Sturm-Liouville Problems with Singular Non-Selfadjoint Boundary Conditions

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Dedicated to our esteemed colleague F.V. Atkinson

Singular boundary conditions are formulated for non-selfadjoint Sturm-Liouville problems which are limit-circle in a very general sense. The characteristic determinant is constructed and it is shown that it can be used to extend the Birkhoff theory for so called 'Birkhoff regular boundary conditions' to the singular case. This is illustrated for a class of singular Birkhoff-regular problems; in particular we prove for this class an asymptotic formula for the eigenvalues and an expansion theorem.

1 Introduction

In 1908 Birkhoff published two seminal papers [2], [3] in which he proved the existence and asymptotic behavior of eigenvalues and the associated expansion theory for a large class of non-selfadjoint regular boundary value problems. The boundary conditions of these problems have come to be known as 'Birkhoff-regular' boundary conditions. In this paper we construct general singular non-selfadjoint boundary conditions for Sturm-Liouville problems and show how these can be used to identify singular problems to which the 'Birkhoff regular' theory can be applied to obtain the existence and asymptotic form of the eigenvalues and an expansion theorem.

The definition of 'Birkhoff-regular' boundary conditions, see the now classic book by Naimark [12], involves the values of solutions and their derivatives at the endpoints of the underlying interval. In general these do not exist at a singular endpoint. But if this singular endpoint is of 'limit-circle' type, in a very general sense made precise below, then the Lagrange form exists and has finite limits. It is these limits upon which our definition is based. These forms involve 'boundary condition functions', u, v. At a regular endpoint these boundary condition functions can be chosen so that the singular conditions reduce to the familiar regular ones, and in this case we show that the singular 'Birkhoff regular' theory presented here reduces to the classical 'Birkhoff regular' theory and is a natural extension of it. (We lament the double meaning of the word 'regular' in this discussion, as in 'regular endpoint' and 'Birkhoff regular' boundary conditions, but both meanings are so well established in the literature that it would be presumptuous of us to try to change either one.)

In the self-adjoint case, general singular boundary conditions, defined in terms of such boundary condition functions u, v, are known, see [18] and the references therein. For singular nonselfadjoint problems the Birkhoff approach has been successfully employed by Stone [14], and by Freiling, Rykhlov and Yurko [8], [7], in each case for a special specific boundary condition at the singular endpoint. The theory of general singular nonselfadjoint boundary conditions presented here seems to be new.

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2 Preliminary Results

We study the Sturm-Liouville differential equation

$$My = -(py')' + qy = \lambda wy \text{ on } J = (a, b),\ -\infty \le a < b \le \infty, \tag{1}$$

under the general assumption

$$\frac{1}{p}, q, w \in L_{loc}(J, \mathbf{C}),\ \lambda =: \rho^2 \in \mathbf{C}. \tag{2}$$

Definition 2.1 (Limit-Circle) The left endpoint a is in the limit circle (LC) case if all solutions of equation (1) are in $L^2((a, c), |w|)$ for some $c \in J$ and for some $\lambda \in \mathbf{C}$. Similarly, the right endpoint b is in the limit-circle case if all solutions of (1) are in $L^2((c, b), |w|)$ for some $c \in J$ and for some $\lambda \in \mathbf{C}$.

Remark 2.2 It follows from (2) that if all solutions of equation (1) are in $L^2((a, c), |w|)$ for some $c \in J$ then this is true for all $c \in J$ and similarly for $L^2((c, b), |w|)$. The next lemma shows that the definition of LC is also independent of $\lambda \in \mathbf{C}$. Thus the LC classification at each endpoint a, b depends only on the coefficients p, q, w, (more accurately, in view of (2) and the basic existence-uniqueness theory for initial value problems, on $1/p, q, w$). For what p, q, w does the LC case hold at a? at b? There is a voluminous literature on this question - *for real-valued coefficients* p, q, w *satisfying the additional restrictions* $p > 0,\ w > 0$ - dating back to the seminal 1910 paper of Hermann Weyl [15]. For this case many sufficient conditions, as well as many necessary conditions, are known. Even some necessary and sufficient conditions are known [9]. Yet the problem is still open for this case, since no necessary and sufficient conditions *which can be checked for each equation* are known. In general, under condition (2), very little about the LC case is known except the results proved or mentioned in [13]. We want to mention that in the case where $p > 0$, and q, w change sign there is a whole section (Section 3.5, p. 147) in Mingarellis book [13] that gives explicit criteria for LC on a half-line..

Lemma 2.3 *Let (1), (2) hold and let* $c \in J$. *If all solutions of (1) are in* $L^2((a, c), |w|)$ *for some* $\lambda = \lambda_0 \in \mathbf{C}$*, then this is true for every* $\lambda \in \mathbf{C}$. *Similarly for the endpoint* b.

Proof. Let

$$Y = \begin{bmatrix} y \\ py' \end{bmatrix},\ P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix},\ W = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}.$$

Then (1) is equivalent to

$$Y' = (P - \lambda W)Y \text{ on } J. \tag{3}$$

Let

$$U = \begin{bmatrix} v & u \\ pv' & pu' \end{bmatrix}$$

be a fundamental matrix of (3) for some particular value of $\lambda = \lambda_0$ normalized so that $\det(U)(t) = 1$, $t \in J$. For all $\lambda \in \mathbf{C}$, let

$$Z(\cdot, \lambda) = U^{-1}\ Y(\cdot, \lambda). \tag{4}$$

Then a computation shows that

$$\begin{aligned} Z'(\cdot, \lambda) &= (\lambda_0 - \lambda)\,(U^{-1}WU)\,Z(\cdot, \lambda)\ = (\lambda_0 - \lambda)\,GZ(\cdot, \lambda) \text{ on } J,\quad \lambda \in \mathbf{C}, \\ G &= [g_{ij}] = \begin{bmatrix} -uvw & -u^2w \\ v^2w & uvw \end{bmatrix}. \end{aligned} \tag{5}$$

By the Cauchy-Schwarz inequality we have

$$\left(\int_a^c |u|\,|v|\,|w|\right)^2 = \left(\int_a^c |u|\,|v|\,|w|^{1/2}|w|^{1/2}\right)^2 \leq \int_a^c |u|^{\,2}\,|w| \int_a^c |v|^{\,2}|w| < \infty. \tag{6}$$

Thus g_{11} and g_{22} are in $L^1((a,c),\mathbf{C})$. It follows similarly that g_{12} and g_{21} are in $L^1(a,c)$. Thus the system (5) is regular at a and therefore all its solutions can be continuously extended to the endpoint a. The proof for the endpoint b is similar. In particular all solutions of (5) are bounded on the closure of the (bounded or unbounded) interval J. Letting $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and reversing the transformation (4) we get the following representation of solutions y of (1) and of their quasi-derivatives py' :

$$y(\cdot,\lambda) = v\,z_1(\cdot,\lambda) + u\,z_2(\cdot,\lambda),\ (py')(\cdot,\lambda) = (pv')z_1(\cdot,\lambda) + (pu')\,z_2(\cdot,\lambda),\ \lambda \in \mathbf{C}. \tag{7}$$

From (7), the boundedness of z_j in (a,c) and the hypothesis on u, v it follows that $y(\cdot,\lambda) \in L^2((a,c),|w|)$, for all $\lambda \in \mathbf{C}$. The proof for the endpoint b is similar and hence omitted. ☐

Remark 2.4 The λ invariance of Lemma 2.3 is established under Hypothesis (2); thus the celebrated LC/LP dichotomy in $L^2((a,c),|w|)$ of Hermann Weyl holds not only for real-valued coefficients $r = 1/p, q, w$ with $p > 0,\, w > 0$ but also when

- the real-valued functions $r = 1/p, q, w$ change sign or are identically zero on one or more subintervals
- and/or when these coefficients are complex-valued.

Lemma 2.5 *Let (1), (2) hold. Then the limits*

$$y(a) = \lim_{t\to a^+} y(t),\ (py')(a) = \lim_{t\to a^+}(py')(t)\ \ (\textit{resp.}\ \ y(b),\ (py')(b)) \tag{8}$$

exist and are finite for all solutions y of (1) if and only if

$$1/p, q, w \in L^1((a,c),\mathbf{C})\ \ (\textit{resp.}\ \ 1/p, q, w \in L^1((c,b),\mathbf{C})). \tag{9}$$

Proof. See [6]. ☐

In the next lemma we also consider the adjoint equation of (1):

$$M^+z = -(\overline{p}z')' + \overline{q}z = \overline{\lambda}\overline{w}z \textit{ on } J. \tag{10}$$

Note that y is a solution of (1) if and only if $z = \overline{y}$ is a solution of (10).

In the formulation of singular boundary conditions below an important role is played by the Lagrange sesquilinear form:

$$[y,z] = y(p\overline{z}') - \overline{z}(py'). \tag{11}$$

This is defined for all y in the expression domain of M and all z in the expression domain of M^+; the former consists of all complex valued functions y such that y and (py') are absolutely continuous on all compact subintervals of J. Note that My exists $a.e.$ for all y in the expression domain of M. A similar definition and observation is made for M^+.

Definition 2.6 Assume that $|w| > 0$ a.e. on J. Let $H = L^2(J,|w|)$. The maximal domains of M and M^+ are defined by

$$\begin{aligned} D_{\max}(M) &= \{f \in H,\ f, (\overline{p}f') \in AC_{loc}(J),\ \frac{1}{|w|}Mf \in H\}, \\ D_{\max}(M^+) &= \{f \in H,\ f, (\overline{p}f') \in AC_{loc}(J),\ \frac{1}{|w|}M^+f \in H\}. \end{aligned}$$

Here $AC_{loc}(J)$ denotes the set of complex valued functions on J which are absolutely continuous on all compact subintervals of J.

Lemma 2.7 *Assume that $|w| > 0$ a.e. on J. Suppose that b (resp. a) is LC for both (1) and (10). Then the limit*

$$[f,g](b) = \lim_{t\to b^-}[f,g](t) \ \ (\textit{resp.}\ [f,g](a) = \lim_{t\to a^+}[f,g](t)) \tag{12}$$

exists and is finite for all $f \in D_{\max}(M)$ and $g \in D_{\max}(M^+)$.

Proof. We first show that

$$\overline{g}Mf - f\overline{M^+g} = [f,g]'. \tag{13}$$

$\overline{g}Mf - f\overline{M^+g}g = \overline{g}\{-(pf')'+qf\} - f\{\overline{-(\overline{p}g')'+\overline{q}g}\} = -\overline{g}(pf')'+f(p\overline{g}')' = \{fp\overline{g'}-\overline{g}(pf')\}' = [f,g]'$. Let $c \in J$. Now

$$\int_c^b |\overline{g}|\,|Mf| = \int_c^b |\overline{g}||w|^{1/2}\frac{1}{|w|^{1/2}}|Mf| \le \left(\int_c^b |\overline{g}|^2\,|w|\right)^{1/2}\left(\int_c^b \frac{1}{|w|}|Mf|^2\right)^{1/2} < \infty. \tag{14}$$

Similarly

$$\int_c^b |f|\,|\overline{M^+g}| < \infty. \tag{15}$$

Integrating (13) from c to β, we obtain

$$\int_c^\beta \{\overline{g}Mf - f\overline{M^+g}\} = [f,g](\beta) - [f,g](c). \tag{16}$$

Now the conclusion follows from (14) and (15) by letting $\beta \to b^-$. The proof for the endpoint a is similar. ☐

Using the finite limits of the Lagrange forms established by Lemma 2.7 one can formulate singular initial (or terminal) value problems at singular endpoints. The next lemma shows that these problems have unique solutions.

Lemma 2.8 *Assume that the left endpoint a is LC and*

$$\frac{1}{p}, q, w \in L_{loc}(J, \mathbf{C}). \tag{17}$$

Assume u, v are solutions of (1) for some $\lambda = \lambda_0 \in \mathbf{C}$, normalized to satisfy

$$[v,\overline{u}](t) = 1,\ t \in J. \tag{18}$$

Let $\lambda \in \mathbf{C}$. Then for any $h, k \in \mathbf{C}$ there exists a unique solution $y = y(\cdot,\lambda)$ of (1) satisfying the initial (or terminal) conditions:

$$[v,\overline{y}](a) = h,\ [y,\overline{u}](a) = k. \tag{19}$$

This solution $y(t,\lambda)$ is an entire function of λ for each fixed $t \in J$.
A similar result holds at the endpoint b.

Proof. We use the notation and proof of Lemma 2.3. From (7) and Cramer's rule we get

$$z_1(t) = [y,\overline{u}](t),\ z_2(t) = [v,\overline{y}](t),\ t \in J. \tag{20}$$

By (5), (6) (and the remark following (6)), the initial value problem $z_1(a) = k,\ z_2(a) = h$ has a unique solution Z defined on the closure of J and therefore (19) has a unique solution y defined on J. Furthermore it follows from (7) that $y(t,\lambda)$ is an entire function of λ since $z_1(t,\lambda)$ and $z_2(t,\lambda)$ are entire functions of λ. ☐

Lemma 2.9 *Let U be defined as in the proof of Lemma 2.3, let $\Psi = \Psi(\cdot, \lambda)$ be the fundamental solution of (5) determined by the initial condition $\Psi(a, \lambda) = I$. Then $\Psi(t, \lambda)$ is defined for $a \leq t \leq b,\ \lambda \in \mathbf{C}$. Let*

$$X = U\Psi = (x_{ij}). \tag{21}$$

Then x_{11}, x_{12} are solutions of (1), and $x_{21} = px'_{11}$, $x_{22} = px'_{12}$. Furthermore we have for all t, $a \leq t \leq b$,

$$\psi_{11}(t) = [x_{11}, \overline{u}](t),\ \psi_{21}(t) = [v, \overline{x}_{11}](t),\ \psi_{12}(t) = [x_{12}, \overline{u}](t),\ \psi_{22}(t) = [v, \overline{x}_{12}](t). \tag{22}$$

Proof. The proof is similar to the proof of Lemma 2.8. From (21) we get

$$\begin{aligned} x_{11} &= v\psi_{11} + u\psi_{21}, \\ x_{21} &= pv'\psi_{11} + pu'\psi_{21}. \end{aligned} \tag{23}$$

Now $\psi_{11}(t) = [x_{11}, \overline{u}](t),\ \psi_{21}(t) = [v, \overline{x}_{11}](t)$ follows from (23), Cramer's rule, and the normalization (18). The other two identities in (22) are established similarly. ☐

Remark 2.10 Let the hypotheses and notation of Lemma 2.9 hold. Since $\Psi' = (\lambda - \lambda_0)\, G\Psi$ and $traceG = 0$, it follows from Abel's Theorem that $\det \Psi(t) = 1,\ a \leq t \leq b$. This, combined with Lemma 2.9, yields an alternate proof and an extension of the Plücker Identity:

$$[x_{11}, \overline{u}](t, \lambda)[v, \overline{x}_{12}](t, \lambda) - [v, \overline{x}_{11}](t, \lambda)[x_{12}, \overline{u}](t, \lambda) = 1,\ a \leq t \leq b,\ \lambda \in \mathbf{C}.$$

3 Construction of the Characteristic Determinant

In this section we construct the characteristic matrix and the associated determinant which characterizes the eigenvalues of singular problems with general limit-circle endpoints, i.e. we assume below that both a and b are LC. The application of the 'Birkhoff regular theory' to singular problems is based on this determinant.

Let $M_2(\mathbf{C})$ denote the set of 2×2 matrices with complex components.

Theorem 3.1 *Let (1) , (2) hold and let a and b be LC. Assume u, v are solutions of (1) for some $\lambda = \lambda_0 \in \mathbf{C}$, normalized to satisfy $[v, \overline{u}](t) = 1,\ t \in J$. Let $\Psi(\cdot, \lambda) = (\psi_{ij})$ be the fundamental matrix of the regular system (5) determined by the initial condition $\Psi(a, \lambda) = I$. Then $\Psi(t, \lambda)$ is defined for all t, $a \leq t \leq b$ and, by Lemma 2.9, $X = U\Psi$ and*

$$\Psi(t, \lambda) = \begin{bmatrix} [x_{11}, \overline{u}](t, \lambda) & [x_{12}, \overline{u}](t, \lambda) \\ [v, \overline{x}_{11}](t, \lambda) & [v, \overline{x}_{12}](t, \lambda) \end{bmatrix},\ a \leq t \leq b. \tag{24}$$

Then a complex number λ is an eigenvalue of the singular boundary value problem consisting of equation (1) with the boundary condition

$$\mathcal{A}Y(a) + \mathcal{B}Y(b) = 0,\ Y = \begin{bmatrix} [y, \overline{u}] \\ [v, \overline{y}] \end{bmatrix},\ \mathcal{A}, \mathcal{B} \in M_2(\mathbf{C}), \tag{25}$$

if and only if

$$\delta(\lambda) = \det\{\mathcal{A} + \mathcal{B}\Psi(b, \lambda)\} = 0. \tag{26}$$

Proof. Consider the boundary value problem consisting of the regular system (5) together with the boundary condition

$$\mathcal{A}Z(a) + \mathcal{B}Z(b) = 0. \tag{27}$$

From the theory of boundary value problems for regular systems it is well known that λ is an eigenvalue of (5), (27) if and only if (26) holds. Letting $Z = U^{-1}Y$ and proceeding as in the proof of Lemma 2.3, we see that Z is a solution of (5), (27) if and only if Y is a solution of (1), (25). This completes the proof. ☐

Theorem 3.2 *Let u, v be as in Theorem 3.1 and let a and b be LC. Let y_1, y_2 be the unique solutions of (1) determined by the singular initial conditions*

$$[y_1,\overline{u}](a,\lambda) = 1 = -[y_2,\overline{v}](a,\lambda),\ [y_2,\overline{u}](a,\lambda) = 0 = [y_1,\overline{v}](a,\lambda),\ \lambda \in \mathbf{C}. \tag{28}$$

Let

$$\Phi(t,\lambda) = \begin{bmatrix} [y_1,\overline{u}](t,\lambda) & [y_2,\overline{u}](t,\lambda) \\ [v,\overline{y_1}](t,\lambda) & [v,\overline{y}_2](t,\lambda) \end{bmatrix},\ a \le t \le b. \tag{29}$$

Then a complex number λ is an eigenvalue of the singular boundary value problem consisting of equation (1) with the boundary condition (25) if and only if

$$\delta(\lambda) = \det\{\mathcal{A} + \mathcal{B}\Phi(b,\lambda)\} = 0. \tag{30}$$

Proof. From Lemma 2.8 it follows that $y_1 = x_{11}$ and $y_2 = x_{12}$. Hence $\Phi(t,\lambda) = \Psi(t,\lambda)$ and Theorem 3.2 follows from Theorem 3.1. ☐

Definition 3.3 The determinant $\delta(\lambda) = \det\{\mathcal{A} + \mathcal{B}\Phi(b,\lambda)\}$, is called the characteristic determinant of the boundary value problem (1), (25).

Remark 3.4 Since $\Phi(b,\lambda) = \Psi(b,\lambda)$ and the latter is an entire function of λ, it follows that $\delta(\lambda)$ is an entire function of λ. Thus the eigenvalues of singular boundary value problems with general LC endpoints and boundary conditions of the form (25) are the zeros of an entire function as in the regular case.

Next we illustrate the construction of the characteristic function $\delta(\lambda)$ for some special boundary conditions (25) with some examples.

Example 3.5 Let y be the unique solution of (1) determined by the singular initial value problem:

$$[y,\overline{u}](a,\lambda) = 0,\ [v,\overline{y}](a,\lambda) = 1,$$

according to Lemma 2.8. Let

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\delta(\lambda) = [y,\overline{u}](b,\lambda),\ \lambda \in \mathbf{C}. \tag{31}$$

Example 3.6 Let y be the unique solution of (1) determined by the singular initial value problem:

$$[y,\overline{u}](a,\lambda) = 1,\ [v,\overline{y}](a,\lambda) = 0,$$

according to Lemma 2.8. Let

$$\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\delta(\lambda) = [y,\overline{v}](b,\lambda),\ \lambda \in \mathbf{C}. \tag{32}$$

The boundary conditions for Examples 3.5 and 3.6 are selfadjoint in the (right-definite) theory when p, q, w are real-valued and w is positive; the next example is not selfadjoint in this sense.

Example 3.7 Let y be determined as in Example 3.5; let

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\delta(\lambda) = 1 - [y,\overline{v}](b,\lambda),\ \lambda \in \mathbf{C}. \tag{33}$$

4 Birkhoff-regular boundary conditions for a class of singular SLP

The main purpose of this section is to illustrate how the Birkhoff theory of non-selfadjoint regular boundary value problems can be combined with the theory of singular problems developed in this paper to obtain a class of non-selfadjoint singular Sturm-Liouville problems which are 'regular' in the sense of Birkhoff and to which the 'Birkhoff regular' theory can be applied. No attempt is made here at finding the largest such class, but our approach shows how one can identify much larger classes of 'Birkhoff regular' singular problems.

Consider

$$M(y) := -y'' + (\frac{A}{x^2} + q_0(x))y = \lambda y =: \rho^2 y,\ x \in (0,1] \tag{34}$$

where

$$q_0(x) = \frac{B}{x} + C(x), \tag{35}$$

A and B are complex constants and C is a complex-valued integrable function.
We assume in the sequel that

$$\eta = \frac{1}{2}\sqrt{1+4A}$$

and q_0 satisfy

$$0 <| \text{ Re } \eta \;|< 1 \text{ and } q_0(x)x^{1\,-\,2\text{Re}\,\eta} \in L[0,1], \tag{36}$$

which ensures in particular that 0 is LC.
Although the proofs given below are for Re $\eta \in (0,1)$, the results hold for $0 <|$ Re $\eta\ |< 1$ since in the case Re $\eta \in (-1,0)$ the proofs are similar. The case Re $\eta = 0$ is not considered in this paper; however see the last paragraph of this section regarding the case $\eta = 0$.

For $k \in \mathbf{Z}$ and $0 < \delta < 1$ we consider the sectors (on the Riemann surfaces of log)

$$\chi_\delta^{(k)} := \{\xi | (k-1+\delta)\pi \le \arg\xi \le (k+1-\delta)\pi\}.$$

By $H_\eta^{(1)}$ and $H_\eta^{(2)}$ we denote the Hankel functions (i.e. Bessel functions of third kind).

Recall (see [1]) that for $\eta \notin \mathbf{Z}$

$$H_\eta^{(1)}(z) = i\ csc(\eta\pi)\{e^{-\eta\pi i}J_\eta(z) - J_{-\eta}(z)\},$$

$$H_\eta^{(2)}(z) = i\ csc(\eta\pi)\{J_{-\eta}(z) - e^{\eta\pi i}J_\eta(z)\},$$

where

$$J_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}$$

denotes a Bessel function of first kind.
In the sequel we use the symbol $\overset{(2)}{=}$ instead of = in order to signify that a given formula is valid and also that the corresponding formula obtained by formal differentiation with respect to x, ignoring the formal differentiation of all error terms, is valid.
Moreover we shall write two formulas in one by the use of double signs together with an index j: the upper signs are to be associated with the value $j = 1$ and the lower signs with $j = 2$.

The solutions of (34) behave for $|\rho| \to \infty$ and $x \to 0+$ like linear combinations of $x^{\frac{1}{2}}H_\eta^{(1)}(x\rho)$ and $x^{\frac{1}{2}}H_\eta^{(2)}(x\rho)$; this was proved by Langer [11] under the assumption that $q_0(x) - \frac{B}{x}$ is real analytic, according to Yurko [17] it is here sufficient that the integrability condition in (36) holds. The following theorem summarizes the estimates derived in [11] (see formulae (11) – (16))

Theorem 4.1 *Let $\ell \in \mathbf{Z}$. The differential equation (34) has for $\rho \neq 0$, and $x \in (0,1]$ a fundamental system of solutions $v_{\ell 1}(\cdot,\rho), v_{\ell 2}(\cdot,\rho)$ satisfying (together with its derivatives) for a fixed, sufficiently large $N > 0$ and $\rho \in \chi_\delta^{(0)}$ the following estimates:*

(i) *For* $|x\rho| \leq N,\ 1 \leq j \leq 2$

$$\begin{aligned} v_{\ell,j}(x,\rho) &\overset{(2)}{=} \sqrt{\pi}\, 2^{-\frac{1}{4}}\ e^{\pm(\eta+\frac{1}{2})i\frac{\pi}{2}}\ x^{\frac{1}{2}}\ \rho^{\frac{1}{4}}\ H_\eta^{(j)}(e^{-\ell\pi i}x\rho) \\ &+x^{\frac{1}{2}-\eta}\rho^{\frac{1}{4}-\eta}m_j(x,\rho) \end{aligned} \tag{37}$$

with

$$m_j(x,\rho) = O(\frac{x\log\rho}{\rho})\ for\ |\rho| \to \infty. \tag{38}$$

(ii) *For* $|x\rho| > N,\ 1 \leq j \leq 2$

$$v_{2\ell,j}(x,\rho) \overset{(2)}{=} 2^{\frac{1}{4}}\rho^{-\frac{1}{4}}\{c_{j,1}^{(\ell)}e^{i\rho x}(1+O(\frac{\log\rho}{x\rho})) + c_{j,2}^{(\ell)}e^{-i\rho x}(1+O(\frac{\log\rho}{x\rho}))\}, \tag{39}$$

and

$$\left\{\begin{array}{l} \left.\begin{array}{lcl} c_{1,1}^{(\ell)} &=& (-1)^{\ell-s+1}\frac{\sin(2s-2\ell-1)\eta\pi}{\sin\eta\pi} \\ c_{2,1}^{(\ell)} &=& (-1)^{\ell-s+1}\frac{i\sin(2s-2\ell)\eta\pi}{\sin\eta\pi} \end{array}\right\},\ x\rho \in \chi_\delta^{(2s-1)} \cup \chi_\delta^{(2s)}, \\ \left.\begin{array}{lcl} c_{1,2}^{(\ell)} &=& (-1)^{\ell-s+1}\frac{i\sin(2s-2\ell)\eta\pi}{\sin\eta\pi} \\ c_{2,2}^{(\ell)} &=& (-1)^{\ell-s}\frac{\sin(2s-2\ell+1)\eta\pi}{\sin\eta\pi} \end{array}\right\},\ x\rho \in \chi_\delta^{(2s)} \cup \chi_\delta^{(2s+1)}. \end{array}\right. \tag{40}$$

Notice that for $s=0$ we have $c_{1,1}^{(0)}=1$, $c_{2,1}^{(0)}=0$, $c_{1,2}^{(0)}=0$, $c_{2,2}^{(0)}=1$. Furthermore

$$\begin{array}{lcl} v_{2\ell,1}(x,\rho) &=& v_{2\ell+1,1}(x,\rho), \\ v_{2\ell-1,2}(x,\rho) &=& v_{2\ell,2}(x,\rho). \end{array}$$

(iii) *The Wronskian W satisfies*

$$W(v_{\ell,1},v_{\ell,2}) = -2^{\frac{3}{2}}i\rho^{\frac{1}{2}}(1+O(\frac{\log\rho}{\rho}))\ (\ell \in \mathbf{Z}). \tag{41}$$

Remark 4.2 We mention here that the factor x in the righthand side of (38) does not appear explicitly in the formulas derived in [11], but it can be seen from the proof therein that the error terms $m_j(x,\rho)$ have the form (38).

Remark 4.3 It is clear from the argument of the Hankel function in formula (37) that the role of $\ell \in \mathbf{Z}$ is to obtain a fundamental system of solutions in each sector $\chi_\delta^{(k)}$.

For the formulation of the subsequent results we introduce the following abbreviations:
For $a \in \mathbf{C}, x \in (0,1]$ and $-\pi+\delta < \arg\rho \leq \pi-\delta$

$$\begin{array}{lcl} [a] &=& a+O(\frac{\log\rho}{\rho}), \\ [[a]]_x &=& a+O(x), \\ [[a]] &=& a+O(\frac{x\log\rho}{\rho}), \\ [a]_\eta &=& a+O(\frac{1}{\rho^{2\eta}}), \\ [[[a]]] &=& [a]+O(\frac{1}{\rho^{2\eta}}), \end{array}$$

which yields $\lim_{x\to 0+}\,[[a]] = \lim_{x\to 0+}\,[[a]]_x \;=\; a$; moreover we set

$$\begin{aligned}
d_{11} &= \sqrt{\pi}\,2^{-\frac{1}{4}-\eta}\frac{1}{\Gamma(\eta+1)}\;e^{(\frac{3}{2}-\eta)\frac{i\pi}{2}}\text{ csc }\eta\pi,\\
d_{12} &= -\sqrt{\pi}\,2^{-\frac{1}{4}+\eta}\;\frac{1}{\Gamma(-\eta+1)}\;e^{(\frac{3}{2}+\eta)\frac{i\pi}{2}}\;\text{ csc }\eta\pi,\\
d_{21} &= ie^{i\eta\pi}d_{11} \qquad\text{and}\qquad d_{22} = ie^{-i\eta\pi}d_{12}.
\end{aligned}$$

From Theorem 4.1 it follows by elementary calculations that the solutions $y_1(\cdot,\rho) := v_{01}(\cdot,\rho)$ and $y_2(\cdot,\rho) := v_{02}(\cdot,\rho)$ of (34) satisfy for $x\in(0,1]$, $j=1,2$ and $\rho\in\chi_\delta^{(0)}$ with $|x\rho|\le N$ the estimates

$$\begin{aligned}
&y_j(x,\rho) = x^{\frac{1}{2}+\eta}\rho^{\eta+\frac{1}{4}}\;[[d_{j1}]] + x^{\frac{1}{2}-\eta}\rho^{\frac{1}{4}-\eta}[[d_{j2}]],\\
&y_j'(x,\rho) = (\tfrac{1}{2}+\eta)x^{-\frac{1}{2}+\eta}\rho^{\eta+\frac{1}{4}}[[d_{j1}]] + x^{-\frac{1}{2}-\eta}\rho^{\frac{1}{4}-\eta}[[(\tfrac{1}{2}-\eta)d_{j2}]].
\end{aligned}\tag{42}$$

Moreover for $\rho\in\chi_\delta^{(0)}$ with $|\rho|\ge N$

$$\begin{aligned}
&y_1(1,\rho) = 2^{\frac{1}{4}}\rho^{-\frac{1}{4}}e^{i\rho}[1],\\
&y_1'(1,\rho) = i2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{i\rho}[1],\\
&y_2(1,\rho) = 2^{\frac{1}{4}}\rho^{-\frac{1}{4}}e^{-i\rho}[1],\\
&y_2'(1,\rho) = -i2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{-i\rho}[1].
\end{aligned}\tag{43}$$

Now we choose $\rho_0>0$ sufficiently large and define

$$u := y_1(\cdot,\rho_0)\rho_0^{\eta-\frac{1}{4}} \text{ and } v := y_2(\cdot,\rho_0)\rho_0^{\eta-\frac{1}{4}},$$

then u and v are, according to Theorem 4.1, linearly independent solutions of (34) for $\lambda = \rho_0^2$. Therefore $u(x)\rho_0^{\frac{1}{4}-\eta}$ and $v(x)\rho_0^{\frac{1}{4}-\eta}$ can be estimated for $x\le\frac{N}{\rho_0}$ by replacing in the right-hand side of the corresponding formula in (42) ρ by ρ_0 and $[[-]]$ by $[[-]]_x$; for convenience we assume below that without loss of generality $\rho_0 = 1$.

Using these estimates and the estimates (42) we get for $\rho\in\chi_\rho^{(0)}$

$$\begin{aligned}
&[y_1(\cdot,\rho),\overline{u}](0) = \lim\nolimits_{x\to 0+}\{y_1(x,\rho)u'(x) - y_1'(x,\rho)u(x)\}\\
&= \lim\nolimits_{x\to 0+}\{x^{-2\eta}\rho^{\frac{1}{4}-\eta}([[d_{12}]]\;[[(\tfrac{1}{2}-\eta)d_{12}]]_x - [[(\tfrac{1}{2}-\eta)d_{12}]]\;[[d_{12}]]_x)\\
&+\rho^{\eta+\frac{1}{4}}[[d_{11}]]\;[[(\tfrac{1}{2}-\eta)d_{12}]]_x\\
&+\rho^{\frac{1}{4}-\eta}(\tfrac{1}{2}+\eta)[[d_{12}]]\;[[d_{11}]]_x\\
&-\rho^{\frac{1}{4}-\eta}[[(\tfrac{1}{2}-\eta)d_{12}]]\;[[d_{11}]]_x\\
&-\rho^{\frac{1}{4}+\eta}(\tfrac{1}{2}+\eta)[[d_{11}]]\;[[d_{12}]]_x\\
&+O(x^{2\eta}\rho^{\eta+\frac{1}{4}})\}.
\end{aligned}$$

Since, according to Lemma 2.7, $[y_1(\cdot,\rho),\overline{u}](0)$ exists, it follows that the first term in the preceding sum is of the form $O(\rho^{\frac{1}{4}-\eta})$ for $0<$ Re $\eta<1$. Hence (for $0<$ Re $\eta<1$)

$$[y_1(\cdot,\rho),\overline{u}](0) = -2\eta\rho^{\eta+\frac{1}{4}} d_{11}d_{12}(1+O(\rho^{-2\eta})) = -2\eta\rho^{\eta+\frac{1}{4}}[d_{11}d_{12}]_\eta. \tag{44}$$

Similarly we derive in this case

$$\begin{aligned}
&[y_1(\cdot,\rho),\overline{v}](0) = -2\eta\rho^{\eta+\frac{1}{4}}[d_{11}d_{22}]_\eta,\\
&[y_2(\cdot,\rho),\overline{u}](0) = -2\eta\rho^{\eta+\frac{1}{4}}[d_{21}d_{12}]_\eta,\\
&[y_2(\cdot,\rho),\overline{v}](0) = -2\eta\rho^{\eta+\frac{1}{4}}[d_{21}d_{22}]_\eta.
\end{aligned} \tag{45}$$

Now we are able to study the boundary value problem

$$\begin{aligned}
&M(y) = \lambda y,\\
&U_1(y) = 0 = U_2(y),
\end{aligned} \tag{46}$$

where $M(y)$ is defined by (34) and

$$\begin{aligned}
U_1(y) &= a_{10}[y,\overline{u}](0) + a_{11}[y,\overline{v}](0) + b_{10}y(1) + b_{11}y'(1),\\
U_2(y) &= a_{20}[y,\overline{u}](0) + a_{21}[y,\overline{v}](0) + b_{20}y(1) + b_{21}y'(1).
\end{aligned}$$

In the sequel we assume without loss of generality that these boundary conditions are normalized with respect to $x = 1$, i.e. we assume below that $b_{21} = 0$ and either $b_{11} = 1$ or $b_{11} = b_{20} = 0$ and $b_{10} = 1$.
We mention that

$$\begin{aligned}
U_1(y) &= a_{10}[y,\overline{u}](0) + a_{11}[y,\overline{v}](0) + \widetilde{b_{10}}[y,\overline{u}](1) + \widetilde{b_{11}}[y,\overline{v}](1),\\
U_2(y) &= a_{20}[y,\overline{u}](0) + a_{21}[y,\overline{v}](0) + \widetilde{b_{20}}[y,\overline{u}](1) + \widetilde{b_{21}}[y,\overline{v}](1)
\end{aligned}$$

for appropriate constants $\widetilde{b}_{11}, \widetilde{b}_{12}, \widetilde{b}_{21}, \widetilde{b}_{22}$; this shows that these boundary conditions are of the general form (25).

The following definition shows how the classical definition of Birkhoff-regular boundary value problems (i.e. problems without singular points) can be generalized to problems of the form (46). A similar definition can be made for more general second order and for higher order boundary value problems with singular endpoints of LC type.

Definition 4.4 The boundary value problem (46) is called Birkhoff-regular if either
Case 1: $b_{11} = 1$ and $\vartheta_1 := a_{20}ie^{i\pi\eta} - a_{21} \neq 0$
or
Case 2: $b_{11} = b_{20} = 0$, $b_{10} = 1$ and $\vartheta_1 \neq 0$
or
Case 3: $a_{20} = a_{21} = 0$, $b_{20} = b_{11} = 1$ and $\vartheta_2 := a_{10}ie^{i\pi\eta} - a_{11} \neq 0$.

Remark 4.5 (i) In the special case $A = B = 0$ (then $\eta = \frac{1}{2}$ and the coefficient q_0 in (34) has no singularity) the conditions of Definition 3.3 correspond essentially to the classical conditions for Birkhoff-regularity (see for example [12], 4.8.).
In this case and also for $\mid$ Re $\eta \mid = \frac{1}{2}$ the boundary conditions $U_i(y)$ can be written in the classical form as

$$\begin{aligned}
U_1(y) &= (a_{10}u'(0) + a_{11}v'(0))\,y(0) - (a_{10}u(0) + a_{11}v(0))y'(0) + b_{10}y(1) + b_{11}y'(1),\\
U_2(y) &= (a_{20}u'(0) + a_{21}v'(0))\,y(0) - (a_{20}u(0) + a_{21}v(0))y'(0) + b_{20}y(1).
\end{aligned} \tag{47}$$

With the corresponding abbreviations introduced in [12], p. 57, we derive in Case 1

$$\frac{\Theta_{-1}}{s} + \Theta_0 + \Theta_1 s := \begin{vmatrix} s - a_{10}u(0) - a_{11}v(0) & a_{10}u(0) + a_{11}v(0) - \frac{1}{s} \\ -a_{20}u(0) - a_{21}v(0) & a_{20}u(0) + a_{21}v(0) \end{vmatrix}.$$

Therefore (46) is Birkhoff-regular in the classical sense if we have

$$\Theta_{-1} = -\Theta_1 = \begin{vmatrix} a_{10}u(0) + a_{11}v(0) & 1 \\ a_{20}u(0) + a_{21}v(0) & 0 \end{vmatrix} \neq 0.$$

From (37), (38) we conclude that $u(0) = -ie^{i\pi\eta}v(0) \neq 0$, therefore $\Theta_{-1} \neq 0 \neq \Theta_1$ is equivalent with the condition of Definition 4.4, Case 1 and, as can be derived similarly, also in Case 2 and Case 3.

(ii) It is worth while to mention that for $\mid \operatorname{Re} \eta \mid = \frac{1}{2}$ there exists another subclass of boundary conditions that we should call Birkhoff-regular (since these boundary conditions are for $\eta = \frac{1}{2}$ regular in the classical sense) although this class does not belong to the three cases considered in Definition 4.4. This results from the fact that the asymptotic estimates derived in Theorem 4.1 do not allow us to normalize the boundary conditions under consideration also with respect to $x = 0$ if $\mid \operatorname{Re}\ \eta \mid \neq \frac{1}{2}$. Notice that periodic and anti-periodic boundary conditions are examples of such Birkhoff-regular boundary conditions (in the sense of Naimark, [12]) that do not satisfy one of the three conditions of Definition 4.4. We mention that the factor $a_{20}u(0) + a_{21}v(0)$ in front of $y'(0)$ in (47) vanishes if and only if $\vartheta_1 = 0$. Since we are here mainly interested in the case $\mid \operatorname{Re}\ \eta \mid \neq \frac{1}{2}$ we do not discuss this special situation in detail.

The concept of Birkhoff-regularity introduced above allows us to investigate singular boundary value problems of the form (46) analogously to the well known non-singular case. The following theorem on the distribution of the eigenvalues of (46) is the main result of this section.

Theorem 4.6 *Assume* $0 < \mid \operatorname{Re} \eta \mid < 1$. *If the boundary value problem (46) is Birkhoff-regular it has a countable set of eigenvalues* λ_k *which are for* $\eta \neq \frac{1}{2}$ *simple for sufficiently large modulus and which can be estimated for* $k \geq k_0$ *by*

$$\lambda_k = (k\pi)^2\{1 + O(k^{-1/2}\log k)\}.$$

In order to prove Theorem 4.6 we derive asymptotic estimates for the characteristic determinant

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1(\cdot,\rho)) & U_1(y_2(\cdot,\rho)) \\ U_2(y_1(\cdot,\rho)) & U_2(y_2(\cdot,\rho)) \end{vmatrix}.$$

Theorem 4.7 *Under the assumptions of Theorem 4.6 we have for* $\rho \in \chi_\delta^{(0)}$

(i) in Case 1

$$\Delta(\rho) = 2^{\frac{5}{4}} i\eta\ \rho^{\eta+1} d_{11}d_{22}\vartheta_1\{ie^{i\pi\eta}e^{i\rho}[[[1]]] + e^{-i\rho}[[[1]]]\} + O(\rho^{\frac{1}{2}}),$$

(ii) in Case 2

$$\Delta(\rho) = 2^{\frac{5}{4}}\eta\ \rho^{\eta} d_{11}d_{22}\vartheta_1\{ie^{i\pi\eta}e^{i\rho}[[[1]]] - e^{-i\rho}[[[1]]]\} + O(\rho^{\frac{1}{2}}).$$

(iii) in Case 3

$$\Delta(\rho) = -2^{\frac{5}{4}}\eta\ \rho^{\eta} d_{11}d_{22}\vartheta_2\{ie^{i\pi\eta}e^{i\rho}[[[1]]] - e^{-i\rho}[[[1]]]\} + 2^{\frac{3}{2}}\rho^{\frac{1}{2}}[i].$$

Proof. We give first the proof for Case 1 - the proof for Case 2 is analogous. From (42) - (45) we obtain in Case 1 with

$$\begin{aligned}
r_{11} &= -2\eta a_{10}[d_{11}d_{12}]_\eta\rho^{\eta+\frac{1}{4}} - 2\eta a_{11}[d_{11}d_{22}]_\eta\rho^{\eta+\frac{1}{4}} + [i]2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{i\rho}, \\
r_{21} &= -2\eta a_{20}[d_{11}d_{12}]_\eta\rho^{\eta+\frac{1}{4}} - 2\eta a_{21}[d_{11}d_{22}]_\eta\rho^{\eta+\frac{1}{4}} + [b_{20}]2^{\frac{1}{4}}\rho^{-\frac{1}{4}}e^{i\rho}, \\
r_{12} &= -2\eta a_{10}[d_{21}d_{12}]_\eta\rho^{\eta+\frac{1}{4}} - 2\eta a_{11}[d_{21}d_{22}]_\eta\rho^{\eta+\frac{1}{4}} - [i]2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{-i\rho}, \\
r_{22} &= -2\eta a_{20}[d_{21}d_{12}]_\eta\rho^{\eta+\frac{1}{4}} - 2\eta a_{21}[d_{21}d_{22}]_\eta\rho^{\eta+\frac{1}{4}} + [b_{20}]2^{\frac{1}{4}}\rho^{-\frac{1}{4}}e^{-i\rho},
\end{aligned}$$

$$\begin{aligned}\Delta(\rho) = &\begin{vmatrix} r_{11} & r_{12} \\ & \\ r_{21} & r_{22} \end{vmatrix} = O(\rho^{\frac{1}{2}}) \\ &+\rho^{\eta+1}\left\{\begin{vmatrix} i2^{\frac{1}{4}} & -2\eta a_{10}d_{12}d_{21} \\ 0 & -2\eta a_{20}d_{12}d_{21} \end{vmatrix} + \begin{vmatrix} i2^{\frac{1}{4}} & -2\eta a_{11}d_{21}d_{22} \\ 0 & -2\eta a_{21}d_{21}d_{22} \end{vmatrix}\right\} e^{i\rho}[[[1]]] \\ &+\rho^{\eta+1}\left\{\begin{vmatrix} -2\eta a_{10}d_{11}d_{12} & -i2^{\frac{1}{4}} \\ -2\eta a_{20}d_{11}d_{12} & 0 \end{vmatrix} + \begin{vmatrix} -2\eta a_{11}d_{11}d_{22} & -i2^{\frac{1}{4}} \\ -2\eta a_{21}d_{11}d_{22} & 0 \end{vmatrix}\right\} e^{-i\rho}[[[1]]] \\ &= O(\rho^{\frac{1}{2}}) + 2^{\frac{5}{4}}\rho^{\eta+1}\eta i d_{22}\vartheta_1(d_{21}e^{i\rho}[[[1]]] + d_{11}e^{-i\rho}[[[1]]]) \\ &= 2^{\frac{5}{4}} i\eta\, \rho^{\eta+1}d_{11}d_{22}\vartheta_1\{ie^{i\pi\eta}e^{i\rho}[[[1]]] + e^{-i\rho}[[[1]]]\} + O(\rho^{\frac{1}{2}}),\end{aligned}$$

since $d_{12} = -ie^{i\eta\pi}d_{22}$ and $d_{21} = ie^{i\eta\pi}d_{11}$. Hence (i) is proved.
In Case 3 we assume (without loss of generality) that $b_{10} = 0$. We obtain,using the preceding notation, and $\vartheta_2 := a_{10}ie^{i\pi\eta} - a_{11} \neq 0$, that in this case

$$\begin{aligned} r_{11} &= 2\eta\rho^{\eta+\frac{1}{4}}d_{22}\vartheta_2[d_{11}]_\eta + [i]2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{i\rho}, \\ r_{12} &= 2\eta\rho^{\eta+\frac{1}{4}}d_{22}\vartheta_2[d_{21}]_\eta - [i]2^{\frac{1}{4}}\rho^{\frac{3}{4}}e^{-i\rho}. \end{aligned}$$

Therefore we get

$$\Delta(\rho) = \begin{vmatrix} r_{11} & r_{12} \\ & \\ e^{i\rho}[1] & e^{-i\rho}[1] \end{vmatrix} 2\rho^{\frac{1}{4}}\rho^{-\frac{1}{2}}$$

$$= 2^{\frac{5}{4}}\ \rho^{\eta}d_{22}\vartheta_2\{-d_{21}e^{i\rho}[[[1]]] + d_{11}e^{-i\rho}[[[1]]]\} + 2^{\frac{3}{2}}\rho^{\frac{1}{2}}[i].$$

On account of $d_{21} = ie^{i\eta\pi}d_{11}$, this proves assertion (iii). ☐

Since we can choose $\delta = \frac{1}{2}$ in Theorem 4.7, it follows that the zeros of $\Delta(\rho)$ (i.e. the square-roots of the eigenvalues of (46)) are asymptotically distributed like the zeros of

$$\{ie^{i\pi\eta}e^{i\rho}[[[1]]] \pm e^{-i\rho}[[[1]]]\} + O(\rho^{\kappa}) = 0,$$

where the upper (lower) sign and $\kappa = -\frac{1}{2} - \eta$ ($\kappa = \frac{1}{2} - \eta$)has to be used in Case 1 (Case 2 and Case 3), respectively.
Therefore we obtain analogously to the proof of Theorem 3.2 in §4 of [12] (see also the results of Langer [10] on the distribution of the zeros of such exponential sums), that the assertion of Theorem 4.6 is valid - we omit details. We mention here only that in many situations (for example if we have estimates for $\Delta(\rho)$ as in Theorem 4.7, Case 3) we can derive more precise asymptotic estimates for the eigenvalues λ_k than those given in Theorem 4.6 (compare the estimates given in [4], Satz 2).

Finally we mention that the case $\eta = 0$ can be treated similarly as above but in this case the solutions of (34) behave near 0 asymptotically like linear combinations of

$$x^{\frac{1}{2}}\rho^{\frac{1}{4}}J_{\frac{1}{2}}(x\rho) \text{ and } \log(x\rho)x^{\frac{1}{2}}\rho^{\frac{1}{4}}J_{\frac{1}{2}}(x\rho).$$

5 An expansion theorem

To complete the investigation of the singular boundary value problem (46) we prove an expansion theorem with respect to its eigenfunctions. The method used is based on estimating the Green's function $G(x, \xi, \lambda)$ with respect to the eigenvalue parameter λ. For $f \in L_2[0, 1]$ the solution of the inhomogeneous equation $M(y) = \rho^2 y + f(x)$ is of the following form if $\lambda = \rho^2$ is no eigenvalue of (46):

$$y(x, \rho) = \int_0^1 G(x, t, \rho^2)f(t)dt.$$

Using an arbitrary fundamental system $w_1(\cdot,\rho), w_2(\cdot,\rho)$ of solutions of (34), the Green's function G is defined by

$$G(x,t,\rho^2) = \frac{H(x,t,\rho)}{\Delta_w(\rho)} \text{ for } (x,t) \in [0,1]^2,\ \rho \in \chi_\delta^{(0)},\ \ \Delta_w(\rho) \neq 0.$$

$$H(x,t,\rho) := \left| \begin{array}{lll} w_1(x,\rho) & w_2(x,\rho) & g(x,t,\rho) \\ U_1(w_1) & U_1(w_2) & U_1(g(\cdot,t,\rho))(t) \\ U_2(w_1) & U_2(w_2) & U_2(g(\cdot,t,\rho))(t) \end{array} \right|,$$

$$\Delta_w(\rho) = \left| \begin{array}{ll} U_1(w_1) & U_1(w_2) \\ U_2(w_1) & U_2(w_2) \end{array} \right|$$

and

$$g(x,t,\rho) := \frac{1}{W(w_1,w_2)} \begin{cases} w_1(x,\rho)w_2(t,\rho) \text{ if } x \le t, \\ w_2(x,\rho)w_1(t,\rho) \text{ if } x > t. \end{cases}$$

We consider the contour integral

$$S_R(f)(x) := \frac{1}{2\pi i} \int_0^1 \int_{\Gamma_{R^2}} G(x,t,\lambda) f(t) d\lambda\, dt \quad (x \in [0,1]), \tag{48}$$

where $R > 0$ is chosen such that there is no eigenvalue on the circle Γ_{R^2} of radius R^2. Since the eigenvalues λ_n $n \in \mathbf{N}$, are poles of the Green's function and since the residues $\mathrm{Res}_{\lambda_n} G(x,t,\lambda)$ can (see [12]) be represented by products of eigen- and associated functions (e.a.f.) of (1.1), (1.2) and of the corresponding adjoint problem, which we denote by φ_n and ψ_n, the integral $S_R(f)(x)$ represents a partial sum of the expansion of f into a series in e.a.f. of (46):

$$S_R(f)(x) = \sum_{n \in M_R} \alpha_n \varphi_n(x)$$

with

$$\alpha_n := \int_0^1 f(t)\overline{\psi_n(t)} dt,\ M_R := \{n \in \mathbf{N} |\ |\lambda_n| < R^2\}.$$

If f has an absolutely continuous derivative f' with $f'' \in L[0,1]$ and if in addition $U_j(f) = 0$ $1 \le j \le 2$ then we infer from (48) by partial integration and using the properties of the Green's function (see for example [12]) that

$$S_R(f)(x) = f(x) + \frac{1}{2\pi i} \int_{\Gamma_{R^2}} \int_0^1 \frac{G(x,t,\lambda)}{\lambda} M(f)(t)\, dt\, d\lambda, \tag{49}$$

provided all integrals exist.

Using here the fundamental systems

$$\begin{array}{l} w_1(x,\rho) = y_1(x,\rho), \\ w_2(x,\rho) = e^{i\rho} y_2(x,\rho), \end{array}$$

for $0 < x \le 1$ and $0 \le \arg\rho \le \frac{\pi}{2},\ \rho \neq 0$ and

$$\begin{array}{l} w_1(x,\rho) = y_2(x,\rho), \\ w_2(x,\rho) = e^{-i\rho}\, y_1(x,\rho), \end{array}$$

for $0 < x \le 1$, and $-\frac{\pi}{2} \le \arg\rho \le 0,\ \rho \neq 0$ and the asymptotic estimates proved in Section 4 for $y_1(x,\rho)$ and $y_2(x,\rho)$, it can be shown by elementary but lengthy calculations that $G(x,\xi,\rho^2)$ satisfies the subsequent asymptotic estimates. Since the method used is well established and similar to that used for example in [5], Sections 6.2 and 6.3 we omit details.

Lemma 5.1 *There exists a strictly increasing sequence* $(R_n)_{n\in\mathbf{N}}$ *with* $\lim_{n\to\infty} R_n = \infty$ *and a constant* $c > 0$ *such that*

(i) in case $0 < |\text{Re}\,\eta| \le \frac{1}{2}$

$$|G(x,\xi,\lambda)| \le \frac{c}{|\lambda|^{\frac{1}{4}}} \tag{50}$$

uniformly for $0 \le x, \xi, \le 1$ *and* $\lambda \in \bigcup_{n\in\mathbf{N}} \Gamma_{R_n^2}$,

(ii) in case $\frac{1}{2} < |\text{ Re}\,\eta| < 1$

$$|G(x,\xi,\lambda)| \le (|x||\xi|)^{\dfrac{1}{2} - |\text{ Re}\,\eta|} \frac{c}{|\lambda|^{\frac{1}{4}}}, \tag{51}$$

uniformly for $0 < x,\ \xi \le 1$ *and* $\lambda \in \bigcup_{n\in\mathbf{N}} \Gamma_{R_n^2}$.

Consequently we can derive from (48),$\cdots$, (51) the following expansion theorem:

Theorem 5.2 *Let* f *have an absolutely continuous derivative* f' *with* $f'' \in L[0,1]$ *and let* $U_j(f) = 0$ *for* $1 \le j \le 2$. *Moreover assume that (36) holds and that (46) is Birkhoff-regular. Then (with* (R_n) *defined as in Lemma 5.1)*

a) *for* $0 < |\text{ Re}\,\eta| \le \frac{1}{2}$

$$\lim_{n\to\infty} S_{R_n}(f) = f(x)$$

uniformly on $[0,1]$*;*

b) for $\frac{1}{2} < |\text{ Re}\,\eta| < 1$ *and under the assumption*

$$\int_0^1 |t|^{\dfrac{1}{2} - |\text{Re}\,\eta|} |f(t)|dt < \infty \tag{52}$$

$$\lim_{n\to\infty} S_{R_n}(f)(x) = f(x)$$

uniformly on any interval $[a,1],\ 0 < a < 1$.

Notice that the assumptions of Theorem 4.7 ensure that, as a consequence of Lemma 5.1, the integrals

$$\int_0^1 G(x,\xi,\lambda)f(\xi)d\xi \text{ and, consequently, } \int_0^1 G(x,\xi,\lambda)\ell(f(\xi)d\xi$$

exist and that the second term in (49) tends to 0 for $R_n \to \infty$ uniformly in $[0,1]$ for $0 < |\text{ Re}\,\eta| \le \frac{1}{2}$ and in $[a,1]$ for $\frac{1}{2} < |\text{ Re}\,\eta| < 1$, respectively.

We mention that in [5], Assumption 6.1, the term "with $l_\nu = -1$" should be omitted and the integrability condition in Theorem 6.9 must be corrected; moreover, for $1 \le \nu \le m$ one should assume therein that

$$(x - x_\nu)^{1 - 2\text{Re}\,\eta_\nu}[\frac{B_\nu}{x - x_\nu} + C_\nu(x)] \in L[x_\nu - \varepsilon, x_\nu + \varepsilon],\ \varepsilon < \varepsilon_0,$$

since [5], Theorem 2.4, is taken from [11], where an equivalent assumption has been used for its proof.

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