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Inverse problems for differential operators with singular boundary conditions

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Dedicated to the memory of F. V. Atkinson

Singular boundary conditions are formulated for Sturm–Liouville operators having singularities and turning points at the end-points of the interval. For boundary-value problems with singular boundary conditions, inverse problems of spectral analysis are studied. We give formulations of the inverse problems both for the case of separated and non-separated singular boundary conditions. For each class of inverse problems we prove a uniqueness theorem and give a procedure for constructing the solution of the inverse problem.

1 Introduction

We consider a class of singular differential equations of the form

$$-\frac{d}{dt}\left(p_2(t)\,\frac{dz}{dt}\right) + p_1(t)z(t) \;=\; \lambda p_0(t)z(t)\,, \quad t \in (t_0, t_1)\,. \tag{1.1}$$

Here λ is the spectral parameter, and the real- or complex-valued functions $p_k(t)$ have zeros or/and singularities at the end-points of the interval (t_0, t_1). More precisely,

$$p_k(t) \;=\; (t-t_0)^{s_{k0}}(t_1-t)^{s_{k1}}p_{k0}(t)\,,$$

where s_{km} are real numbers, $p_{10}(t) \in C[t_0, t_1]$, $p_{k0}(t) \in C^2[t_0, t_1]$, $p_{k0}(t) > 0$ for $k = 0, 2$, and $t \in [t_0, t_1]$. We assume that $s_{2m} < s_{0m} + 2$, $s_{2m} \le s_{1m} + 2$, $m = 0, 1$, i.e., we consider the case of the so-called regular singularities. Irregular singularities possess different qualitative properties and require different investigations.

Since the solutions of Eq. (1.1) have singularities at the end-points of the interval, and since in general the values of the solutions and of their derivatives at the end-points are not defined, an important question is how to introduce singular two-point boundary conditions in the general case under consideration. For some particular cases, this problem was studied in [1]–[6] and other works. For example, in [1] singular boundary conditions were constructed in the case when the end-points are of "limit-circle" type. In a preceding paper [7] we demonstrated how one can define two-point singular boundary conditions in the general case—see also Section 2 where we construct singular boundary conditions and formulate the corresponding boundary-value problems.

The main purpose of this paper is to study inverse spectral problems for differential operators with singular boundary conditions (see Sections 3 and 4). For the classical Sturm–Liouville operators, inverse problems have been studied fairly completely in many works (see, for example, [8]–[11] and the references therein). Singular boundary conditions produce new qualitative modifications in the investigation of the inverse problems. To study the inverse problem for singular boundary conditions, in this paper we develop the ideas of the method of spectral mappings [12]. This gives us an opportunity to construct the inverse problem theory for operators

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with singular boundary conditions. Section 3 is devoted to inverse problems for the case of separated singular boundary conditions, and in Section 4 we study the inverse problem for boundary-value problems with non-separated singular boundary conditions. For each class of these inverse problems we prove a uniqueness theorem and give a procedure for constructing the solution of the inverse problem. For studying the inverse problem for non-separated singular boundary conditions in Section 4 we essentially use the results obtained in Section 3 for the case of separated singular boundary conditions.

For simplicity, we confine ourselves to the case when there are no singularities and turning points inside the interval. We note that direct and inverse spectral problems for differential equations having singularities and/or turning points inside the interval (with classical boundary conditions at the end-points) were studied in [13]–[24] and other works.

2 Boundary-value problems with singular boundary conditions

Denote

$$r(t) \;=\; \frac{p_0(t)}{p_2(t)}\,, \quad \chi(t) \;=\; \frac{p_1(t)}{p_2(t)} + \frac{\ddot{p}_2(t)}{2p_2(t)} - \left(\frac{\dot{p}_2(t)}{2p_2(t)}\right)^2, \quad T \;=\; \int_{t_0}^{t_1} (r(\xi))^{1/2}\,d\xi\,,$$

$s_m = s_{0m} - s_{2m}$. Then $s_m > -2$, and there exist the finite limits

$$\chi_0 \;=\; \lim_{t\to t_0+0}(t-t_0)^2\chi(t)\,, \quad \chi_1 \;=\; \lim_{t\to t_1-0}(t_1-t)^2\chi(t)\,.$$

Denote

$$\nu_m \;=\; \frac{2}{s_m+2}\,(\chi_m+1/4)^{1/2}\,, \quad m \;=\; 0\,,1\,.$$

For definiteness, let $\nu_m > 0$, $\nu_m \notin \mathbf{N}$, $m = 0,1$ (other cases require minor modifications). We transform Eq. (1.1) by means of the replacement

$$x \;=\; \int_{t_0}^{t} (r(\xi))^{1/2}\,d\xi\,, \quad y(x) \;=\; (p_0(t)p_2(t))^{1/4}z(t)$$

to the differential equation

$$-y''(x) + q(x)y(x) \;=\; \lambda y(x)\,, \quad x \in (0,T)\,, \tag{2.1}$$

where $q(x) = \ddot{r}(t)\big(4r^2(t)\big)^{-1} - 5(\dot{r}(t))^2\big(16r^3(t)\big)^{-1} + \chi(t)(r(t))^{-1}$ is real-valued. The function $q(x)$ is continuous for $x \in (0,T)$, and it has quadratic singularities at the ends of the interval:

$$q(x) \;=\; \begin{cases} \dfrac{\omega_0}{x^2} + q_0(x)\,, & x \in (0,T/2]\,,\\[2mm] \dfrac{\omega_1}{(T-x)^2} + q_0(x)\,, & x \in (T/2,T)\,, \end{cases}$$

where $\omega_m = \nu_m^2 - 1/4,\ m = 0,1$. We assume that $q_0(x)x^{-2\theta_0}(T-x)^{-2\theta_1} \in \mathcal{L}(0,T)$, where $\theta_m := \nu_m - 1/2$, $m = 0,1$.

In [25] fundamental systems of solutions $\{S_{jm}(x,\lambda)\}_{j=1,2}$, $m = 0,1$, of Eq. (2.1) were constructed with the following properties:

(a) For each fixed $x \in (0,T)$, the functions $S_{jm}^{(\nu)}(x,\lambda)$, $\nu = 0,1$, are entire in λ.

(b) Denote $\mu_{jm} = (-1)^j\nu_m + 1/2$. Then for $j = 1,2$,

$$S_{j0}(x,\lambda) \;=\; O(x^{\mu_{j0}}) \text{ as } x \longrightarrow 0 \qquad \text{and} \qquad S_{j1}(x,\lambda) \;=\; O((T-x)^{\mu_{j1}}) \text{ as } x \longrightarrow T\,.$$

(c) The following relation

$$\langle S_{1m}(x,\lambda), S_{2m}(x,\lambda)\rangle \;\equiv\; 1\,, \quad m \;=\; 0\,,1\,, \tag{2.2}$$

holds, where $\langle y(x), \widetilde{y}(x)\rangle := y(x)\widetilde{y}'(x) - y'(x)\widetilde{y}(x)$ is the Wronskian of y and $\widetilde{y}$.

(d) Let $\lambda = \rho^2$. Here and below we assume that $\arg\rho \in (-\pi/2, \pi/2]$. For $|\rho| \to \infty$, $|\rho|\, x \ge 1$, $|\rho|(T-x) \ge 1$, $\nu = 0, 1$, $j = 1, 2$,

$$S_{j0}^{(\nu)}(x,\lambda) \;=\; d_{j0}\rho^{-\mu_{j0}}\big((-i\rho)^\nu \exp(-i\rho x)[1]_0 + (i\rho)^\nu \exp(-i\pi\mu_{j0})\exp(i\rho x)[1]_0\big)\,, \tag{2.3}$$

$$\begin{aligned} S_{j1}^{(\nu)}(x,\lambda) \;=\; &(-1)^{j-1} d_{j1}\rho^{-\mu_{j1}}\,\big((i\rho)^\nu \exp(-i\rho(T-x))[1]_0 \\ &+(-i\rho)^\nu \exp(-i\pi\mu_{j1})\exp(i\rho(T-x))[1]_0\big)\,, \end{aligned} \tag{2.4}$$

where $d_{1m} = 1,\ d_{2m} = -(4i\sin\pi\nu_m)^{-1},\ m = 0,1,\ [1]_0 = 1 + O\big((\rho x)^{-\beta}\big) + O\big((\rho(T-x))^{-\beta}\big)$ and $\beta := \min(1, 2\nu_0, 2\nu_1)$, i.e., $f(x,\rho) = [1]_0$ means $|f(x,\rho) - 1| \le C\big(|\rho x|^{-\beta} + |\rho(T-x)|^{-\beta}\big)$.

We will call the functions $S_{jm}(x,\lambda)$ the Bessel-type solutions for Eq. (2.1).

Let us introduce the linear forms

$$\sigma_{km}(y) \;:=\; (-1)^{k-1}\langle y(x), S_{3-k,m}(x,\lambda)\rangle\big|_{x=mT}\,, \quad k = 1, 2, \quad m = 0, 1\,.$$

It follows from Eq. (2.2) that

$$\sigma_{km}(S_{jm}) \;=\; \delta_{jk}\,, \quad j, k = 1, 2, \quad m = 0, 1\,, \tag{2.5}$$

where δ_{jk} is the Kronecker symbol.

We note that for the classical Sturm–Liouville equation one has $\nu_m = 1/2$ (i.e., $\omega_m = 0$), $m = 0, 1$. Hence in this case $\sigma_{km}(y) = y^{(k-1)}(mT)$, $k = 1, 2$, $m = 0, 1$, i.e., the boundary functionals have the classical form.

The linear forms $\sigma_{km}(y)$ allow us to introduce singular two-point boundary conditions of the general form for Eq. (2.1):

$$a_{k1}^0\sigma_{10}(y) + a_{k2}^0\sigma_{20}(y) + a_{k1}^1\sigma_{11}(y) + a_{k2}^1\sigma_{21}(y) \;=\; 0\,, \quad k = 1, 2\,,$$

where we assume that

$$\operatorname{rank}\begin{bmatrix} a_{11}^0 & a_{12}^0 & a_{11}^1 & a_{12}^1 \\ a_{21}^0 & a_{22}^0 & a_{21}^1 & a_{22}^1 \end{bmatrix} \;=\; 2\,.$$

Remark 2.1 Similarly one can introduce singular boundary conditions also for Eq. (1.1). Denote

$$\{z(t), \widetilde{z}(t)\} \;:=\; p_2(t)\bigg(z(t)\,\frac{d\widetilde{z}(t)}{dt} - \widetilde{z}(t)\,\frac{dz(t)}{dt}\bigg)\,.$$

Then

$$\{z(t), \widetilde{z}(t)\} \;=\; \langle y(x), \widetilde{y}(x)\rangle\,,$$

where $y(x) = (p_0(t)p_2(t))^{1/4}z(t)$, $\widetilde{y}(x) = (p_0(t)p_2(t))^{1/4}\widetilde{z}(t)$, $x = \int_{t_0}^t (r(\xi))^{1/2}\,d\xi$. Moreover, if $z(t)$ and $\widetilde{z}(t)$ are solutions of Eq. (1.1) then the expression $\{z(t), \widetilde{z}(t)\}$ does not depend on t. Let

$$s_{jm}(t,\lambda) \;:=\; (p_0(t)p_2(t))^{1/4}S_{jm}(x,\lambda)\,, \qquad j = 1, 2, \quad m = 0, 1\,.$$

$$\tau_{km}(z) := (-1)^{k-1}\{z(t), s_{3-k,m}(t,\lambda)\}\big|_{t=t_m}\,, \quad k = 1, 2, \quad m = 0, 1\,.$$

Then the functions $s_{jm}(t,\lambda)$ are solutions of Eq. (1.1) and $\tau_{km}(z) = \sigma_{km}(y)$, $k = 1, 2$, $m = 0, 1$.

The linear forms $\tau_{km}(z)$ allow us to introduce singular two-point boundary conditions of the general form for Eq. (1.1):

$$a_{k1}^0\tau_{10}(z) + a_{k2}^0\tau_{20}(z) + a_{k1}^1\tau_{11}(z) + a_{k2}^1\tau_{21}(z) \;=\; 0\,, \quad k = 1, 2\,.$$

For definiteness we will study below boundary-value problems for Eq. (2.1). Similar results are also valid for Eq. (1.1).

3 Inverse problems for separated boundary conditions

In this section we construct the inverse problem theory for spectral problems defined by the differential equation (2.1) and separated singular boundary conditions (defined by Eq. (3.1)). We study below three inverse problems of recovering operators from their spectral characteristics, namely:

(i) from the Weyl function;
(ii) from the so-called spectral data;
(iii) from two spectra.

For each class of inverse problems we prove the corresponding uniqueness theorems and show connections between the different spectral characteristics. Moreover, we provide a procedure for constructing the solution of the inverse problem.

3.1 Properties of the spectrum

Let us consider the boundary-value problem L defined by Eq. (2.1) and the *separated boundary conditions*

$$U(y) \;:=\; \sigma_{20}(y) - a_1\sigma_{10}(y) \;=\; 0\,, \quad V(y) \;:=\; \sigma_{21}(y) + a_2\sigma_{11}(y) \;=\; 0\,, \tag{3.1}$$

where a_1 and a_2 are real or complex numbers. Notice that this is a generalized notation of *separated boundary conditions* which coincides in the case $\nu_m = 1/2$, $m = 1, 2$, with the classical notation. We resign to associate an operator to L—for this purpose one has to introduce weighted normed subspaces of the space of all solutions of Eq. (2.1) satisfying Eq. (3.1).

Denote by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ the solutions of Eq. (2.1) under the initial conditions $\sigma_{10}(\varphi) = 1$, $\sigma_{20}(\varphi) = a_1$, and $\sigma_{11}(\psi) = 1$, $\sigma_{21}(\psi) = -a_2$, respectively. Clearly,

$$\varphi(x,\lambda) \;=\; S_{10}(x,\lambda) + a_1 S_{20}(x,\lambda)\,, \quad \psi(x,\lambda) \;=\; S_{11}(x,\lambda) - a_2 S_{21}(x,\lambda)\,, \tag{3.2}$$

$$U(\varphi) \;=\; 0\,,\; U_0(\varphi) \;=\; 1\,, \qquad V(\psi) \;=\; 0\,,\; V_0(\psi) \;=\; 1\,, \tag{3.3}$$

where $U_0(y) := \sigma_{10}(y)$, $V_0(y) := \sigma_{11}(y)$. Denote

$$\Delta(\lambda) \;=\; \langle \psi(x,\lambda), \varphi(x,\lambda)\rangle\,. \tag{3.4}$$

By virtue of Liouville's formula for the Wronskian [26, p. 83] $\langle \psi(x,\lambda), \varphi(x,\lambda)\rangle$ does not depend on x. Moreover, the function $\Delta(\lambda)$ is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}$. The function $\Delta(\lambda)$ is called the *characteristic function* of L.

Lemma 3.1 *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary-value problem L. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n) \;=\; \beta_n\varphi(x,\lambda_n)\,, \quad \beta_n \;\neq\; 0\,.$$

Proof. 1) Since the functions $S_{j0}(x,\lambda)$, $j = 1, 2$, form a fundamental system of solutions for Eq. (2.1), one has

$$S_{k1}(x,\lambda) \;=\; \alpha_{k1}(\lambda)S_{10}(x,\lambda) + \alpha_{k2}(\lambda)S_{20}(x,\lambda)\,, \quad 0 \;<\; x \;<\; T\,, \quad k \;=\; 1\,,2\,. \tag{3.5}$$

Using Eqs. (2.2), (2.5) and (3.5) we calculate

$$\begin{cases} \alpha_{11}(\lambda) \;=\; \sigma_{10}(S_{11}) \;=\; \sigma_{21}(S_{20})\,, & \alpha_{12}(\lambda) \;=\; \sigma_{20}(S_{11}) \;=\; -\sigma_{21}(S_{10})\,, \\ \alpha_{21}(\lambda) \;=\; \sigma_{10}(S_{21}) \;=\; -\sigma_{11}(S_{20})\,, & \alpha_{22}(\lambda) \;=\; \sigma_{20}(S_{21}) \;=\; \sigma_{11}(S_{10})\,, \end{cases} \tag{3.6}$$

$$\det[\alpha_{kj}(\lambda)]_{k,j=1,2} \;=\; \det[\sigma_{k0}(S_{j1})]_{k,j=1,2} \;=\; \det[\sigma_{k1}(S_{j0})]_{k,j=1,2} \;=\; 1\,. \tag{3.7}$$

It follows obviously from Eqs. (3.1), (3.2), (3.4) and (3.6) that

$$\Delta(\lambda) \;=\; V(\varphi) \;=\; -U(\psi)\,. \tag{3.8}$$

2) Let λ_0 be a zero of $\Delta(\lambda)$. Then, by virtue of Eqs. (3.3), (3.4) and (3.8), $\psi(x,\lambda_0) = \beta_0\varphi(x,\lambda_0)$, and the functions $\psi(x,\lambda_0)$ and $\varphi(x,\lambda_0)$ satisfy the boundary conditions (3.1). Hence, λ_0 is an eigenvalue, and $\psi(x,\lambda_0)$ and $\varphi(x,\lambda_0)$ are eigenfunctions related to λ_0.

3) Let λ_0 be an eigenvalue of L, and let y_0 be a corresponding eigenfunction. Then $U(y_0) = V(y_0) = 0$. Clearly $\sigma_{10}(y_0) \neq 0$ (if $\sigma_{10}(y_0) = 0$ then $\sigma_{20}(y_0) = 0$, and consequently $y_0(x) \equiv 0$). Without loss of generality we assume that $\sigma_{10}(y_0) = 1$. Then $\sigma_{20}(y_0) = a_1$, hence $y_0(x) \equiv \varphi(x,\lambda_0)$. Therefore, Eq. (3.8) yields $\Delta(\lambda_0) = V(\varphi(x,\lambda_0)) = V(y_0(x)) = 0$. We have also proved that for each eigenvalue there exists only one (up to a multiplicative constant) eigenfunction. □

Theorem 3.2 *The boundary-value problem L has a countable set of eigenvalues $\{\lambda_n\}_{n\geq 0}$. For $n \to \infty$, define*

$$\rho_n \;:=\; \sqrt{\lambda_n} \;=\; \frac{\pi}{T}\left(n + p + \frac{\mu_{10}+\mu_{11}}{2} + O\left(\frac{1}{n^\beta}\right)\right), \tag{3.9}$$

where $\beta := \min(1, 2\nu_0, 2\nu_1)$, and $p \in \mathbf{Z}$ does not depend on $q_0(x)$, a_1 and a_2, and depends only on ν_0 and ν_1.

P r o o f. It follows from Eqs. (3.1), (3.2), (3.6) and (3.8) that

$$\Delta(\lambda) \;=\; -\alpha_{12}(\lambda) + a_1\alpha_{11}(\lambda) + a_2\alpha_{22}(\lambda) - a_1a_2\alpha_{21}(\lambda)\,. \tag{3.10}$$

It was shown in [7] that for $|\rho| \to \infty$, $k,j = 1,2$,

$$\alpha_{kj}(\lambda) \;=\; 2i(-1)^{k-j+1}d_{3-j,0}d_{k1}\rho^{1-\kappa_{jk}}\big(\exp(-i\rho T)[1] - \exp(-i\pi\kappa_{jk})\exp(i\rho T)[1]\big)\,, \tag{3.11}$$

where $\kappa_{jk} = \mu_{3-j,0} + \mu_{k1}$, $[1] = 1 + O\big(\rho^{-\beta}\big)$. In particular, Eq. (3.11) yields

$$|\alpha_{kj}(\lambda)| \;\leq\; C\left|\rho^{1-\kappa_{jk}}\right|\exp(|\mathrm{Im}\,\rho|\,T)\,. \tag{3.12}$$

Here and below the symbol C denotes various positive constants in the estimates. Substituting Eqs. (3.11) and (3.12) into Eq. (3.10) we obtain

$$\Delta(\lambda) \;=\; -\,2i\rho^{\nu_0+\nu_1}\big(\exp(-i\rho T)[1] - \exp(-i\pi(\mu_{10}+\mu_{11}))\exp(i\rho T)[1]\big) \quad \text{as} \quad |\rho| \longrightarrow \infty\,, \tag{3.13}$$

$$|\Delta(\lambda)| \;\leq\; C\,|\rho|^{\nu_0+\nu_1}\exp(|\mathrm{Im}\,\rho|\,T)\,. \tag{3.14}$$

Similarly one can calculate

$$\dot{\Delta}(\lambda) \;=\; -\,T\rho^{\nu_0+\nu_1-1}\big(\exp(-i\rho T)[1] + \exp(-i\pi(\mu_{10}+\mu_{11}))\exp(i\rho T)[1]\big)\,, \quad \text{as} \quad |\rho| \longrightarrow \infty\,, \tag{3.15}$$

where $\dot{\Delta}(\lambda) := \frac{d}{d\lambda}\,\Delta(\lambda)$. Using Eq. (3.13) and Rouche's theorem [27, p. 125] we arrive by the usual way (see [28, Ch. 1] and the proof of [7, Theorem 1]) at Eq. (3.9).

Fix $\delta > 0$. Denote $G_\delta := \{\rho \in \mathbf{C} : \; |\rho - \rho_n| \geq \delta,\; n \geq 0\}$. By the well-known method [11, Ch. 1] one can get the following estimate from below:

$$|\Delta(\lambda)| \;\geq\; C\,|\rho|^{\nu_0+\nu_1}\exp(|\mathrm{Im}\,\rho|\,T)\,, \quad \rho \in G_\delta\,. \tag{3.16}$$

Furthermore, let $\big\{\lambda_n^0\big\}_{n\geq 0}$, $\lambda_n^0 = \big(\rho_n^0\big)^2$, be the eigenvalues for the case $q_0(x) \equiv 0$, $a_1 = a_2 = 0$, and let $\Delta^0(\lambda)$ be the corresponding characteristic function. Notice that both ρ_n and ρ_n^0 satisfy estimates of the form (3.9). Then, by virtue of Eq. (3.13),

$$\big|\Delta(\lambda) - \Delta^0(\lambda)\big| \;\leq\; C\,|\rho|^{\nu_0+\nu_1-\beta}\exp(|\mathrm{Im}\,\rho|\,T)\,. \tag{3.17}$$

It follows from Eqs. (3.16) and (3.17), and Rouche's theorem that

$$\rho_n \;=\; \rho_n^0 + O\left(\frac{1}{n^\beta}\right) \quad \text{as} \quad n \longrightarrow \infty\,, \tag{3.18}$$

and consequently, the integer p from Eq. (3.9) does not depend on $q_0(x)$, a_1 and a_2. Hence Theorem 3.2 is proved. □

It follows from Eqs. (3.9) and (3.15) that $\dot{\Delta}(\lambda_n) \neq 0$ for sufficiently large n, i.e., the zeros of $\Delta(\lambda)$ are simple for $n \geq n^*$. For convenience we assume below that all eigenvalues are simple—the general case requires minor modifications.

The authors are grateful to a referee who read the first version of our manuscript carefully; he pointed out that the boundary conditions (3.1) are—even in the case of real coefficients—in general non-selfadjoint (with respect to an adequate scalar product) and provided the following simple example that shows that there may exist non-real eigenvalues:

Choose $a_1 = a_2 = 0$ and $q(x) = \frac{2}{x^2}$, $0 < x \leq 1$, i.e., $\nu_0 = 3/2$ and $\nu_1 = 1/2$; then

$$\Delta(\rho^2) \;=\; c\left[\rho\sin\rho + \cos\rho - \rho^2 cos\rho\right] \quad \text{with} \quad c \neq 0\,,$$

which has for example a non-real zero approximately at

$$\lambda \;=\; \rho^2\,, \quad \rho \;=\; 1.08177098 + 0.753012221\,i\,.$$

Notice that in this case the boundary conditions are not separated in the classical sense.

3.2 The inverse problem from the Weyl function

Let $\Phi(x,\lambda)$ be the solution of Eq. (2.1) under the conditions $U(\Phi) = 1$ and $V(\Phi) = 0$. We set $M(\lambda) := U_0(\Phi)$. The function $M(\lambda)$ is called the Weyl function for the boundary-value problem L. The notion of the Weyl function introduced here is a generalization of the notion of the Weyl function for the classical Sturm–Liouville operators (see [11] and [29]). Clearly,

$$\Phi(x,\lambda) \;=\; -\frac{\psi(x,\lambda)}{\Delta(\lambda)} \;=\; S_{20}(x,\lambda) + M(\lambda)\varphi(x,\lambda)\,, \tag{3.19}$$

$$\langle\varphi(x,\lambda), \Phi(x,\lambda)\rangle \;\equiv\; 1\,, \tag{3.20}$$

$$M(\lambda) \;=\; -\frac{d(\lambda)}{\Delta(\lambda)}\,, \tag{3.21}$$

where $d(\lambda) := U_0(\psi)$. Thus, the Weyl function is meromorphic with simple poles in the eigenvalues $\lambda = \lambda_n$, $n \geq 0$.

Inverse Problem 1. Given the Weyl function $M(\lambda)$, construct $q(x)$, a_1 and a_2.

Let us prove the uniqueness theorem for the solution of the Inverse Problem 1. For this purpose we agree that together with L we consider a boundary-value problem $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x)$, $\widetilde{a}_1$ and $\widetilde{a}_2$. Everywhere below if a certain symbol α denotes an object related to L, then the corresponding symbol $\widetilde{\alpha}$ with tilde denotes the analogous object related to $\widetilde{L}$.

Theorem 3.3 *If $M(\lambda) = \widetilde{M}(\lambda)$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0,T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$. Thus, the specification of the Weyl function uniquely determines L.*

P r o o f. It follows from Eqs. (3.2) and (3.6) that

$$d(\lambda) \;=\; \alpha_{11}(\lambda) - a_2\alpha_{21}(\lambda)\,. \tag{3.22}$$

Substituting Eqs. (3.11) and (3.12) into Eq. (3.22) we calculate

$$d(\lambda) \;=\; -\,2id_{20}\rho^{\nu_1-\nu_0}\big(\exp(-i\rho T)[1] - \exp(-i\pi(\mu_{20}+\mu_{11}))\exp(i\rho T)[1]\big) \quad \text{as} \quad |\rho| \longrightarrow \infty\,, \tag{3.23}$$

$$|d(\lambda)| \;\leq\; C\,|\rho|^{\nu_1-\nu_0}\exp(|\mathrm{Im}\,\rho|\,T)\,. \tag{3.24}$$

Using Eqs. (3.16), (3.21) and (3.24) we infer

$$|M(\lambda)| \;\leq\; C\,|\rho|^{-2\nu_0}\,, \quad \rho \in G_\delta\,. \tag{3.25}$$

Moreover, Eqs. (3.13), (3.21) and (3.23) yield

$$\begin{cases} M(\lambda) \;=\; -d_{20}\rho^{-2\nu_0}[1] & \text{as} \quad |\rho| \longrightarrow \infty\,, \quad \arg\rho \in (0,\pi/2)\,, \\ M(\lambda) \;=\; -d_{20}\rho^{-2\nu_0}\exp(-2\pi i\nu_0)[1] & \text{as} \quad |\rho| \longrightarrow \infty\,, \quad \arg\rho \in (-\pi/2,0)\,. \end{cases} \tag{3.26}$$

Since $M(\lambda) = \widetilde{M}(\lambda)$, it follows from Eq. (3.26) that $\nu_0 = \widetilde{\nu}_0$.

Furthermore, the functions $\Delta(\lambda)$ and $d(\lambda)$ have no common zeros (otherwise $U_0(\psi) = U(\psi) = 0$ which is impossible); hence $\lambda_n = \widetilde{\lambda}_n$ for all $n \geq 0$. It follows from Eq. (3.14) that $\Delta(\lambda)$ is entire in λ of order $1/2$, and consequently by Hadamard's factorization theorem [27, p. 289] $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) \;=\; A\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right) \tag{3.27}$$

(the case when $\Delta(0) = 0$ requires minor modifications). The constants A and ν_1 can be uniquely determined with the help of the asymptotics Eq. (3.13). Indeed, since $\lambda_n = \widetilde{\lambda}_n$ for all $n \geq 0$, it follows from Eq. (3.27) that $\widetilde{\Delta}(\lambda)/\Delta(\lambda) \equiv \widetilde{A}/A$. On the other hand, in view of Eq. (3.13) and the equality $\widetilde{\nu}_0 = \nu_0$, we get

$$\frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} \;=\; \rho^a[1]\,, \quad a \;=\; \widetilde{\nu}_1 - \nu_1 \quad \text{as} \quad |\rho| \longrightarrow \infty\,, \quad \arg\rho \;=\; \pi/2\,,$$

and consequently, $\widetilde{\nu}_1 = \nu_1$ and $\widetilde{\Delta}(\lambda) \equiv \Delta(\lambda)$. By virtue of Eq. (3.21) this yields $d(\lambda) \equiv \widetilde{d}(\lambda)$. Since $\nu_m = \widetilde{\nu}_m$, $m = 0, 1$, it follows from Eqs. (2.3), (2.4) and (3.2) that for $|\rho|\,x \geq 1$, $|\rho|(T-x) \geq 1$, $\nu = 0, 1$,

$$\begin{cases} \left|\varphi^{(\nu)}(x,\lambda)\right| \;\leq\; C\,|\rho|^{\theta_0+\nu}\exp(|\tau|\,x)\,, \quad \left|\psi^{(\nu)}(x,\lambda)\right| \;\leq\; C\,|\rho|^{\theta_1+\nu}\exp(|\tau|(T-x))\,, \\ \left|\varphi^{(\nu)}(x,\lambda) - \widetilde{\varphi}^{(\nu)}(x,\lambda)\right| \;\leq\; C\,|\rho|^{\theta_0+\nu-\beta}\exp(|\tau|\,x)\,, \\ \left|\psi^{(\nu)}(x,\lambda) - \widetilde{\psi}^{(\nu)}(x,\lambda)\right| \;\leq\; C\,|\rho|^{\theta_1+\nu-\beta}\exp(|\tau|(T-x))\,, \end{cases} \tag{3.28}$$

where $\tau := \operatorname{Im}\rho$.

Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix} \widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{bmatrix} \;=\; \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{3.29}$$

Using Eqs. (3.20) and (3.29) we calculate for $j = 1, 2$:

$$\begin{cases} P_{j1}(x,\lambda) \;=\; \varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda)\,, \\ P_{j2}(x,\lambda) \;=\; \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}(x,\lambda)\,, \end{cases} \tag{3.30}$$

$$\begin{cases} \varphi(x,\lambda) \;=\; P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda)\,, \\ \Phi(x,\lambda) \;=\; P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda)\,. \end{cases} \tag{3.31}$$

It follows from Eqs. (3.19), (3.20) and (3.30) that

$$\begin{aligned} P_{11}(x,\lambda) \;=\; & 1 + \frac{\psi(x,\lambda)}{\Delta(\lambda)}\big(\widetilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda)\big) + \frac{\varphi(x,\lambda)}{\Delta(\lambda)}\big(\psi'(x,\lambda) - \widetilde{\psi}'(x,\lambda)\big) \\ & + \frac{\widetilde{\Delta}(\lambda) - \Delta(\lambda)}{\Delta(\lambda)\widetilde{\Delta}(\lambda)}\,\varphi(x,\lambda)\widetilde{\psi}'(x,\lambda)\,, \end{aligned}$$

$$P_{12}(x,\lambda) \;=\; \frac{1}{\Delta(\lambda)}\big(\varphi(x,\lambda)\widetilde{\psi}(x,\lambda) - \psi(x,\lambda)\widetilde{\varphi}(x,\lambda)\big) + \frac{\Delta(\lambda) - \widetilde{\Delta}(\lambda)}{\Delta(\lambda)\widetilde{\Delta}(\lambda)}\,\varphi(x,\lambda)\widetilde{\psi}(x,\lambda)\,.$$

By virtue of Eqs. (3.16), (3.17) and (3.28), this yields for $|\rho|\,x \geq 1$, $|\rho|(T-x) \geq 1$, $\rho \in G_\delta$,

$$|P_{11}(x,\lambda) - 1| \;\leq\; \frac{C}{|\rho|^\beta}\,, \quad |P_{12}(x,\lambda)| \;\leq\; \frac{C}{|\rho|^\beta}\,, \tag{3.32}$$

$$|P_{22}(x,\lambda) - 1| \;\leq\; \frac{C}{|\rho|^\beta}\,, \quad |P_{21}(x,\lambda)| \;\leq\; C\,|\rho|^{1-\beta}\,. \tag{3.33}$$

According to Eqs. (3.19) and (3.30),

$$P_{11}(x,\lambda) \;=\; \varphi(x,\lambda)\widetilde{S}_{20}'(x,\lambda) - S_{20}(x,\lambda)\widetilde{\varphi}'(x,\lambda) + (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda)\,,$$
$$P_{12}(x,\lambda) \;=\; S_{20}(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\widetilde{S}_{20}(x,\lambda) + (M(\lambda) - \widetilde{M}(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,.$$

Since $M(\lambda) \equiv \widetilde{M}(\lambda)$, it follows that for each fixed $x \in (0,T)$, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. Together with Eq. (3.32) this yields $P_{11}(x,\lambda) \equiv 1$ and $P_{12}(x,\lambda) \equiv 0$. Substituting into Eq. (3.31) we get $\varphi(x,\lambda) \equiv \widetilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \widetilde{\Phi}(x,\lambda)$ for all x and λ, and consequently, $L = \widetilde{L}$. ☐

3.3 The inverse problem from the spectral data

The Weyl function $M(\lambda)$ is meromorphic with simple poles in the points $\lambda_n,\ n \geq 0$. Denote

$$\alpha_n \;=\; \operatorname*{Res}_{\lambda=\lambda_n} M(\lambda)\,. \tag{3.34}$$

The data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ are called the spectral data of L.

Inverse Problem 2. Given the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$, construct $q(x)$, a_1 and a_2.

Let us first prove a uniqueness theorem for the solution of the Inverse Problem 2.

Theorem 3.4 *If $\lambda_n = \widetilde{\lambda}_n$ and $\alpha_n = \widetilde{\alpha}_n$ for all $n \geq 0$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0,T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$. Thus, the specification of the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ uniquely determines L.*

P r o o f. It follows from Eq. (3.21) that

$$\alpha_n \;=\; -\frac{d(\lambda_n)}{\dot{\Delta}(\lambda_n)}\,, \tag{3.35}$$

hence $\alpha_n \neq 0$ for all n. Using Eqs. (3.9), (3.15), (3.23) and (3.35) we calculate

$$\alpha_n \;=\; \alpha n^{1-2\nu_0}\big(1 + O\big(n^{-\beta}\big)\big) \quad \text{as} \quad n \;\longrightarrow\; \infty\,, \tag{3.36}$$

where

$$\alpha \;=\; -id_{20}T^{2\nu_0-2}\pi^{1-2\nu_0}(1 - \exp(-2\pi i\nu_0)) \;\neq\; 0\,. \tag{3.37}$$

Let us show that

$$M(\lambda) \;=\; \sum_{n=0}^{\infty} \frac{\alpha_n}{\lambda - \lambda_n}\,. \tag{3.38}$$

By virtue of Eqs. (3.9) and (3.36), the series in Eq. (3.38) converges absolutely for each $\lambda \neq \lambda_n$. In order to prove Eq. (3.38) we consider the contour integral

$$J_N(\lambda) \;:=\; \frac{1}{2\pi i}\int_{\Gamma_N} \frac{M(\mu)}{\lambda - \mu}\,d\mu\,, \quad \lambda \in int\,\Gamma_N\,,$$

where the contours $\Gamma_N := \{\lambda : |\lambda| = R_N\}$, $R_N \to \infty$ (with counterclockwise circuit) are chosen such that $\Gamma_N \subset G_\delta$ and $int\,\Gamma_N \cap \{\lambda_n : n \geq 0\} = \big\{\lambda_n : n = \overline{0,N}\,\big\}$. In view of Eq. (3.25), $\lim_{N\to\infty} J_N(\lambda) = 0$. On the other hand, the residue theorem and Eq. (3.34) yield

$$J_N(\lambda) \;=\; -M(\lambda) + \sum_{n=0}^{N} \frac{\alpha_n}{\lambda - \lambda_n}\,,$$

and consequently Eq. (3.38) is proved.

Since $\lambda_n = \widetilde{\lambda}_n$ and $\alpha_n = \widetilde{\alpha}_n$ for all $n \geq 0$, it follows from Eq. (3.38) that $M(\lambda) \equiv \widetilde{M}(\lambda)$. According to Theorem 3.3 this implies $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$. Theorem 3.4 is proved. □

Remark 3.5 By virtue of Eq. (3.38), the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$, i.e., the Inverse Problem 1 is equivalent to the Inverse Problem 2.

3.4 The inverse problem from two spectra

Let $\{\mu_n\}_{n\geq 0}$ be the eigenvalues of the boundary-value problem L_1 for Eq. (2.1) with the boundary conditions $U_0(y) = V(y) = 0$.

Inverse Problem 3. Given two spectra $\{\lambda_n,\, \mu_n\}_{n\geq 0}$, construct $q(x)$, a_1 and a_2.

Theorem 3.6 *If $\lambda_n = \widetilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$ for all $n \geq 0$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$.*

Proof. According to Lemma 2.1, the set $\{\lambda_n\}_{n\geq 0}$ coincides with the set of zeros of the function $\Delta(\lambda) = -U(\psi)$. Similarly, the set $\{\mu_n\}_{n\geq 0}$ coincides with the set of zeros of the function $d(\lambda) = U_0(\psi)$. Since $\Delta(\lambda)$ and $d(\lambda)$ are entire in λ of order $1/2$, it follows that $\Delta(\lambda)$ and $d(\lambda)$ are uniquely determined up to multiplicative constants by their zeros. Taking the asymptotics (3.13) and (3.23) into account we obtain $\nu_m = \widetilde{\nu}_m$, $m = 0, 1$, $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$ and $d(\lambda) \equiv \widetilde{d}(\lambda)$. Together with Eq. (3.21) this yields $M(\lambda) \equiv \widetilde{M}(\lambda)$. By Theorem 3.3 we get $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$. □

Remark 3.7 It follows from the proofs of Theorems 3.3 and 3.6 that the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$, i.e., the Inverse Problem 1 is equivalent to the Inverse Problem 3.

Remark 3.8 Similar results are also valid for any other class of separated singular boundary conditions. Let us briefly formulate the corresponding results for one of these classes which will be used in Section 4.

Let $\{z_n\}_{n\geq 0}$ be the eigenvalues of the boundary-value problem Q_1 for Eq. (2.1) with the boundary conditions $U(y) = V_0(y) = 0$. The set $\{z_n\}_{n\geq 0}$ coincides with the set of zeros of the entire function $\gamma(\lambda) := V_0(\varphi)$. It follows from Eqs. (3.2) and (3.6) that $\gamma(\lambda) = \alpha_{22}(\lambda) - a_1\alpha_{21}(\lambda)$, and consequently, by virtue of Eq. (3.11),

$$\gamma(\lambda) \;=\; -\,2id_{21}\rho^{\nu_0-\nu_1}\big(\exp(-i\rho T)[1] - \exp(-i\pi(\mu_{10}+\mu_{21}))\exp(i\rho T)[1]\big) \quad \text{as} \quad |\rho| \longrightarrow \infty\,. \tag{3.39}$$

Denote $\gamma_0(\lambda) := V_0(S_2)$. Let $\{z_n^0\}_{n\geq 0}$ be the zeros of $\gamma_0(\lambda)$, i.e., $\{z_n^0\}_{n\geq 0}$ are the eigenvalues of the boundary-value problem Q_0 for Eq. (2.1) with the boundary conditions $U_0(y) = V_0(y) = 0$. The function $M_0(\lambda) = \gamma_0(\lambda)/\gamma(\lambda)$ is the Weyl function for Q_1, and

$$M_0(\lambda) \;=\; \sum_{n=0}^{\infty} \frac{\alpha_n^0}{\lambda - z_n}\,, \qquad \alpha_n^0 \;:=\; \operatorname*{Res}_{\lambda=z_n} M_0(\lambda)\,.$$

Clearly,

$$\alpha_n^0 \;=\; \frac{\gamma_0(z_n)}{\dot{\gamma}(z_n)}\,, \tag{3.40}$$

The data $\{z_n, \alpha_n^0\}_{n\geq 0}$ are the spectral data for Q_1. The following uniqueness theorem is an analog of Theorems 3.3, 3.4 and 3.6 for the boundary-value problem Q_1.

Theorem 3.9 1) *If $M_0(\lambda) = \widetilde{M}_0(\lambda)$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$ and $a_1 = \widetilde{a}_1$.*
2) *If $z_n = \widetilde{z}_n$ and $\alpha_n^0 = \widetilde{\alpha}_n^0$ for all $n \geq 0$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$ and $a_1 = \widetilde{a}_1$.*
3) *If $z_n = \widetilde{z}_n$ and $z_n^0 = \widetilde{z}_n^0$ for all $n \geq 0$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$ and $a_1 = \widetilde{a}_1$.*

3.5 Solution of the the inverse problem

In this subsection we give a constructive procedure for the solution of the inverse problem of recovering L from the given spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$. For this purpose we develop the ideas of the method of spectral mappings (see [11] and [12]). The central role here is played by the so-called main equation of the inverse problem which connects the spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation which is a linear equation in a suitable Banach space. Moreover, we prove the unique solvability of the main equation. Using the solution of the main equation we provide explicit formulae for the solution of the inverse problem under consideration.

Let $\{\lambda_n, \alpha_n\}_{n\geq 0}$ be the spectral data of L and $\rho_n := \sqrt{\lambda_n}$. Using Eqs. (3.13) and (3.36) we can find ν_m, $m = 0, 1$. Indeed, by virtue of Eq. (3.36),

$$\nu_0 \;=\; \frac{1}{2}\left(1 - \lim_{n\to\infty}\frac{\ln \alpha_n}{\ln n}\right). \tag{3.41}$$

It follows from Eqs. (3.13) and (3.27) that

$$\nu_0 + \nu_1 \;=\; \lim \frac{\ln F(\rho)}{\ln \rho} \quad \text{as} \quad |\rho| \longrightarrow \infty\,, \quad \arg\rho \in (0, \pi/2)\,, \tag{3.42}$$

where

$$F(\rho) \;:=\; \exp(i\rho T)\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right).$$

Let us choose a model boundary-value problem $\widetilde{L}$ such that $\nu_m = \widetilde{\nu}_m$, $m = 0, 1$, with arbitrary $\widetilde{q}_0(x)$, $\widetilde{a}_1$ and $\widetilde{a}_2$. Denote

$$\xi_n \;:=\; |\rho_n - \widetilde{\rho}_n| + (n+1)^{2\theta_0}\,|\alpha_n - \widetilde{\alpha}_n| \quad \text{and} \quad D(x, \lambda, \mu) \;:=\; \frac{\langle \varphi(x,\lambda), \varphi(x,\mu)\rangle}{\lambda - \mu}\,.$$

It follows from Eqs. (3.18) and (3.36) that $\xi_n = O\big(n^{-\beta}\big)$, and consequently,

$$\sup_{n\geq 0}\sum_{k=0}^{\infty}\frac{\xi_k}{|n-k|+1} \;<\; \infty\,. \tag{3.43}$$

Denote

$$\begin{aligned}
&\lambda_{n0} \;=\; \lambda_n\,, \quad \lambda_{n1} \;=\; \widetilde{\lambda}_n\,,\\
&\alpha_{n0} \;=\; \alpha_n\,, \quad \alpha_{n1} \;=\; \widetilde{\alpha}_n\,,\\
&\varphi_{ni}(x) \;=\; \varphi(x, \lambda_{ni})\,, \quad \widetilde{\varphi}_{ni}(x) \;=\; \widetilde{\varphi}(x, \lambda_{ni})\,,\\
&P_{ni,kj}(x) \;=\; D(x, \lambda_{ni}, \lambda_{kj})\alpha_{kj}\,, \;\; \widetilde{P}_{ni,kj}(x) \;=\; \widetilde{D}(x, \lambda_{ni}, \lambda_{kj})\alpha_{kj}\,, \;\; n\,,\, k \,\geq\, 0\,, \;\; i\,, j \,=\, 0\,, 1\,.
\end{aligned}$$

Fix $\varepsilon > 0$. Let $x \in [\varepsilon, T - \varepsilon]$, $n \geq 0$ and $i, \nu = 0, 1$. It follows from Eqs. (3.9) and (3.28) that

$$\big|\varphi_{ni}^{(\nu)}(x)\big| \;\leq\; C(n+1)^{\theta_0 + \nu}\,. \tag{3.44}$$

Moreover, for a fixed $a > 0$,

$$\big|\varphi^{(\nu)}(x,\lambda)\big| \;\leq\; C(n+1)^{\theta_0+\nu}\,, \quad |\rho - \rho_{n1}| \;\leq\; a\,.$$

Applying Schwarz's lemma [27, p. 130] in the ρ-plane to the circle $|\rho - \rho_{n1}| \leq a$ and to the function $f(\rho) := \varphi^{(\nu)}(x, \lambda) - \varphi^{(\nu)}(x, \lambda_{n1})$ with fixed ν, n, x and a, we get

$$\big|\varphi^{(\nu)}(x,\lambda) - \varphi^{(\nu)}(x,\lambda_{n1})\big| \;\leq\; C(n+1)^{\theta_0+\nu}\,|\rho - \rho_{n1}|\,, \quad |\rho - \rho_{n1}| \;\leq\; a\,.$$

In particular, this yields

$$\big|\varphi_{n0}^{(\nu)}(x) - \varphi_{n1}^{(\nu)}(x)\big| \;\leq\; C(n+1)^{\theta_0+\nu}\,|\rho_n - \widetilde{\rho}_n|\,. \tag{3.45}$$

By similar arguments one gets that the following estimates are valid for $x \in [\varepsilon, T-\varepsilon]$, $n, k \geq 0$, and $i, j, \nu = 0, 1$:

$$\begin{cases} |P_{ni,kj}(x)| \;\leq\; \dfrac{C(n+1)^{\theta_0}}{(k+1)^{\theta_0}(|n-k|+1)}\,, \\ |P_{ni,k0}(x) - P_{ni,k1}(x)| \;\leq\; \dfrac{C\xi_k(n+1)^{\theta_0}}{(k+1)^{\theta_0}(|n-k|+1)}\,, \\ |P_{n0,kj}(x) - P_{n1,kj}(x)| \;\leq\; \dfrac{C\xi_n(n+1)^{\theta_0}}{(k+1)^{\theta_0}(|n-k|+1)}\,, \\ |P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x)| \;\leq\; \dfrac{C\xi_n\xi_k(n+1)^{\theta_0}}{(k+1)^{\theta_0}(|n-k|+1)}\,. \end{cases} \tag{3.46}$$

The analogous estimates are also valid for $\widetilde{\varphi}_{ni}(x)$ and $\widetilde{P}_{ni,kj}(x)$.

Lemma 3.10 *The following relations hold:*

$$\widetilde{\varphi}(x,\lambda) \;=\; \varphi(x,\lambda) + \sum_{k=0}^{\infty}\left(\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}_{k0}(x)\rangle\alpha_{k0}}{\lambda-\lambda_{k0}}\,\varphi_{k0}(x) - \frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}_{k1}(x)\rangle\alpha_{k1}}{\lambda-\lambda_{k1}}\,\varphi_{k1}(x)\right), \tag{3.47}$$

$$\begin{aligned} &\frac{\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle}{\lambda-\mu} - \frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\mu)\rangle}{\lambda-\mu} \\ &+ \sum_{k=0}^{\infty}\left(\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}_{k0}(x)\rangle\alpha_{k0}}{\lambda-\lambda_{k0}}\,\frac{\langle\varphi_{k0}(x),\varphi(x,\mu)\rangle}{\lambda_{k0}-\mu} - \frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}_{k1}(x)\rangle\alpha_{k1}}{\lambda-\lambda_{k1}}\,\frac{\langle\varphi_{k1}(x),\varphi(x,\mu)\rangle}{\lambda_{k1}-\mu}\right) = 0\,. \end{aligned} \tag{3.48}$$

Both series converge absolutely and uniformly with respect to $x \in [\varepsilon, T-\varepsilon]$, λ and μ on compact sets.

P r o o f. 1) Denote $\lambda' = \min_{ni}\lambda_{ni}$ and take a fixed $h > 0$.

Let $I := \{\lambda :\ |\mathrm{Im}\,\lambda| \leq h,\ \mathrm{Re}\,\lambda \geq \lambda' - h\}$, and let $\gamma := \partial I$ be the boundary of I.

Denote $\Gamma'_N = \Gamma_N \cap I$, $\Gamma''_N = \Gamma_N \setminus \Gamma'_N$, $\gamma'_N = \gamma \cap int\,\Gamma_N$. In the λ-plane we consider closed contours $\gamma_N = \gamma'_N \cup \Gamma'_N$, $\gamma^0_N = \gamma'_N \cup \Gamma''_N$ (with counterclockwise circuit). It follows from Eqs. (3.30) and (3.19) that for each fixed $x \in (0,T)$, the functions $P_{jk}(x,\lambda)$ are meromorphic in λ with simple poles $\{\lambda_n\}_{n\geq 0}$ and $\{\widetilde{\lambda}_n\}_{n\geq 0}$. By Cauchy's integral formula [27, p. 84],

$$P_{1k}(x,\lambda) - \delta_{1k} \;=\; \frac{1}{2\pi i}\int_{\gamma^0_N}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\lambda-\xi}\,d\xi\,, \quad k \;=\; 1\,,2\,, \quad \lambda \in int\,\gamma^0_N\,. \tag{3.49}$$

Hence

$$P_{1k}(x,\lambda) - \delta_{1k} \;=\; \frac{1}{2\pi i}\int_{\gamma_N}\frac{P_{1k}(x,\xi)}{\lambda-\xi}\,d\xi - \frac{1}{2\pi i}\int_{\Gamma_N}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\lambda-\xi}\,d\xi\,, \tag{3.50}$$

where Γ_N is used with counterclockwise circuit. Substituting into Eq. (3.31) we obtain

$$\varphi(x,\lambda) \;=\; \widetilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma_N}\frac{\widetilde{\varphi}(x,\lambda)P_{11}(x,\xi) + \widetilde{\varphi}'(x,\lambda)P_{12}(x,\xi)}{\lambda-\xi}\,d\xi + \varepsilon_N(x,\lambda)\,,$$

where

$$\varepsilon_N(x,\lambda) \;=\; -\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\widetilde{\varphi}(x,\lambda)(P_{11}(x,\xi)-1) + \widetilde{\varphi}'(x,\lambda)P_{12}(x,\xi)}{\lambda-\xi}\,d\xi\,.$$

By virtue of Eq. (3.32) we have that

$$\lim_{N\to\infty}\varepsilon_N(x,\lambda) \;=\; 0 \tag{3.51}$$

uniformly with respect to $x \in [\varepsilon, T-\varepsilon]$ and λ on compact sets. Taking Eq. (3.30) into account we calculate

$$\begin{aligned} \varphi(x,\lambda) \;=\; \widetilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma_N}\Big(&\widetilde{\varphi}(x,\lambda)\Big(\varphi(x,\xi)\widetilde{\Phi}'(x,\xi) - \Phi(x,\xi)\widetilde{\varphi}'(x,\xi)\Big) \\ &+ \widetilde{\varphi}'(x,\lambda)\Big(\Phi(x,\xi)\widetilde{\varphi}(x,\xi) - \varphi(x,\xi)\widetilde{\Phi}(x,\xi)\Big)\Big)\,\frac{d\xi}{\lambda-\xi} + \varepsilon_N(x,\lambda)\,. \end{aligned}$$

In view of Eq. (3.19), this yields

$$\widetilde{\varphi}(x,\lambda) \;=\; \varphi(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma_N} \frac{\langle \widetilde{\varphi}(x,\lambda), \widetilde{\varphi}(x,\xi)\rangle}{\lambda-\xi}\,\widehat{M}(\xi)\varphi(x,\xi)\,d\xi + \varepsilon_N(x,\lambda)\,, \tag{3.52}$$

where $\widehat{M}(\lambda) = M(\lambda) - \widetilde{M}(\lambda)$, since the terms with $S_{20}(x,\xi)$ vanish by Cauchy's theorem. It follows from Eq. (3.34) that

$$\operatorname*{Res}_{\xi=\lambda_{kj}} \frac{\langle \widetilde{\varphi}(x,\lambda), \widetilde{\varphi}(x,\xi)\rangle}{\lambda-\xi}\,\widehat{M}(\xi)\varphi(x,\xi) \;=\; (-1)^j\,\frac{\langle \widetilde{\varphi}(x,\lambda), \widetilde{\varphi}_{kj}(x)\rangle \alpha_{kj}}{\lambda-\lambda_{kj}}\,\varphi_{kj}(x)\,.$$

Calculating the integral in Eq. (3.52) by the residue theorem [27, p. 112] and using Eq. (3.51) we arrive at Eq. (3.47).

2) Since

$$\frac{1}{\lambda-\mu}\left(\frac{1}{\lambda-\xi} - \frac{1}{\mu-\xi}\right) \;=\; \frac{1}{(\lambda-\xi)(\xi-\mu)}\,,$$

we have by Cauchy's integral formula

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda-\mu} \;=\; \frac{1}{2\pi i}\int_{\gamma_N^0} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi\,, \quad k\,,\, j \;=\; 1\,,\, 2\,, \quad \lambda\,,\, \mu \in int\,\gamma_N^0\,.$$

Acting in the same way as above and using Eqs. (3.32) and (3.33) we obtain

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda-\mu} \;=\; \frac{1}{2\pi i}\int_{\gamma_N} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi + \varepsilon_{Njk}(x,\lambda,\mu)\,, \tag{3.53}$$

where $\lim_{N\to\infty} \varepsilon_{Njk}(x,\lambda,\mu) = 0$, $j,k = \overline{1,n}$.

From Eqs. (3.30) and (3.20) it follows that

$$\begin{cases} P_{11}(x,\lambda)\varphi'(x,\lambda) - P_{21}(x,\lambda)\varphi(x,\lambda) \;=\; \widetilde{\varphi}'(x,\lambda)\,, \\ P_{22}(x,\lambda)\varphi(x,\lambda) - P_{12}(x,\lambda)\varphi'(x,\lambda) \;=\; \widetilde{\varphi}(x,\lambda)\,, \end{cases} \tag{3.54}$$

$$P(x,\lambda)\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} \;=\; \big\langle y(x), \widetilde{\Phi}(x,\lambda)\big\rangle \begin{bmatrix} \varphi(x,\lambda) \\ \varphi'(x,\lambda) \end{bmatrix} - \langle y(x), \widetilde{\varphi}(x,\lambda)\rangle \begin{bmatrix} \Phi(x,\lambda) \\ \Phi'(x,\lambda) \end{bmatrix}, \tag{3.55}$$

for any $y(x) \in C^1[0,1]$. Taking Eqs. (3.53) and (3.55) into account, we calculate

$$\begin{aligned} &\frac{P(x,\lambda) - P(x,\mu)}{\lambda-\mu}\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} \\ &= \frac{1}{2\pi i}\int_{\gamma_N}\left(\big\langle y(x), \widetilde{\Phi}(x,\xi)\big\rangle \begin{bmatrix} \varphi(x,\xi) \\ \varphi'(x,\xi) \end{bmatrix} - \langle y(x),\ \widetilde{\varphi}(x,\xi)\rangle \begin{bmatrix} \Phi(x,\xi) \\ \Phi'(x,\xi) \end{bmatrix}\right) \frac{d\xi}{(\lambda-\xi)(\xi-\mu)} \\ &\quad + \varepsilon_N^0(x,\lambda,\mu)\,, \end{aligned} \tag{3.56}$$

with $\lim_{N\to\infty} \varepsilon_N^0(x,\lambda,\mu) = 0$. According to Eq. (3.29),

$$P(x,\lambda)\begin{bmatrix} \widetilde{\varphi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) \end{bmatrix} \;=\; \begin{bmatrix} \varphi(x,\lambda) \\ \varphi'(x,\lambda) \end{bmatrix}.$$

Therefore,

$$\det\left(P(x,\lambda)\begin{bmatrix} \widetilde{\varphi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) \end{bmatrix}, \begin{bmatrix} \varphi(x,\mu) \\ \varphi'(x,\mu) \end{bmatrix}\right) \;=\; \langle \varphi(x,\lambda), \varphi(x,\mu)\rangle\,.$$

Furthermore, using Eq. (3.54) we get

$$\det\left(P(x,\mu)\begin{bmatrix}\widetilde{\varphi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda)\end{bmatrix},\begin{bmatrix}\varphi(x,\mu)\\ \varphi'(x,\mu)\end{bmatrix}\right) = \widetilde{\varphi}(x,\lambda)(P_{11}(x,\mu)\varphi'(x,\mu)-P_{21}(x,\mu)\varphi(x,\mu)) - \widetilde{\varphi}'(x,\lambda)(P_{22}(x,\mu)\varphi(x,\mu)-P_{12}(x,\mu)\varphi'(x,\mu)) = \langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\mu)\rangle\,.$$

Thus,

$$\det\left((P(x,\lambda)-P(x,\mu))\begin{bmatrix}\widetilde{\varphi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda)\end{bmatrix},\begin{bmatrix}\varphi(x,\mu)\\ \varphi'(x,\mu)\end{bmatrix}\right) = \langle\varphi(x,\lambda),\varphi(x,\mu)\rangle-\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\mu)\rangle\,.$$

Consequently, Eq. (3.56) for $y(x)=\widetilde{\varphi}(x,\lambda)$ yields

$$\frac{\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle}{\lambda-\mu}-\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\mu)\rangle}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma_N}\left(\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\Phi}(x,\xi)\rangle\langle\varphi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}-\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\xi)\rangle\langle\Phi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}\right)d\xi + \varepsilon_N^1(x,\lambda,\mu)\,,$$

with $\lim_{N\to\infty}\varepsilon_N^1(x,\lambda,\mu)=0$. By virtue of Eqs. (3.19), (3.34) and the residue theorem, we arrive at Eq. (3.48). □

It follows from the definition of $\widetilde{P}_{ni,kj}(x)$, $P_{ni,kj}(x)$, $\varphi_{ni}(x)$ and $\widetilde{\varphi}_{ni}(x)$, and from Eqs. (3.47) and (3.48) that

$$\widetilde{\varphi}_{ni}(x) = \varphi_{ni}(x)+\sum_{k=0}^{\infty}\left(\widetilde{P}_{ni,k0}(x)\varphi_{k0}(x)-\widetilde{P}_{ni,k1}(x)\varphi_{k1}(x)\right), \tag{3.57}$$

$$P_{ni,\ell j}(x)-\widetilde{P}_{ni,\ell j}(x)+\sum_{k=0}^{\infty}\left(\widetilde{P}_{ni,k0}(x)P_{k0,\ell j}(x)-\widetilde{P}_{ni,k1}(x)P_{k1,\ell j}(x)\right) = 0\,. \tag{3.58}$$

For each fixed $x\in(0,T)$, the relation (3.57) can be considered as a system of linear equations with respect to $\varphi_{ni}(x)$, $n\ge 0$, $i=0,1$. But the series in (3.57) converges only “with brackets”. Therefore, it is not convenient to use (3.57) as a main equation of the inverse problem. Below we will transfer Eq. (3.57) to a linear equation in a corresponding Banach space of sequences (see Eqs. (3.61) or (3.62)).

Let V be a set of indices $u=(n,i)$, $n\ge 0$, $i=0,1$. For each fixed $x\in(0,T)$, we define the vector

$$\psi(x) = [\psi_u(x)]_{u\in V} = \begin{bmatrix}\psi_{n0}(x)\\ \psi_{n1}(x)\end{bmatrix}_{n\ge 0} = [\psi_{00},\psi_{01},\psi_{10},\psi_{11},\psi_{20},\psi_{21},\dots]^T$$

by the formulae

$$\psi_{n0}(x) = \frac{\chi_n}{(n+1)^{\theta_0}}\left(\varphi_{n0}(x)-\varphi_{n1}(x)\right) \quad\text{and}\quad \psi_{n1}(x) = \frac{1}{(n+1)^{\theta_0}}\varphi_{n1}(x)\,,$$

where

$$\chi_n = \begin{cases}\xi_n^{-1}\,, & \xi_n \ne 0\,,\\ 0\,, & \xi_n = 0\,.\end{cases}$$

We also define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v\in V} = \begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix}_{n,k\ge 0}, \quad u=(n,i)\,,\quad v=(k,j)\,,$$

by the formulae

$$\begin{aligned} H_{n0,k0}(x) &= \xi_k \chi_n \frac{(k+1)^{\theta_0}}{(n+1)^{\theta_0}} \big(P_{n0,k0}(x) - P_{n1,k0}(x)\big) , \\ H_{n0,k1}(x) &= \chi_n \frac{(k+1)^{\theta_0}}{(n+1)^{\theta_0}} \big(P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x)\big) , \\ H_{n1,k0}(x) &= \xi_k \frac{(k+1)^{\theta_0}}{(n+1)^{\theta_0}} P_{n1,k0}(x) , \\ H_{n1,k1}(x) &= \frac{(k+1)^{\theta_0}}{(n+1)^{\theta_0}} \big(P_{n1,k0}(x) - P_{n1,k1}(x)\big) . \end{aligned}$$

Analogously we define $\widetilde{\psi}(x)$ and $\widetilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$ by $\widetilde{\varphi}_{ni}(x)$ and $P_{ni,kj}(x)$ by $\widetilde{P}_{ni,kj}(x)$. It follows from Eqs. (3.44)–(3.46) that

$$\big|\psi_{ni}^{(\nu)}(x)\big| \le C(n+1)^{\nu} , \quad |H_{ni,kj}(x)| \le \frac{C\xi_k}{(|n-k|+1)} . \tag{3.59}$$

Similarly,

$$\big|\widetilde{\psi}_{ni}^{(\nu)}(x)\big| \le C(n+1)^{\nu} , \quad \big|\widetilde{H}_{ni,kj}(x)\big| \le \frac{C\xi_k}{(|n-k|+1)} . \tag{3.60}$$

Let us consider the Banach space m of bounded sequences $\beta = [\beta_u]_{u\in V}$ with the norm $\|\beta\|_m = \sup_{u\in V} |\beta_u|$. It follows from Eqs. (3.43), (3.59) and (3.60) that for each fixed $x \in (0,T)$, the operators $E+\widetilde{H}(x)$ and $E-H(x)$ (here E is the identity operator), acting from m to m, are linear bounded operators, and

$$\|H(x)\| , \big\|\widetilde{H}(x)\big\| \le C \sup_{n\ge 0} \sum_{k=0}^{\infty} \frac{\xi_k}{|n-k|+1} < \infty .$$

Theorem 3.11 *For each fixed $x \in (0,T)$, the vector $\psi(x) \in m$ satisfies the equation*

$$\widetilde{\psi}(x) = \big(E + \widetilde{H}(x)\big)\psi(x) \tag{3.61}$$

in the Banach space m. Moreover, the operator $E + \widetilde{H}(x)$ has a bounded inverse operator, i.e., Eq. (3.61) *is uniquely solvable.*

P r o o f. We rewrite Eq. (3.57) in the form

$$\begin{aligned} \widetilde{\varphi}_{n0}(x) - \widetilde{\varphi}_{n1}(x) &= \varphi_{n0}(x) - \varphi_{n1}(x) + \sum_{k=0}^{\infty} \Big(\big(\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x)\big)(\varphi_{k0}(x) - \varphi_{k1}(x)) \\ &\qquad + \big(\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n0,k1}(x) + \widetilde{P}_{n1,k1}(x)\big)\varphi_{k1}(x)\Big) , \\ \widetilde{\varphi}_{n1}(x) &= \varphi_{n1}(x) + \sum_{k=0}^{\infty} \Big(\widetilde{P}_{n1,k0}(x)(\varphi_{k0}(x) - \varphi_{k1}(x)) + \big(\widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n1,k1}(x)\big)\varphi_{k1}(x)\Big) . \end{aligned}$$

Taking into account our notations, we obtain

$$\widetilde{\psi}_{ni}(x) = \psi_{ni}(x) + \sum_{k,j} \widetilde{H}_{ni,kj}(x)\psi_{kj}(x) , \quad (n,i) , (k,j) \in V , \tag{3.62}$$

which is equivalent to Eq. (3.61). The series in Eq. (3.62) converges absolutely for each fixed $x \in (0,T)$. Similarly, Eq. (3.58) takes the form

$$H_{ni,kj}(x) - \widetilde{H}_{ni,kj}(x) + \sum_{\ell,s} \widetilde{H}_{ni,\ell s}(x) H_{\ell s,kj}(x) = 0 , \quad (n,i) , (k,j) , (\ell,s) \in V ,$$

or

$$\big(E+\widetilde{H}(x)\big)(E-H(x)) \;=\; E\,.$$

Interchanging places for L and $\widetilde{L}$, we obtain analogously

$$\psi(x) \;=\; (E-H(x))\widetilde{\psi}(x)\,, \quad (E-H(x))\big(E+\widetilde{H}(x)\big) \;=\; E\,.$$

Hence the operator $\big(E+\widetilde{H}(x)\big)^{-1}$ exists and it is a linear bounded operator. □

Eq. (3.61) is called the *main equation* of the inverse problem. Solving Eq. (3.61) we find the vector $\psi(x)$, and consequently, the functions $\varphi_{ni}(x)$, $n \ge 0$, $i = 0, 1$. Since $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ are solutions of Eq. (2.1), we can construct the potential $q(x)$ by the formula

$$q(x) \;=\; \lambda_{ni} + \varphi''_{ni}(x)/\varphi_{ni}(x) \tag{3.63}$$

or by the formula

$$q(x) \;=\; \lambda + \varphi''(x,\lambda)/\varphi(x,\lambda)\,, \tag{3.64}$$

where $\varphi(x,\lambda)$ is calculated via Eq. (3.47). Then one can construct the functions $S_{jm}(x,\lambda)$, $j = 1, 2$, $m = 0, 1$, and calculate the coefficients a_1 and a_2 by the formulae

$$a_1 \;=\; \sigma_{20}(\varphi)\,, \quad a_2 \;=\; -\left(\frac{\sigma_{21}(\varphi)}{\sigma_{11}(\varphi)}\right)\bigg|_{\lambda=\lambda_n}\,. \tag{3.65}$$

Thus, we obtain the following algorithm for the solution of the inverse problem.

Algorithm 3.12 *Let the numbers* $\{\lambda_n,\, \alpha_n\}_{n\ge 0}$ *be given.*

(1) *Calculate* ν_m, $m = 0, 1$, *via Eqs.* (3.41) *and* (3.42).
(2) *Choose* $\widetilde{L}$ *such that* $\widetilde{\nu}_m = \nu_m$, $m = 0, 1$, *and construct* $\widetilde{\psi}(x)$ *and* $\widetilde{H}(x)$.
(3) *Find* $\psi(x)$ *by solving Eq.* (3.61).
(4) *Calculate* $q(x)$ *via Eq.* (3.63) *or Eq.* (3.64) *and find* a_1 *and* a_2 *via Eq.* (3.65).

4 The inverse problem for non-separated boundary conditions

Let us consider the boundary-value problem Q for Eq. (2.1) with the following singular boundary conditions

$$\begin{cases} W_1(y) \;:=\; \sigma_{20}(y) - a_1\sigma_{10}(y) + b\sigma_{11}(y) \;=\; 0\,, \\ W_2(y) \;:=\; \sigma_{21}(y) + a_2\sigma_{11}(y) - b\sigma_{10}(y) \;=\; 0\,, \end{cases} \tag{4.1}$$

where $q(x)$, a_1, a_2 and b are real with $b \ne 0$. In this section we study the inverse problem for the boundary-value problem Q. We prove the uniqueness theorem and give a constructive procedure for the solution of the inverse problem. The results obtained here are a generalization of the results from [30] for the classical Sturm–Liouville operators with the boundary conditions $y'(0) - a_1y(0) + by(T) = y'(T) + a_2y(T) - by(0) = 0$.

The set of eigenvalues $\{s_n\}_{n\ge 0}$ of Q coincides with the set of zeros of the entire function

$$s(\lambda) \;:=\; \det[W_k(S_{j0})]_{k,j=1,2}\,.$$

Taking Eqs. (2.5), (3.2), (3.6), (3.7) and (4.1) into account, we calculate

$$s(\lambda) \;=\; -\sigma_{21}(\varphi) - a_2\sigma_{11}(\varphi) + b^2\sigma_{11}(S_2) + 2b\,. \tag{4.2}$$

Using Eqs. (3.2), (3.6), (3.11) and (3.12) we obtain

$$s(\lambda) \;=\; 2i\rho^{\nu_0+\nu_1}\big(\exp(-i\rho T)[1] - \exp(-i\pi(\mu_{10}+\mu_{11}))\exp(i\rho T)[1]\big) \quad \text{as} \quad |\rho| \;\longrightarrow\; \infty\,, \tag{4.3}$$

$$|s(\lambda)| \;\le\; C\,|\rho|^{\nu_0+\nu_1}\exp(|\mathrm{Im}\,\rho|\,T)\,. \tag{4.4}$$

Denote

$$s^+(\lambda) \; = \; s(\lambda) - 4b\,, \tag{4.5}$$

$$\begin{cases} p(\lambda) \; = \; \big(\sigma_{21}(\varphi) + a_2\sigma_{11}(\varphi) - b^2\sigma_{11}(S_2)\big)/2\,, \\ u(\lambda) \; = \; \big(\sigma_{21}(\varphi) + a_2\sigma_{11}(\varphi) + b^2\sigma_{11}(S_2)\big)/2\,. \end{cases} \tag{4.6}$$

Using Eqs. (4.2) and (4.5) we calculate

$$p(\lambda) - b \; = \; -s(\lambda)/2\,, \quad p(\lambda) + b \; = \; -s^+(\lambda)/2\,, \tag{4.7}$$

and consequently,

$$p^2(\lambda) - b^2 \; = \; s(\lambda)s^+(\lambda)/4\,. \tag{4.8}$$

Furthermore, it follows from Eq. (4.6) that

$$\begin{cases} u(\lambda) + p(\lambda) \; = \; \sigma_{21}(\varphi) + a_2\sigma_{11}(\varphi)\,, \\ u(\lambda) - p(\lambda) \; = \; b^2\sigma_{11}(S_2)\,. \end{cases} \tag{4.9}$$

It follows from Eqs. (3.2) and (3.7) that

$$\sigma_{11}(\varphi)\sigma_{21}(S_2) - \sigma_{11}(S_2)\sigma_{21}(\varphi) \; = \; 1\,.$$

Together with Eq. (4.9) this yields

$$u^2(\lambda) - p^2(\lambda) \; = \; -b^2 + b^2\gamma(\lambda)\mu(\lambda)\,, \tag{4.10}$$

where $\mu(\lambda) := \sigma_{21}(S_2) + a_2\sigma_{11}(S_2)$ and $\gamma(\lambda)$ was defined in Subsection 3.4.

Since $\{z_n\}_{n\geq 0}$ is the sequence of zeros of $\gamma(\lambda)$, we infer from Eq. (4.10) that $u^2(z_n) = p^2(z_n) - b^2$, and by virtue of Eq. (4.8) that

$$u^2(z_n) \; = \; s(z_n)s^+(z_n)/4\,. \tag{4.11}$$

Denote $\omega_n := \arg u(z_n) \in [0,\ 2\pi)$. Then Eq. (4.11) yields

$$u(z_n) \; = \; \frac{\omega_n}{2}\sqrt{|\; s(z_n)s^+(z_n)\;|}\,. \tag{4.12}$$

The inverse problem is formulated as follows:

Inverse problem 4. Given λ_n, z_n, ω_n, $n \geq 0$, and b, construct Q.

Let us prove the following uniqueness theorem for the solution of the Inverse Problem 4.

Theorem 4.1 *If $s_n = \widetilde{s}_n$, $z_n = \widetilde{z}_n$, $\omega_n = \widetilde{\omega}_n$, for all $n \geq 0$, and $b = \widetilde{b}$, then $Q = \widetilde{Q}$, i.e., $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$.*

P r o o f. By virtue of Eq. (4.4), the function $s(\lambda)$ is entire in λ of order $1/2$, and consequently, by Hadamard's factorization theorem, $s(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$s(\lambda) \; = \; A_1 \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{s_n}\right).$$

Similarly,

$$\gamma(\lambda) \; = \; A_2 \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{z_n}\right).$$

The constants ν_0, ν_1, A_1 and A_2 are uniquely determined with the help of the asymptotics (3.39) and (4.3). Thus, the functions $s(\lambda)$ and $\gamma(\lambda)$ are uniquely determined by their zeros. Under the assumptions of Theorem 4.1, one has

$$\nu_m \; = \; \widetilde{\nu}_m\,, \;\; m \; = \; 0\,,1\,, \quad s(\lambda) \; \equiv \; \widetilde{s}(\lambda)\,, \quad \gamma(\lambda) \; \equiv \; \widetilde{\gamma}(\lambda)\,. \tag{4.13}$$

Since $b = \widetilde{b}$, it follows from Eqs. (4.5), (4.7) and (4.13) that

$$s^{+}(\lambda) \equiv \widetilde{s}^{+}(\lambda), \quad p(\lambda) \equiv \widetilde{p}(\lambda). \tag{4.14}$$

Using Eq. (4.12) we obtain $u(z_n) = \widetilde{u}(z_n)$, $n \geq 0$. Together with Eqs. (4.14) and Eq. (4.9) this yields

$$\gamma_0(z_n) = \widetilde{\gamma}_0(z_n), \quad n \geq 0, \tag{4.15}$$

where $\gamma_0(\lambda) := \sigma_{11}(S_2)$ was defined in Subsection 3.4. It follows from Eqs. (3.40), (4.13) and (4.15) that

$$\alpha_n^0 = \widetilde{\alpha}_n^0, \quad n \geq 0.$$

Applying Theorem 3.9 we get $q(x) = \widetilde{q}(x)$ a.e. on $(0, T)$, $a_1 = \widetilde{a}_1$ and $a_2 = \widetilde{a}_2$. □

Remark 4.2 The proof of Theorem 4.1 is constructive and it gives us a procedure for constructing the solution of the Inverse Problem 4.

Remark 4.3 Similar results are valid for all other classes of non-separated singular boundary conditions, in particular for

$$\begin{aligned} W_1^0(y) &:= a_{11}^0\sigma_{10}(y) + a_{12}^0\sigma_{20}(y) + a_{11}^1\sigma_{11}(y) + a_{12}^1\sigma_{21}(y) = 0, \\ W_2^0(y) &:= a_{21}^0\sigma_{10}(y) + a_{21}^1\sigma_{11}(y) = 0. \end{aligned}$$

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