi \ne \sigma_\eta$ for $\xi \ne \eta$. For studying the inverse problem we will use spectral properties of a chain of $n-1$ boundary value problems B_m , $m = \overline{1, n-1}$

for equation (1) with m boundary conditions at zero: $U_\xi(y)=0$, $\xi=\overline{1,m}$, and with boundary conditions at infinity on the growth of the solutions.

Let $\lambda=\rho^n$. It is known that the ρ- plane can be partitioned into sectors S_ν of angle $\frac{\pi}{n}$ $\Big(S_\nu := \{\rho:\ \arg\rho\in\left(\frac{\nu\pi}{n},\frac{(\nu+1)\pi}{n}\right)\},\ \nu=\overline{0,2n-1}\Big)$ in each of which the roots $R_1,R_2,\dots,R_n$ of the equation $R^n-1=0$ can be numbered in such a way that

$$Re(\rho R_1)<Re(\rho R_2)<\ldots<Re(\rho R_n),\quad \rho\in S_\nu. \tag{3}$$

Let the functions $\Phi_m(x,\lambda)$, $m=\overline{1,n}$ be the solutions of equation (1) satisfying the conditions $U_\xi(\Phi_m)=\delta_{\xi m}$, $\xi=\overline{1,m}$ and $\Phi_m(x,\lambda)=O(\exp(\rho R_m x))$, $x\to\infty$, $\rho\in S_\nu$ in each sector with property (3). Here and in the sequel, $\delta_{\xi m}$ is the Kronecker symbol. These conditions uniquely determine the solutions $\Phi_m(x,\lambda)$.

Denote $M_{mk}(\lambda)=U_k(\Phi_m)$, $k=\overline{m+1,n}$. The functions $M_{mk}(\lambda)$ are called *the Weyl-Yurko functions*, and the matrix

$$M(\lambda)=[M_{mk}(\lambda)]_{m,k=\overline{1,n}},$$

where $M_{mk}(\lambda)=\delta_{mk}$, $k=\overline{1,m}$, is called *the Weyl matrix* or the spectrum of L. We note that the Weyl matrix is a triangular matrix and $\det M(\lambda)=1$.

Formulation of the inverse problem. Given the Weyl matrix $M(\lambda)$, construct the differential expression ℓ and the linear forms U (i.e. construct the pair $L=(\ell,U)$).

Properties of the Weyl matrix. Let $\Gamma_\pm := \{\lambda:\ \pm\lambda\ge 0\}$, and let $\Pi_\pm$ be the λ-plane with a cut along $\Gamma_\pm$.

For definiteness it will be assumed below that $\sigma_\xi=n-\xi$.

Theorem 1. *The Weyl matrix $M(\lambda)$ has the following properties:*

1) $M_{mk}(\lambda)=\delta_{mk}$, $m\ge k$.

2) The Weyl functions $M_{mk}(\lambda)$ are analytic in $\Pi_{(-1)^{n-m}}$ with the exception of at most countable bounded sets Λ'_{mk} of poles and are continuous in $\bar{\Pi}_{(-1)^{n-m}}$ with the exception of bounded sets Λ_{mk}.

3) $M_{mk}(\lambda)=O(\rho^{m-k})$ as $|\lambda|\to\infty$.

4) The functions $M_{mk}(\lambda)-M_{m,m+1}(\lambda)M_{m+1,k}(\lambda)$ are analytic for $\lambda\in\Gamma_{(-1)^{n-m}}\setminus\Lambda$, where $\Lambda=\bigcup_{m,k}\Lambda_{mk}$.

Denote by W_ν the set of functions $f(x)$ such that $f(x),f'(x),\dots,f^{(\nu-1)}(x)$ are absolutely continuous and $f^{(k)}(x)\in L(0,\infty)$, $k=\overline{0,\nu}$. Let $N\ge 0$ be a fixed integer. We say that $L\in V_N$ if $p_\nu(x)\in W_{\nu+N}$, $\nu=\overline{0,n-2}$. We shall solve the inverse problem in the classes V_N. We define $p_n(x)=1$, $p_{n-1}(x)=0$, and $u_{\xi\nu}=\delta_{\nu,n-\xi}$, $\nu\ge n-\xi$. Let

$$\langle y(x),z(x)\rangle_\ell := \sum_{\nu,j=0}^{n-1}\mathcal{L}_{\nu j}(x)y^{(\nu)}(x)z^{(j)}(x),$$

$$\left.\begin{array}{c}\mathcal{L}_{\nu j}(x)=\displaystyle\sum_{s=j}^{n-\nu-1}(-1)^s\binom{s}{j}p_{s+\nu+1}^{s-j}(x),\quad \nu+j\le n-1,\\ \mathcal{L}_{\nu j}(x)=0,\quad \nu+j>n-1.\end{array}\right\}$$

We consider the differential equation and the linear forms

$$\ell^* z := (-1)^n z^{(n)} + \sum_{\nu=0}^{n-2} (-1)^\nu (p_\nu(x) z)^{(\nu)} = \lambda z, \tag{4}$$

$$U_\xi^*(z) = z^{(n-\xi)}(0) + \sum_{\nu=0}^{n-\xi-1} u_{\xi\nu}^* z^{(\nu)}(0), \quad \xi = \overline{1,n},$$

where the linear forms U_ξ^* are determined from the relation

$$\langle y, z\rangle_{\ell\,|x=0} = \sum_{k=1}^{n} (-1)^{k-1} U_k(y) U_{n-k+1}^*(z).$$

Notice that $\bar{\ell}^*$ is the adjoint of ℓ. It is clear that for any sufficiently smooth functions $y(x)$ and $z(x)$

$$\ell y\, z - y\, \ell^* z = \frac{d}{dx} \langle y, z\rangle_\ell\,.$$

Solution of the inverse problem. First we formulate the uniqueness theorem for the inverse problem considered. For this purpose together with $L = (\ell, U)$ we consider a pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$ of the same form but with different coefficients. Everywhere below if a symbol α denotes an object related to L, then $\tilde{\alpha}$ denotes the analogous object related to $\tilde{L}$.

Theorem 2. *If* $M(\lambda) = \tilde{M}(\lambda)$, *then* $L = \tilde{L}$.

Thus, the specification of the Weyl matrix $M(\lambda)$ determines the differential equation and the linear forms uniquely. A counterexample [5, Ch.1] shows that dropping one element of the Weyl matrix violates the uniqueness of the solution of the inverse problem.

The central role for the solution of the inverse problem is played by the so-called *main equation* of the inverse problem (see (5) below), which is a linear integral singular equation. We give a derivation of the main equation and prove its solvability in a suitable Banach space. Using the main equation we give a constructive procedure for the solution of the inverse problem along with necessary and sufficient conditions for its solvability.

Let $M(\lambda)$ be the Weyl matrix for the pair $L = (\ell, U)$. Let $\tilde{L} = (\tilde{\ell}, \tilde{U})$ be a known pair. In the λ - plane we consider the contour $\gamma = \gamma_{-1} \cup \gamma_0 \cup \gamma_1$ (with a counterclockwise circuit), where γ_0 is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ (i.e. $\Lambda \cup \tilde{\Lambda} \cup \{0\} \subset \mathrm{int}\gamma_0$), and $\gamma_{\pm 1}$ is a two-sided cut along the ray $\{\lambda :\ \pm\lambda > 0,\ \lambda \notin \mathrm{int}\gamma_0\}$. Denote $J_\gamma = \{\lambda :\ \lambda \notin \gamma \cup \mathrm{int}\gamma_0\}$. Let the functions $\Phi_m^*(x, \lambda)$, $m = \overline{1,n}$ be the solutions of equation (4) under the conditions $U_\xi^*(\Phi_m^*) = \delta_{\xi m}$, $\xi = \overline{1,m}$, and $\Phi_m^*(x, \lambda) = O(\exp(\rho R_m^* x))$, $x \to \infty$, $\rho \in S$, where $R_m^* = -R_{n-m+1}$. Denote

$$Y = [\delta_{j,k-1}]_{j=\overline{1,n-1}, k=\overline{1,n}}, \quad I_{n-1} = [\delta_{jk}]_{j,k=\overline{1,n-1}}, \quad A_0(\lambda) = (M(\lambda) - \tilde{M}(\lambda)) \tilde{M}^{-1}(\lambda),$$

and for $\lambda \in \gamma$ set

$$Q(\lambda) = I_{n-1} - \frac{1}{2}\, \chi_{+1}(\lambda) \chi_{-1}(\lambda) Y A_0(\lambda) Y^T,$$

$$\varphi(x, \lambda) = [\chi((-1)^{n-k+1}\lambda) \Phi_k(x, \lambda)]_{k=\overline{2,n}}, \quad g(x, \lambda) = [(-1)^{k-1} \tilde{\Phi}_{n-k+1}^*(x, \lambda)]_{k=\overline{1,n}}^T A(\lambda) Y^T,$$

where $\chi_{\pm 1}(\lambda)=1$ for $\lambda\in\gamma_0\cup\gamma_{\pm 1}$, $\chi_{\pm 1}(\lambda)=0$ for $\lambda\in\gamma_{\mp 1}$, and

$$A(\lambda)=\begin{cases} A_0(\lambda) & \lambda\in\gamma_0,\\ [\delta_{j,\nu-1}\chi_{(-1)^{n-j}}(\lambda)(M_{j,j+1}(\lambda)-\tilde M_{j,j+1}(\lambda)]_{j,\nu=\overline{1,n}} & \lambda\in\gamma_1\cup\gamma_{-1}\,.\end{cases}$$

Theorem 3. *For each fixed* $x\ge 0$, *the vector* $\varphi(x,\lambda)$ *is a solution of the linear integral equation*

$$\tilde\varphi(x,\lambda)=Q(\lambda)\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \frac{H(x,\lambda,\mu)}{\mu-\lambda}\varphi(x,\mu)\,d\mu,\quad \lambda\in\gamma, \tag{5}$$

where $H(x,\lambda,\mu)=\langle\tilde\varphi(x,\lambda),g(x,\mu)\rangle_{\tilde\ell}$.

Equation (5) is called the *main equation* of the inverse problem.

Denote $\Omega(x,\lambda)=\mathrm{diag}\,[\rho^{n-k}\exp(-\rho R_k x)]_{k=\overline{2,n}}$, $\gamma''=\{\lambda:\ \lambda\in\gamma_1\cup\gamma_{-1},\ d(\lambda,\gamma_0)\ge\alpha_0>0\}$, $\gamma'=\gamma\setminus\gamma''$, where $d(\lambda,\gamma_0):=\inf|\lambda-\mu|$, $\mu\in\gamma_0$. Thus, $\gamma=\gamma'\cup\gamma''$. We introduce the Banach space $B=L_2^{n-1}(\gamma')\oplus L_\infty^{n-1}(\gamma'')$ of vector-valued functions $z(\lambda)=[z_j(\lambda)]_{j=\overline{1,n-1}}$, $\lambda\in\gamma$ with the norm

$$\|z\|_B=\sum_{j=1}^{n-1}\Big(\|z_j\|_{L_2(\gamma')}+\|z_j\|_{L_\infty(\gamma'')}\Big).$$

Now we establish the unique solvability of the main equation.

Theorem 4. *For each fixed* $x\ge 0$, *the main equation (5) has a unique solution in the class* $\Omega(x,\lambda)\varphi(x,\lambda)\in B$. *Moreover,* $\sup_x\|\Omega(x,\lambda)\varphi(x,\lambda)\|_B<\infty$.

Let us denote

$$\kappa_{\nu s}(x)=\frac{1}{2\pi i}\int_\gamma \tilde g^{(\nu)}(x,\mu)\varphi^{(s)}(x,\mu)\,d\mu,\quad \nu+s\le n-1,$$

$$\left.\begin{array}{c} t_{j\nu}(x)=-\displaystyle\sum_{\beta=\nu+1}^{j}\binom{j}{\beta}\binom{\beta-1}{\nu}\kappa_{\beta-\nu-1,j-\beta}(x),\quad j>\nu,\\ t_{j\nu}(x)=\delta_{j\nu},\quad j\le\nu;\quad j,\nu=\overline{0,n}, \end{array}\right\}$$

$$\xi_\nu(x)=\sum_{s=0}^{n-\nu-1}\sum_{j=\nu+1}^{n-s}\Big(\binom{j+s}{j}\binom{j-1}{\nu}\tilde p_{j+s}(x)\kappa_{j-\nu-1,s}(x)$$

$$+\delta_{s0}(-1)^{j-\nu}\sum_{r=0}^{j-\nu-1}\binom{j-\nu-1}{r}\tilde p_j^{(j-\nu-1-r)}(x)\kappa_{r0}(x)\Big),\quad \nu=\overline{0,n-2},$$

$$\varepsilon_\nu(x)=\xi_\nu(x)-\sum_{j=\nu+1}^{n-2}\varepsilon_j(x)t_{j\nu}(x),\quad \nu=\overline{0,n-2}. \tag{6}$$

The following theorem gives us explicit formulae for calculating the coefficients of the differential equation and the linear forms $L=(\ell,U)$.

Theorem 5. *The following relations hold*

$$p_\nu(x) = \tilde{p}_\nu(x) + \varepsilon_\nu(x), \quad \tilde{u}_{\xi\nu} = \sum_{j=0}^{n-1} u_{\xi j} t_{j\nu}(0). \tag{7}$$

Thus, we arrive at the following algorithm for the solution of the inverse problem.

Algorithm 1. Let the Weyl-Yurko matrix $M(\lambda)$ for the pair $L = (\ell, U)$ be given.
1) Choose $\tilde{L} = (\tilde{\ell}, \tilde{U})$.
2) Construct the matrices $H(x,\lambda,\mu)$, $Q(\lambda)$, $\tilde{\varphi}(x,\lambda)$, $x \geq 0$, $\lambda, \mu \in \gamma$.
3) Find $\varphi(x,\lambda)$, $x \geq 0$, $\lambda \in \gamma$ by solving the main equation (5).
4) Calculate $\varepsilon_\nu(x)$ by (6).
5) Construct $L = (\ell, U)$ by (7).

Now let us state necessary and sufficient conditions for the solvability of the inverse problem considered. For definiteness, let $\arg\rho \in (0, 2\pi/n)$. We denote by $\mathcal{M}$ the set of matrices $M(\lambda)$ having properties 1-4 of Theorem 1.

Theorem 6. *A matrix $M(\lambda) \in \mathcal{M}$ is the Weyl matrix for a certain $L \in V_N$ if and only if the following conditions hold:*
1) (Asymptotics) There exists $\tilde{L} \in V_N$ such that $M_{m,m+1}(\lambda) - \tilde{M}_{m,m+1}(\lambda) = O(\rho^{-n-2})$ as $|\lambda| \to \infty$.
2) (Condition S) For each fixed $x \geq 0$, equation (5) has a unique solution in the class $\Omega(x,\lambda)\varphi(x,\lambda) \in B$, and $\sup_x \|\Omega(x,\lambda)\varphi(x,\lambda)\|_B < \infty$.
3) $\varepsilon_\nu(x) \in W_{\nu+N}$, $\nu = \overline{0, n-2}$, where the functions $\varepsilon_\nu(x)$ are defined by (6).
Under these conditions the differential equation and the linear forms $L = (\ell, U)$ are constructed via (7).

Connection with the Gelfand-Levitan equation. Let us consider the Sturm-Liouville selfadjoint equation

$$-y'' + q(x)y = \lambda y, \ x > 0 \tag{8}$$

with the condition $y'(0) - hy(0) = 0$, where $q(x) \in L(0,\infty)$ and h are real. Let $\varphi(x,\lambda)$ be a solution of (8) under the conditions $\varphi(0,\lambda) = 1$ and $\varphi'(0,\lambda) = h$, and let $\sigma(\lambda)$ be the spectral function. Then the main equation of the inverse problem, after contracting the contour to the real axis, becomes

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int_{-\infty}^{\infty} \Big(\int_0^x \tilde{\varphi}(t,\lambda)\tilde{\varphi}(t,\mu)\, dt \Big) \varphi(x,\mu)\, d\tau(\mu), \tag{9}$$

where $\tau(\lambda) = \sigma(\lambda) - \tilde{\sigma}(\lambda)$. Let for definiteness, $\tilde{q}(x) = \tilde{h} = 0$. Then $\tilde{\varphi}(x,\lambda) = \cos\sqrt{\lambda}t$. ¿From (9), using the cosine Fourier transform and the representation

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x G(x,t)\cos\sqrt{\lambda}t\, dt,$$

we obtain the Gelfand-Levitan equation

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \quad 0 < t < x,$$

$$F(x,t)=\int_{-\infty}^{\infty}\cos\sqrt{\mu}x\cos\sqrt{\mu}t\,d\tau(\mu).$$

Remark 1. One of the conditions in Theorem 6 under which an arbitrary matrix $M(\lambda)$ is the Weyl matrix for a non-selfadjoint differential operator (1) is the requirement of the unique solvability of the main equation. It is difficult to verify this condition in the general case. Therefore, an important problem is that of obtaining sufficient conditions for the solvability of the main equation, and the extraction of classes of operators for which unique solvability can be proved. One of such classes is the class of selfadjoint operators. In [18] this class was investigated, and the unique solvability of the main equation was proved. We also note that if only the discrete spectrum is perturbed, then the main equation becomes a linear algebraic system, and condition S is the condition that the determinant of this system is nonzero.

Remark 2. Similar arguments can be used for studying the inverse problem for equation (1) on a finite interval $x\in(0,b)$; for details see [5, Ch.1]. Here we only show how to introduce the Weyl matrix and how to formulate the inverse problem. For this purpose together with the linear forms $U_{\xi 0}$ we consider the linear forms $U_{\xi 1}$:

$$U_{\xi 1}(y)=y^{(\tau_\xi)}(b)+\sum_{\nu=0}^{\tau_\xi-1}u_{\xi\nu 1}y^{(\nu)}(b),\quad \xi=\overline{1,n},$$

where $0\le\tau_\xi\le n-1$, $\tau_\xi\ne\tau_\eta$ for $\xi\ne\eta$. Let the functions $\Phi_m(x,\lambda),\ m=\overline{1,n}$ be the solutions of equation (1) satisfying the conditions $U_\xi(\Phi_m)=\delta_{\xi m}$, $U_{\eta 1}(\Phi_m)=0$, $\xi=\overline{1,m},\ \eta=\overline{1,n-m}$. Denote $M_{mk}(\lambda)=U_k(\Phi_m),\quad k=\overline{m+1,n}$. The matrix

$$M(\lambda)=[M_{mk}(\lambda)]_{m,k=\overline{1,n}},$$

where $M_{mk}(\lambda)=\delta_{mk},\quad k=\overline{1,m}$, is called *the Weyl matrix* for (1) on a finite interval. The inverse problem is formulated as follows: given the Weyl matrix, construct the coefficients of the differential system and the linear forms. The solution of this inverse problem along with necessary and sufficient conditions of its solvability see in [1] and [5].

Remark 3. We mentioned above that there is another generalization of the Weyl function for the Sturm-Liouville equation. Let us compare the concept of the Weyl matrix introduced above and the concept of the Weyl matrix from [2, Sect. 21.4] (where a slightly different notation is used). For simplicity, let $n=4$ and $U_\xi(y)=y^{(\xi-1)}(0)$. Let the functions $\Phi_k(x,\lambda),\ k=\overline{1,4}$ be solutions of equation (1) for $n=4$ under the conditions $\Phi_k^{(\nu-1)}(0,\lambda)=\delta_{\nu k}$, $\nu=\overline{1,k}$, $\Phi_k(x,\lambda)=O(\exp(\rho R_k x)),\quad x\to\infty,\quad \rho\in S_\mu$ in each sector with property (3), and let $M_{k\nu}(\lambda)=\Phi_k^{(\nu-1)}(0,\lambda)$, $\nu>k$. Thus,

$$\Phi_1(0,\lambda)=1,\quad \Phi_1(x,\lambda)=O(\exp(\rho R_1 x)),\ x\to\infty,$$

$$\Phi_2(0,\lambda)=0,\ \Phi_2'(0,\lambda)=1,\quad \Phi_2(x,\lambda)=O(\exp(\rho R_2 x)),\ x\to\infty,$$

$$\Phi_3(0,\lambda)=\Phi_3'(0,\lambda)=0,\ \Phi_3''(0,\lambda)=1,\quad \Phi_3(x,\lambda)=O(\exp(\rho R_3 x)),\ x\to\infty,$$

$$\Phi_4(0,\lambda)=\Phi_4'(0,\lambda)=\Phi_4''(0,\lambda)=0,\ \Phi_4'''(0,\lambda)=1,$$

$$M_{12}(\lambda)=\Phi_1'(0,\lambda),\ M_{13}(\lambda)=\Phi_1''(0,\lambda),\ M_{14}(\lambda)=\Phi_1'''(0,\lambda),$$

$$M_{23}(\lambda)=\Phi_2''(0,\lambda),\ M_{24}(\lambda)=\Phi_2'''(0,\lambda),$$
$$M_{34}(\lambda)=\Phi_3'''(0,\lambda),$$

and the Weyl-Yurko matrix $M(\lambda)$ has the form

$$M(\lambda)=\begin{bmatrix} 1 & M_{12}(\lambda) & M_{13}(\lambda) & M_{14}(\lambda) \\ 0 & 1 & M_{23}(\lambda) & M_{24}(\lambda) \\ 0 & 0 & 1 & M_{34}(\lambda) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly,

$$\begin{bmatrix} \Phi_1(x,\lambda) \\ \Phi_2(x,\lambda) \\ \Phi_3(x,\lambda) \\ \Phi_4(x,\lambda) \end{bmatrix}=\begin{bmatrix} 1 & M_{12}(\lambda) & M_{13}(\lambda) & M_{14}(\lambda) \\ 0 & 1 & M_{23}(\lambda) & M_{24}(\lambda) \\ 0 & 0 & 1 & M_{34}(\lambda) \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \\ C_3(x,\lambda) \\ C_4(x,\lambda) \end{bmatrix},$$

where $C_k(x,\lambda),\ k=\overline{1,4}$ are solutions of (1) for $n=4$ under the initial conditions $C_k^{(\nu-1)}(0,\lambda)=\delta_{\nu k}$, $\nu,k=\overline{1,4}$. For the functions $\Phi_k(x,\lambda)$ we have the scale of growth: the smallest exponents is for Φ_1 , the biggest one is for Φ_4. Moreover, Φ_1 and Φ_2 vanish at infinity, and Φ_3 and Φ_4 do not.

Let now $\psi_k(x,\lambda),\ k=\overline{1,2}$ be solutions of equation (1) for $n=4$ under the conditions

$$\psi_1(0,\lambda)=0,\ \psi_1'(0,\lambda)=-1,\ \psi_1(x,\lambda)\in L_2(0,\infty),$$

$$\psi_2(0,\lambda)=1,\ \psi_2'(0,\lambda)=0,\ \psi_2(x,\lambda)\in L_2(0,\infty),$$

and let

$$m_{11}(\lambda)=\psi_1''(0,\lambda),\quad m_{12}(\lambda)=\psi_1'''(0,\lambda),$$
$$m_{21}(\lambda)=\psi_2''(0,\lambda),\quad m_{22}(\lambda)=\psi_2'''(0,\lambda).$$

The matrix

$$m(\lambda)=\begin{bmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{bmatrix}$$

is called the m - matrix; see [2] for more details. Clearly,

$$\begin{bmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{bmatrix}=\begin{bmatrix} -C_2(x,\lambda) \\ C_1(x,\lambda) \end{bmatrix}+m(\lambda)\begin{bmatrix} C_3(x,\lambda) \\ C_4(x,\lambda) \end{bmatrix},$$

and

$$\psi_1(x,\lambda)=-\Phi_2(x,\lambda),\quad \psi_2(x,\lambda)=\Phi_1(x,\lambda)-M_{12}(\lambda)\Phi_2(x,\lambda),$$
$$m_{11}(\lambda)=-M_{23}(\lambda),\quad m_{12}(\lambda)=-M_{24}(\lambda),$$
$$m_{21}(\lambda)=M_{13}(\lambda)-M_{12}(\lambda)M_{23}(\lambda),\quad m_{22}(\lambda)=M_{14}(\lambda)-M_{12}(\lambda)M_{24}(\lambda).$$

In the selfadjoint case the specification of the m - matrix $m(\lambda)$ is equivalent to the specification of the spectral matrix $\sigma(\lambda)$, and

$$m(\lambda)=\int_{-\infty}^{\infty}\frac{d\sigma(\mu)}{\mu-\lambda}.$$

The spectral matrix $\sigma(\lambda)$ and the m - matrix $m(\lambda)$ are used for studying *direct* problems of spectral analysis for equation (1) in the self-adjoint case (see [2] for more details), but they are not suitable for solving general inverse problems. Moreover, the specification of the m - matrix does not uniquely determine the operator. For studying inverse problems the concept of the Weyl matrix introduced above is more natural (see [1] and [5] for more details). Similar arguments are used for the case of a finite interval.

We mention that for the selfadjoint case there are the following connections between $M_{kj}(\lambda)$:

$$M_{12}(\lambda) = M_{34}(\lambda), \quad M_{24}(\lambda) = -M_{13}(\lambda) + M_{12}(\lambda)M_{23}(\lambda).$$

3. Inverse problem for systems of differential equations. Many papers are devoted to inverse problems for systems of differential equations (see [10], [19]-[30] and the references therein). Some systems can be treated similarly to the Sturm-Liouville operator. As example we mention inverse problems for Dirac and AKNS systems and their generalizations ([10], [21], [23], [24], [26], [27]). For such systems the transformation operator method can be used for solving the inverse problem. The transformation operator method can be also used for systems with analytic potentials similarly to higher-order differential operators (see, for example, [25]). But in the general case, the inverse problems for systems of the form (2) deal with more serious difficulties like those for operators (1) with integrable coefficients. There are only a number of works where inverse problems for such systems are studied. We mark the important works [19], [20], [28] in which by using the Riemann problem the inverse scattering problem for system (2) on the line was studied. The inverse problem theory for system (2) on the half-line and on a finite interval in the general case has not been constructed yet.

In this paper we study the inverse problem for non-selfadjoint system (2) on the half-line in the general case, i.e. with arbitrary roots of the characteristic polynomial and with arbitrary behavior of the spectrum. As the main spectral characteristics for (2) we introduce and study the so-called Weyl matrix which is an analog of the Weyl matrix introduced in Section 2 for operators (1). Developing the method of spectral mappings we give a constructive procedure for the solution of the inverse problem of recovering system (2) from the given Weyl matrix, prove the uniqueness theorem and provide necessary and sufficient conditions for the solvability of this inverse problem.

Let us consider system (2) on the half-line $x > 0$. The matrix $Q(x)$ is called the potential. We shall say that $l \in V_N$ if $q_{kj}(x) \in W_N$, $k, j = \overline{1,n}$. Below we will consider the operator l in the classes V_N, $N \geq 1$. Let for definiteness $q_k \neq 0$ and $q_k \neq q_j$ for $k \neq j$, $k, j = \overline{1,n}$. Denote $\beta_k = 1/q_k$. It is known that the ρ - plane can be partitioned into sectors $S_j = \{\rho : \ \arg\rho \in (\theta_j, \theta_{j+1})\}$, $j = \overline{0, 2r-1}$, $0 \leq \theta_0 < \theta_1 < \ldots < \theta_{2r-1} \leq 2\pi$ in which there exist permutations $i_k = i_k(S_j)$ of the numbers $1, \ldots, n$, such that for the numbers $R_k = R_k(S_j)$ of the form $R_k = \beta_{i_k}$ one has

$$\mathrm{Re}(\rho R_1) < \ldots < \mathrm{Re}(\rho R_n), \quad \rho \in S_j. \tag{10}$$

Let a matrix $h = [h_{\xi\nu}]_{\xi,\nu=\overline{1,n}}$, $\det h \neq 0$, be given, where $h_{\xi\nu}$ are complex numbers. We introduce the linear forms $U(Y) = [U_\xi(Y)]^T_{\xi=\overline{1,n}}$ by the formula $U(Y) = hY(0)$, i.e. $U_\xi(Y) = [h_{\xi 1}, \ldots, h_{\xi n}]Y(0)$. Denote $\Omega^0_{mk}(j_1, \ldots, j_m) = \det[h_{\xi, j_\nu}]_{\xi=\overline{1,m-1},k;\ \nu=\overline{1,m}}$. Suppose that

$$\Omega^0_{mm}(i_1,\ldots,i_m)\neq 0,\quad m=\overline{1,n-1},\quad ,j=\overline{0,2r-1}, \tag{11}$$

where $i_k=i_k(S_j)$ is the above-mentioned permutation for the sector S_j . Condition (11) is called the *regularity condition* for the pair $L=(l,U)$. Systems, which do not satisfy the regularity condition, possess qualitatively different properties for investigating inverse problems, and are not considered in this paper. Without loss of generality we assume that the following normalizing conditions are fulfilled: $\det h=1$, and for a fixed sector (for definiteness, for the sector S_0) one has $\Omega^0_m(i_1,\ldots,i_m)=1$, $m=\overline{1,n-1}$. Everywhere below we shall assume that the regularity conditions and the normalizing conditions are fulfilled.

The Weyl matrix. Let the vector-functions $\Phi_m(x,\rho)=[\Phi_{km}(x,\rho)]^T_{k=\overline{1,n}}$, $m=\overline{1,n}$ be the solutions of (2) satisfying the conditions $U_\xi(\Phi_m)=\delta_{\xi m}$, $\xi=\overline{1,m}$, and also $\Phi_m(x,\rho)=O(\exp(\rho R_m x))$, $x\to\infty$, $\rho\in S_j$ in each sector S_j with property (10). These conditions uniquely determine the solutions $\Phi_m(x,\rho)$. Denote $M_{m\xi}(\rho)=U_\xi(\Phi_m)$, $\xi>m$, $M(\rho)=[M_{m\xi}(\rho)]_{m,\xi=\overline{1,n}}$, $M_{m\xi}(\rho)=\delta_{\xi m}$ for $\xi\le m$, $\Phi(x,\rho)=[\Phi_1(x,\rho),\ldots,\Phi_n(x,\rho)]=[\Phi_{km}(x,\rho)]_{k,m=\overline{1,n}}$. The functions $\Phi_m(x,\rho)$ and $M_{m\xi}(\rho)$ are called the Weyl solutions and the Weyl functions respectively. The matrix $M(\rho)$ is called the Weyl matrix or the spectrum of $L=(l,U)$. We note that the Weyl matrix is a triangular matrix and $\det M(\rho)\equiv 1$.

Formulation of the inverse problem. Fix Q_0, i.e. the numbers $\beta_k=1/q_k$, $k=\overline{1,n}$, are known and fixed. The inverse problem is formulated as follows: Given the Weyl matrix $M(\rho)$, construct the pair $L=(l,U)$.

Properties of the Weyl matrix. Denote $\Phi^*(x,\rho)=(Q_0\Phi(x,\rho))^{-1}$, $\mathcal{N}(\rho)=M^T(\rho)$, $\mathcal{N}^*(\rho)=(\mathcal{N}(\rho))^{-1}$. For $\xi=\overline{0,n-2}$ we introduce functions $B^\xi_{mk}(\rho)$, $m=\overline{1,n-\xi-1}$, $k=\overline{m+\xi+1,n}$ by the recurrence formulae:

$$B^0_{mk}(\rho)=M_{mk}(\rho),\quad B^\xi_{mk}(\rho)=B^{\xi-1}_{mk}(\rho)-B^{\xi-1}_{m,m+\xi}(\rho)B^0_{m+\xi,k}(\rho).$$

Denote $\Gamma_j=\{\rho:\ \arg\rho=\theta_j\}$, $j=\overline{0,2r-1}$, $\Gamma_{2r}:=\Gamma_0$. We cut the ρ - plane along the rays Γ_j and denote by $\Gamma^\pm_j=\{\rho:\ \arg\rho=\theta_j\pm 0\}$ the sides of the cuts. We put $\overline{S}_j=S_j\cup\Gamma^+_j\cup\Gamma^-_{j+1}$ and denote by $\Sigma=\mathbf{C}\setminus\Big(\bigcup\limits_{j=0}^{2r-1}\Gamma_j\Big)=\bigcup\limits_{j=0}^{2r-1}S_j$ the ρ - plane without the cuts along the rays Γ_j , and denote by $\overline{\Sigma}=\bigcup\limits_{j=0}^{2r-1}\bar{S}_j$ the closure of Σ (we distiunguish between the sides of the cuts).

Fix $j=\overline{0,2r-1}$. Notice that for $\rho\in\Gamma_j$ strict inequalities in (10) in some places become equalities. Let $m_i=m_i(j)$, $p_i=p_i(j)$, $i=\overline{1,s}$ be such that for $\rho\in\Gamma_j$:

$$\mathrm{Re}(\rho R_{m_i-1})<\mathrm{Re}(\rho R_{m_i})=\ldots=\mathrm{Re}(\rho R_{m_i+p_i})<\mathrm{Re}(\rho R_{m_i+p_i+1}),\ i=\overline{1,s},$$

where $R_k=R_k(S_j)$. Denote $N_j:=\{m:\ m=\overline{m_1,m_1+p_1-1},\ldots,\overline{m_s,m_s+p_s-1}\}$, $J_m:=\{j:\ m\in N_j\}$, $\gamma_m=\bigcup\limits_{j\in J_m}\Gamma_j$, and let $\Sigma_m=\mathbf{C}\setminus\gamma_m$ be the ρ - plane without the cuts along the rays from γ_m , and let $\overline{\Sigma}_m$ be the closure of Σ_m (we distinguish between the sides of the cuts). Clearly, the domain $\Sigma_m = \bigcup_\nu S_{m\nu}$ consist of sectors $S_{m\nu}$, each of which is a union of several sectors S_j with the same collection $\{R_\xi\}_{\xi=\overline{1,m}}$.

Denote by $\mathcal{M}$ the set of functions $M(\rho) = [M_{mk}(\rho)]_{m,k=\overline{1,n}}$ with the properties:
1) $M_{mk}(\rho) \equiv \delta_{mk}$ for $m \ge k$;
2) the functions $M_{mk}(\rho)$, $k > m$ are analytic in Σ_m with the exeption of an at most countable bounded set Λ'_m of poles and are continuous in $\overline{\Sigma}_m$ with the exeption of a bounded set Λ_m (in general the sets Λ'_m and Λ_m are different for each matrix $M(\rho)$ from $\mathcal{M}$);
3) the functions $B^{m-\nu}_{\nu k}(\rho)$ are analytic on $\Gamma_j \setminus \Lambda'_m$ for $j \notin J_m$, $1 \le \nu \le m \le n-1$, $m+1 \le k \le n$.
4) For $|\rho| \to \infty$, $\rho \in \overline{S}_j$,

$$M_{mk}(\rho) = \mu^0_{mk}(S_j) + O(\rho^{-1}), \quad \mu^0_{mk}(S_j) = \frac{\Omega^0_{mk}(i_1, \dots, i_m)}{\Omega^0_{mm}(i_1, \dots, i_m)}, \quad i_k = i_k(S_j).$$

Theorem 7. *If $M(\rho)$ is the Weyl matrix for $L = (l, U)$, then $M(\rho) \in \mathcal{M}$.*

Solution of the inverse problem. Let us first formulate the uniqueness theorem.

Theorem 8. *The specification of the Weyl matrix $M(\rho)$ uniquely determines the potential $Q(x)$ and the matrix h.*

We note that the matrix h is uniquely determined by the specification of the numbers $\mu^0_{mk}(S_j)$ for all m, k, j.

Let us now give a *constructive procedure* for the solution of the inverse problem along with necessary and sufficient conditions for its solvability. For this purpose together with $L = (l, U)$ we consider a pair $\tilde{L} = (\tilde{l}, \tilde{U})$ of the same form but with different matrices $\tilde{Q}, \tilde{h}$ (we remind that the matrix Q_0 is known a priori and fixed). As before everywhere below if a symbol α denotes an object related to L, then $\tilde{\alpha}$ denotes the analogous object related to $\tilde{L}$, Let a pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$ be taken such that

$$M(\rho) - \tilde{M}(\rho) = O(\rho^{-1}), \quad |\rho| \to \infty, \tag{12}$$

i.e. $\mu^0_{mk}(S_j) = \tilde{\mu}^0_{mk}(S_j)$ for all m, k, j. We note that condition (12) is equivalent to the condition $h = \tilde{h}$. For example, one can take $\tilde{Q}(x) \equiv 0$.

In the ρ - plane we consider the contour $\omega = \omega^0 \cup \omega^+ \cup \omega^-$ (with a counterclockwise circuit), where ω^0 is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ (i.e. $\Lambda \cup \tilde{\Lambda} \cup \{0\} \subset \operatorname{int}\omega^0$), and $\omega^\pm = \bigcup_{j=0}^{2r-1} \omega^\pm_j$, $\omega^\pm_j = \Gamma^\pm_j \setminus \operatorname{int}\omega^0$.

We define the matrices $A(\rho) = [a_{mk}(\rho)]_{m,k=\overline{1,n}}$ and $\tilde{A}(\rho) = [\tilde{a}_{mk}(\rho)]_{m,k=\overline{1,n}}$ by the formulae $A(\rho) = \mathcal{N}^*(\rho)(\mathcal{N}(\rho) - \tilde{\mathcal{N}}(\rho))$, $\tilde{A}(\rho) = \tilde{\mathcal{N}}^*(\rho)(\mathcal{N}(\rho) - \tilde{\mathcal{N}}(\rho))$, where $\mathcal{N}(\rho) = (M(\rho))^T$, $\mathcal{N}^*(\rho) = (M^*(\rho))^T$.

Fix $j = \overline{0, 2r-1}$. Let $N_j = \{\overline{m_1, m_1 + p_1 - 1}, \dots, \overline{m_s, m_s + p_s - 1}\}$, $m_i - 1, m_i + p_i \notin N_j$, $i = \overline{1, s}$. Consider the matrices $A^{(j)}(\rho) = [a^{(j)}_{k\xi}(\rho)]_{k,\xi=\overline{1,n}}$, $\tilde{A}^{(j)}(\rho) = [\tilde{a}^{(j)}_{k\xi}(\rho)]_{k,\xi=\overline{1,n}}$, $\rho \in \Gamma^\pm_j$, where $a^{(j)}_{k\xi}(\rho) = a_{k\xi}(\rho)$, $\tilde{a}^{(j)}_{k\xi}(\rho) = \tilde{a}_{k\xi}(\rho)$ for $m_i \le \xi < k \le m_i + p_i$, $i = \overline{1, s}$, and $a^{(j)}_{k\xi}(\rho) = \tilde a^{(j)}_{k\xi}(\rho) = 0$ otherwise. For $\rho \in \omega$ we introduce the matrices $A_0(\rho)$ and $\tilde A_0(\rho)$ by the formulae

$$A_0(\rho) = \begin{cases} A^{(j)}(\rho), & \rho \in \omega_j^{\pm}, \\ A(\rho), & \rho \in \omega^0, \end{cases} \qquad \tilde A_0(\rho) = \begin{cases} \tilde A^{(j)}(\rho), & \rho \in \omega_j^{\pm}, \\ \tilde A(\rho), & \rho \in \omega^0. \end{cases}$$

In the contour ω it is convenient to stick together the sides of the cuts. Therefore, in the ρ - plane we consider the contour $\omega^* := \omega^0 \cup \omega^1$, where $\omega^1 = \bigcup\limits_{j=0}^{2r-1} \omega_j^1$, $\omega_j^1 := \{\rho : \rho \in \Gamma_j \setminus \omega^0\}$ (with the orientation towards the growth of the modulus of ρ). For $\rho \in \omega^*$ we define the matrices $\varphi(x,\rho)$, $\tilde\varphi(x,\rho)$, $\tilde G(x,\rho)$, $\tilde S(\rho)$, $\tilde r(x,\mu,\rho)$ $D(x,\rho)$ by the formulae

$$\varphi(x,\rho) = \begin{cases} [\Phi^+(x,\rho), \Phi^-(x,\rho)], & \rho \in \omega^1, \\ \Phi(x,\rho), & \rho \in \omega^0, \end{cases} \qquad \tilde\varphi(x,\rho) = \begin{cases} [\tilde\Phi^+(x,\rho), \tilde\Phi^-(x,\rho)], & \rho \in \omega^1, \\ \tilde\Phi(x,\rho), & \rho \in \omega^0, \end{cases}$$

$$\tilde G(x,\rho) = \begin{bmatrix} -(A_0(\rho)\tilde\Phi^*(x,\rho))^+ Q_0 \\ (A_0(\rho)\tilde\Phi^*(x,\rho))^- Q_0 \end{bmatrix}, \ \rho \in \omega^1, \quad \tilde G(x,\rho) = A_0(\rho)\tilde\Phi^*(x,\rho) Q_0, \ \rho \in \omega^0,$$

$$\tilde S(\rho) = \begin{bmatrix} I - \frac12 A_0^+(\rho) & \frac12 \Big(A_0(\rho)\tilde{\mathcal N}^*(\rho)\Big)^+ \tilde{\mathcal N}^-(\rho) \\ \frac12 \Big(A_0(\rho)\tilde{\mathcal N}^*(\rho)\Big)^- \tilde{\mathcal N}^+(\rho) & I - \frac12 A_0^-(\rho) \end{bmatrix}, \ \rho \in \omega^1, \quad f^{\pm} := f_{|\omega^{\pm}},$$

$$\tilde S(\rho) = I - \frac12 A_0(\rho), \ \rho \in \omega^0, \quad \tilde r(x,\mu,\rho) = \frac{\tilde G(x,\mu)\tilde\varphi(x,\rho)}{\mu - \rho}, \ \rho,\mu \in \omega^*,$$

$$D(x,\rho) = \operatorname{diag}[D_k(x,\rho)]_{k=\overline{1,n}}, \ D_k(x,\rho) = \exp(-\rho R_k x), \ R_k = R_k(S_j) \text{ for } \rho \in \omega^0,$$

$$D(x,\rho) = \operatorname{diag}[D_k(x,\rho)]_{k=\overline{1,2n}}, \ D_k(x,\rho) = \exp(-\rho R_k x) \ (k \le n),$$

$$D_k(x,\rho) = \exp(-\rho R_{k-n} x) \ (k > n), \ R_k = R_k(S_j) \text{ for } \rho \in \omega_j^1,$$

where I is the identity matrix. We consider the Banach space $\mathcal B_p := \{f(\rho) : \ f(\rho)\rho^{-1} \in L_p(\omega^*)\}$, $p > 1$ with the norm $\|f\|_{\mathcal B_p} := \|f(\rho)\rho^{-1}\|_{L_p(\omega^*)}$.

Theorem 9. *Let $M(\rho)$ be the Weyl matrix for a pair $L = (l,U)$. Let a pair $\tilde L = (\tilde l, \tilde U)$ be taken such that (12) holds. Then*

$$\tilde\varphi(x,\rho) = \varphi(x,\rho)\tilde S(\rho) + \frac{1}{2\pi i}\int_{\omega^*} \varphi(x,\mu)\tilde r(x,\mu,\rho)\, d\mu, \quad \rho \in \omega^*. \tag{13}$$

For each fixed $x \ge 0$, equation (13) has a unique solution $\varphi(x,\rho)$ in the class $\varphi(x,\rho)D(x,\rho) \in \mathcal B_p$ for each $p > 1$, and $\sup\limits_{x\ge0} \|\varphi(x,\rho)D(x,\rho)\|_{\mathcal B_p} < \infty$.

Equation (13) is called the *main equation* of the inverse problem. Let us now formulate necessary and sufficient conditions for the solvability of the inverse problem.

Theorem 10. *For a matrix $M(\rho) \in \mathcal M$ to be the Weyl matrix for a pair $L = (l,U)$, $l \in V_N$, it is necessary and sufficient that the following conditions are fulfilled:*

1) (Asymptotics) There exists a pair $\tilde L = (\tilde l, \tilde U)$ such that (12) holds.

2) (Condition P) For each fixed $x \geq 0$, *the main equation (13) has a unique solution* $\varphi(x,\rho)$ *in the class* $\varphi(x,\rho)D(x,\rho) \in \mathcal{B}_p$, $p > 1$, *and* $\sup_{x\geq 0} \|\varphi(x,\rho)D(x,\rho)\|_{\mathcal{B}_p} < \infty$.

3) $\varepsilon(x) \in W_N$, *where*

$$\varepsilon(x) = \frac{1}{2\pi i}\int_{\omega} \Big(\Phi(x,\mu)A_0(\mu)\tilde{\Phi}^*(x,\mu)Q_0 - Q_0\Phi(x,\mu)A_0(\mu)\tilde{\Phi}^*(x,\mu)\Big)\,d\mu. \tag{14}$$

Under these conditions the pair $L = (l, U)$ *is constructed by the formulae*

$$Q(x) = \tilde{Q}(x) + \varepsilon(x), \quad h = \tilde{h}. \tag{15}$$

Algorithm 2. Let the Weyl matrix $M(\lambda)$ for the pair $L = (l, U)$ be given.

1) Choose $\tilde{L} = (\tilde{l}, \tilde{U})$ such that (12) holds.

2) Construct the matrices $\tilde{\varphi}(x,\rho)$, $\tilde{S}(\rho)$, $\tilde{r}(x,\mu,\rho)$, $x \geq 0$, $\mu,\rho \in \omega^*$.

3) Find $\varphi(x,\rho)$, $x \geq 0$, $\rho \in \omega^*$ by solving the main equation (13), i.e. find $\Phi(x,\rho)$ for $x \geq 0$, $\rho \in \omega$.

4) Calculate $\varepsilon_\nu(x)$ by (14).

5) Construct $L = (l, U)$ by (15).

Remark 4. Analogously one can study the inverse problem for system (1) on a finite interval.

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