SAMPLING THEOREMS ASSOCIATED WITH DIFFERENTIAL OPERATORS ITERATED FROM LOWER ORDER ONES

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Abstract. In this paper we derive sampling representations for integral transforms, which arise from differential operators of orders greater than one, iteratively from sampling series associated with lower order operators. We use the sampling theorems associated with first and second order differential operators as basic steps.

1. Introduction

The classical sampling theorem of Whittaker, Kotel'nikov and Shannon (WKS) states that if $f(t)$ is any element of the Paley-Wiener space PW_π^2, then

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(n-\lambda)}{\pi(n-\lambda)}, \quad \lambda \in \mathbb{C} \tag{1.1}$$

Here, PW_π^2 is the Hilbert space of all $L^2(\mathbb{R})-$ entire functions f which have exponential type π. The convergence of (1.1) is uniform on $\mathbb{R}$ and on compact subsets of $\mathbb{C}$. See e.g. [8, 23, 26, 28].

The connection between sampling theory and eigenvalue problems was first observed by Weiss in his note [27] which is followed by various publications on the subject as it is seen below. But, to the best we know, Campbell, [9], was the first to derive (1.1) using the first order eigenvalue problem

$$-iy' = \lambda y, \qquad y(-\pi) = y(\pi). \tag{1.2}$$

Also, Haddad, Yao and Thomas, [20], derived expansion (1.1) using Green's function of the first order problem (1.2). The work of Everitt and Poulkou, [17], provided a general result in this respect. The papers [9, 14, 15, 16, 18, 24, 29, 33] introduced sampling theorems associated with second order differential operators. The sampling theory associated with higher order operators is also investigated extensively, see e.g. [1, 2, 7, 32]. The use of Green's function in sampling theory is studied in [2, 4, 6, 30]. See also the monographs [21, 31].

The purpose of this paper is to establish sampling theorems associated with higher order differential operators from those derived using lower order ones. This idea goes back to Higgins in [22] who established sampling theorems associated with n-th order problems which are iterates of first order ones. We use below the

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sampling results associated with first and second order eigenvalue problems as basic steps and then we derive those associated with higher order ones. The next section is a preliminary section. It contains briefly the mathematical tools used throughout the paper as well as the basic sampling theorems. Section three is devoted to deriving sampling theorems associated with $n-$th order eigenvalue problems which are iterates of first order ones. A major difference here is that while all eigenvalues of the basic problem are simple, the eigenvalues of the $n-$th order problem may be double. In section four we give sampling representations associated with iterates of operators defined by general regular second order eigenvalue problems.

2. Preliminaries

Let $\mathfrak{H}$ be a Hilbert space and let $A : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a compact self-adjoint operator. It is known [19, pp.108-114] that A has a denumerable set of real eigenvalues $\{\mu_k\}_{k=1}^{\infty}$ that tends to 0 as k tend to ∞ and to each eigenvalue there corresponds a finite number of orthonormal eigenfunctions. Therefore we can assume that every eigenvalue is repeated according to its multiplicity and $\{\phi_k\}_{k=1}^{\infty}$ is a corresponding set of orthonormal eigenfunctions. Then it is known [19, pp.108-114] that for $x \in \mathfrak{H}$,

- $x = h + \sum\limits_{k=1}^{\infty} \langle x, \phi_k \rangle \, \phi_k, \; h \in \ker A,$
- $Ax = \sum\limits_{k=1}^{\infty} \mu_k \, \langle x, \phi_k \rangle \, \phi_k.$

The convergence in both series is in the norm of $\mathfrak{H}$. For $n \in \mathbb{Z}^+$, the operator $A^n : \mathfrak{H} \longrightarrow \mathfrak{H}$ defined by

$$A^n x = \sum_k \mu_k^n \, \langle x, \phi_k \rangle \, \phi_k, \quad x \in \mathfrak{H}. \tag{2.1}$$

is compact and self adjoint. It has the sequence of eigenvalues $\{\mu_k^n\}_{k=1}^{\infty}$ and the corresponding sequence of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$. In particular, if $\mathfrak{H} = L^2(a,b)$ and $k(x,\xi) \in L^2\left((a,b)\times(a,b)\right)$, then the Fredholm integral operator $A : L^2(a,b) \longrightarrow L^2(a,b)$ defined by

$$Af(x) = \int_a^b k(x,\xi) f(\xi)\, d\xi, \quad f(\cdot) \in L^2(a,b), \tag{2.2}$$

is compact. Moreover, if $k(x,\xi) = \overline{k}(\xi,x)$, then A is self-adjoint [10, pp.27-28]. Consequently, if $\{\mu_k\}_{k=1}^{\infty}$, $\{\phi_k\}_{k=1}^{\infty}$ are the sequences of eigenvalues and their corresponding orthonormal eigenfunctions, then,

$$Af(x) = \sum_k \mu_k \left(\int_a^b f(t) \overline{\phi_k}(t)\, dt \right) \phi_k(x), \quad f(\cdot) \in L^2(a,b).$$

The resolvent kernel, R_k, of $k(x,\xi)$ is defined to be the unique L^2-solution of the integral equation

$$R_k(x,\xi,\lambda) = k(x,\xi) + \lambda \int_a^b R_k(x,t,\lambda) k(t,\xi)\, dt, \; \lambda \neq \mu_k, \tag{2.3}$$

cf. [10, pp.27-28]. According to the previous discussion, for $n \in \mathbb{Z}^+$, the operator $A^n : L^2(a,b) \longrightarrow L^2(a,b)$ defined by

$$A^n f(x) = \sum_{k=1}^{\infty} \mu_k^n \left(\int_a^b f(t)\overline{\phi_k}(t)\,dt \right) \phi_k(x), \quad f(\cdot) \in L^2(a,b),$$

has the sequence of eigenvalues $\{\mu_k^n\}_{k=1}^{\infty}$ and the corresponding sequence of eigenfunction $\{\phi_k\}_{k=1}^{\infty}$. It is not hard to see that A^n is also an integral operator with the kernel

$$k_n(x,\xi) = \int_a^b k_{n-1}(x,s)k(s,\xi)\,ds, \quad k_1(x,\xi) = k(x,\xi), \qquad n \geq 2. \tag{2.4}$$

Before we state the basic sampling results we are using in this paper, we discuss the relationship between Green's functions of eigenvalue problems and the resolvent kernels associated with them. We will prove for the convenience of the reader the well known fact that they are exactly the same. Consider the eigenvalue problem

$$\ell_0(y) : p_0(x)y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x)y(x) = \lambda y,\ a \leq x \leq b, \tag{2.5}$$

$$N_\nu(y) := \sum_{j=1}^{n} \alpha_{\nu j}y^{(j-1)}(a) + \beta_{\nu j}y^{(j-1)}(b) = 0,\ \nu = 1, \ldots, n, \tag{2.6}$$

where the coefficients $p_k(x)$, $k = 0, 1, \ldots, n$, have continuous derivatives up to the order $(n-k)$ inclusive on the interval $[a,b]$ and $N_1, N_2, \ldots, N_n$ are linearly independent forms in the variables $y(a), y'(a), \ldots, y^{(n-1)}(a), y(b), y'(b), \ldots, y^{(n-1)}(b)$. Let L be the operator generated be $\ell_0(\cdot)$ and $N_\nu(\cdot)$, $1 \leq \nu \leq n$. Assume that zero is not an eigenvalue of (2.5)–(2.6). Let $G(x,\xi)$ and $G(x,\xi,\lambda)$ be Green's functions associated with L and $L - \lambda I$, I is the identity, respectively, where λ is not an eigenvalue of (2.5)–(2.6) see [25, pp. 28-32]. Let $R_G(x,\xi,\lambda)$ be the resolvent kernel associated with $G(x,\xi)$, λ is not an eigenvalue of (2.5)–(2.6). We can see that $G(x,\xi,\lambda) \equiv R_G(x,\xi,\lambda)$ because $G(x,\xi,\lambda)$ satisfies the integral equation (2.3). Indeed, from the properties of Green's function, for a fixed $\xi \in [a,b]$, if $\lambda \in \mathbb{C}$ is not an eigenvalue

$$LG(x,\xi,\lambda) = \lambda G(x,\xi,\lambda), \quad LG(x,\xi) = 0. \tag{2.7}$$

Thus

$$L\left\{G(x,\xi,\lambda) - G(x,\xi)\right\} = \lambda G(x,\xi,\lambda). \tag{2.8}$$

Applying the Fredholm integral operator whose kernel is $G(x,\xi)$ on both sides of (2.8) and using [25, Theorem 2, pp. 31–33], we obtain

$$G(x,\xi,\lambda) - G(x,\xi) = \lambda \int_a^b G(x,\xi,\lambda)G(x,\xi) \tag{2.9}$$

proving that $G(x,\xi,\lambda) \equiv R_G(x,\xi,\lambda)$.

In the rest of this section we state the basic sampling theorems which we will employ to derive the new results. These basic theorems include two cases, the first order case and the regular second order case. We start with the first order setting. Let $\mathcal{L}$ be the operator generated by the differential expression,

$$\ell_1(y) := \quad -iy' = \lambda y, \qquad -\infty < -\sigma \le x \le \sigma < \infty, \tag{2.10}$$

and the boundary condition

$$U(y) := \quad y(-\sigma) - e^{i\theta} y(\sigma) = 0 \qquad \theta \in \mathbb{R}, \tag{2.11}$$

The domain of $\mathcal{L}$ is the set of all functions which are absolutely continuous and whose first derivatives are in $L^2(-\sigma,\sigma)$ and satisfy (2.11). A fundamental set of solutions (f.s.) of (2.10) consists of

$$\varphi(x,\lambda) := e^{i\lambda(x+\sigma)}. \tag{2.12}$$

The eigenvalues of (2.10)–(2.11) are the zeros of the characteristic determinant (CD)

$$\Delta(\lambda) := U(\varphi) = 1 - e^{i(2\sigma\lambda+\theta)}. \tag{2.13}$$

Hence, the eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \frac{n\pi}{\sigma} - \frac{\theta}{2\sigma},\ \varphi(x,\lambda_n) = e^{i\lambda_n(x+\sigma)}, \quad n \in \mathbb{Z}. \tag{2.14}$$

Green's function of problem (2.10)–(2.11) is given for $\lambda \ne \lambda_n$ by

$$G(x,\xi,\lambda) = \frac{i}{\Delta(\lambda)} \begin{cases} e^{i(\lambda(x-\xi))}, & -\sigma \le \xi \le x \le \sigma, \\ \\ e^{i(\lambda(x-\xi+2\sigma)+\theta)}, & -\sigma \le x \le \xi \le \sigma. \end{cases} \tag{2.15}$$

Theorem A. *If $f(\cdot) \in L^2(-\sigma,\sigma)$ and $\xi_0 \in [-\sigma,\sigma]$ is fixed, then the transforms,*

$$F(\lambda) = \int_{-\sigma}^{\sigma} f(x)\varphi(x,\lambda)\,dx;\ F^*(\lambda) = \int_{-\sigma}^{\sigma} f(x)\Delta(\lambda)G(x,\xi_0,\lambda)\,dx, \quad \lambda \in \mathbb{C} \tag{2.16}$$

have the sampling representations,

$$\begin{pmatrix} F(\lambda) \\ F^*(\lambda) \end{pmatrix} = \sum_{j\in\mathbb{Z}} \begin{pmatrix} F\left(\frac{j\pi}{\sigma} - \frac{\theta}{2\sigma}\right) \\ F^*\left(\frac{j\pi}{\sigma} - \frac{\theta}{2\sigma}\right) \end{pmatrix} \frac{e^{i(\sigma\lambda+\theta/2)}\sin(\sigma\lambda+\theta/2)}{\sigma\left(\lambda - \left(\frac{j\pi}{\sigma} - \frac{\theta}{2\sigma}\right)\right)}. \tag{2.17}$$

Series (2.17) *converges absolutely and uniformly on compact subsets of* $\mathbb{C}$ *and uniformly on* $\mathbb{R}$.
See [1, 6, 7, 9, 17, 30] for proofs, discussions and generalizations.

Now let $\mathcal{D}$ be the operator generated by the differential expression,

$$\ell_2(y): \quad y''(x) - p(x)y(x) = -\mu y, \quad x \in [a,b],\ \mu \in \mathbb{C}, \tag{2.18}$$

and the boundary conditions,

$$V_1(y) := \alpha_{11}y(a) + \alpha_{12}y'(a) + \beta_{11}y(b) + \beta_{12}y'(b) = 0, \tag{2.19}$$

$$V_2(y) := \alpha_{21}y(a) + \alpha_{22}y'(a) + \beta_{21}y(b) + \beta_{22}y'(b) = 0, \tag{2.20}$$

where $p(\cdot)$ is a real-valued continuous function an $[a, b]$ and α_{ij}, $\beta_{ij} \in \mathbb{R}$ and $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}$ and such that $V_1(\cdot)$, $V_2(\cdot)$ are two linearly independent forms. The domain of $\mathcal{D}$, $D_{\mathcal{D}}$, consists of those functions y which satisfy (2.19)–(2.20), have absolutely continuous first order derivatives on $[a, b]$ and their second order derivatives lie in $L^2(a, b)$. Let $\varphi_1(x, \mu)$ and $\varphi_2(x, \mu)$ be the f.s. of (2.18) such that,

$$\varphi_1(a, \mu) = 1 \qquad \varphi_1'(a, \mu) = 0,$$
$$\varphi_2(a, \mu) = 0 \qquad \varphi_2'(a, \mu) = 1.$$

Problem (2.18)–(2.20) has a countable set of real eigenvalues $\{\mu_k\}_{k=1}^{\infty}$. We may assume without any loss of generality that all eigenvalues are nonnegative. The sequence $\{\mu_k\}_{k=1}^{\infty}$ has ∞ as the unique limit point, [25]. These eigenvalues μ_k are the zeros of the CD

$$\Omega(\mu) := \begin{vmatrix} V_1(\varphi_1(x, \mu)) & V_1(\varphi_2(x, \mu)) \\ V_2(\varphi_1(x, \mu)) & V_2(\varphi_2(x, \mu)) \end{vmatrix}. \tag{2.21}$$

The eigenfunctions corresponding to the eigenvalue μ are generated by the two solutions, $\phi(x, \mu)$ and $\chi(x, \mu)$. Unlike problem (2.10)–(2.11), the eigenvalues of problem (2.18)–(2.20) may be double. The derivation of the sampling results in this case depends on the choice of the kernels that generate all eigenfunctions. This problem is treated in [5, 15, 16]. According to [16] the kernels $\phi(x, \mu)$ and $\chi(x, \mu)$ are defined to be

$$\phi(x, \mu) := \frac{1}{\Omega'(\mu)} \begin{vmatrix} \varphi_1(x, \mu) & \varphi_2(x, \mu) \\ V_2(\varphi_1(x, \mu)) & V_2(\varphi_2(x, \mu)) \end{vmatrix}, \tag{2.22}$$

$$\chi(x, \mu) := \frac{1}{\Omega'(\mu)} \begin{vmatrix} V_1(\varphi_1(x, \mu)) & V_1(\varphi_2(x, \mu)) \\ \varphi_1(x, \mu) & \varphi_2(x, \mu) \end{vmatrix}, \qquad \mu \in \mathbb{C}. \tag{2.23}$$

According to [16, p. 219], these functions are entire. Moreover $\Omega'(\mu)$ has only simple zeros at the double zeros eigenvalues of (2.18)–(2.20). In this case $\phi(x, \mu)$ and $\chi(x, \mu)$ are defined to be

$$\phi(x, \mu) := \frac{1}{\Omega''(\mu)} \begin{vmatrix} \varphi_1(x, \mu) & \varphi_2(x, \mu) \\ V_2(\frac{\partial}{\partial \mu}\varphi_1(x, \mu)) & V_2(\frac{\partial}{\partial \mu}\varphi_2(x, \mu)) \end{vmatrix}, \tag{2.24}$$

$$\chi(x, \mu) := \frac{1}{\Omega''(\mu)} \begin{vmatrix} V_1(\frac{\partial}{\partial \mu}\varphi_1(x, \mu)) & V_1(\frac{\partial}{\partial \mu}\varphi_2(x, \mu)) \\ \varphi_1(x, \mu) & \varphi_2(x, \mu) \end{vmatrix}, \qquad \mu \in \mathbb{C}. \tag{2.25}$$

If μ_k is an eigenvalue of problem (2.18)–(2.20), then we distinguish between two cases. If μ_k is simple, then $\phi(x,\mu_k)$ and $\chi(x,\mu_k)$ are linearly dependent. When μ_k is double, then $\phi(x,\mu_k)$ and $\chi(x,\mu_k)$ are linearly independent, cf. [15, 16] for details. Green's function of problem (2.18)–(2.20) is given by, [25, pp.36-37],

$$\begin{aligned} H(x,\xi,\mu) &= \frac{(-1)^n}{\Omega(\mu)}\begin{vmatrix} \varphi_1(x,\mu) & \varphi_2(x,\mu) & h(x,\xi,\mu) \\ V_1(\varphi_1) & V_1(\varphi_2) & V_1(h) \\ V_2(\varphi_1) & V_2(\varphi_2) & V_2(h) \end{vmatrix}, \qquad (2.26) \\ h(x,\xi,\mu) &= \frac{\pm 1}{2W(\xi)}\begin{vmatrix} \varphi_1(x,\mu) & \varphi_2(x,\mu) \\ \varphi_1(\xi,\mu) & \varphi_2(\xi,\mu) \end{vmatrix}, \end{aligned}$$

where, the positive sign is taken if $x \geq \xi$ and the negative sign if $x < \xi$, and $W(x) := \varphi_1(x,\mu)\varphi_2'(x,\mu) - \varphi_1'(x,\mu)\varphi_2(x,\mu)$.

Theorem B. *If $f(\cdot) \in L^2(a,b)$ and $\xi_0 \in [a,b]$ is fixed, then the transforms,*

$$\begin{pmatrix} \mathcal{F}(\mu) \\ \mathcal{F}^*(\mu) \\ \mathcal{F}^{**}(\mu) \end{pmatrix} = \int_a^b f(x) \begin{pmatrix} \phi(x,\mu) \\ \chi(x,\mu) \\ \Omega(\mu)H(x,\xi_0,\mu) \end{pmatrix} dx, \quad \mu \in \mathbb{C} \qquad (2.27)$$

have the sampling representations,

$$\begin{pmatrix} \mathcal{F}(\mu) \\ \mathcal{F}^*(\mu) \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} \mathcal{F}(\mu_n) \\ \mathcal{F}^*(\mu_n) \end{pmatrix} \frac{\mathcal{D}(\mu)}{(\mu-\mu_n)\,\mathcal{D}'(\mu_n)}, \qquad (2.28)$$

and

$$\mathcal{F}^{**}(\mu) = \sum_{n=1}^{\infty} \mathcal{F}^{**}(\mu_n) \frac{\Omega(\mu)}{(\mu-\mu_n)\,\Omega'(\mu_n)}, \qquad (2.29)$$

where μ_n are the eigenvalues of the problem (2.18)–(2.20) *and $\mathcal{D}(\mu) := \frac{\Omega(\mu)}{\Omega'(\mu)}$. Series* (2.28) *converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and uniformly on $\mathbb{R}$.*
See [16, 6] for proofs.

In the following we denote by $C^j[a,b]$ to the set of all functions $y(\cdot)$ which have continuous derivatives up to the $n-$th order inclusive on the interval $[a,b]$ and $\omega_1,\ldots,\omega_n$ denote the $n-$th roots of 1.

3. ITERATIONS FROM FIRST ORDER OPERATORS

This section includes sampling theorems associated with the operator $\mathcal{L}_n := \mathcal{L}^n$, where $n \in \mathbb{Z}^+$, iterated from those associated with the first order boundary value problem $\mathcal{L}$. We will derive sampling theorems for integral transforms whose kernels are either solutions or Green's function of $\mathcal{L}_n$. The first theorem is similar to that derived by Higgins in [22]. For this aim we investigate the relationship between $\mathcal{L}$, $\mathcal{L}_n$ and the associated sampling from many aspects. We compare the fundamental solutions, Green's functions, characteristic determinants and sampling results.

For $\ell \in \{\ell_1, \ell_2\}$ and $i \in \mathbb{N}$ we define the iterated differential expressions ℓ^i recursively by

$$\ell^0(y) := y, \;\; \ell^1(y) := \ell(y) \;\text{ and }\; \ell^i(y) = \ell^{i-1}(\ell(y)), \;\; i \in \mathbb{N}.$$

First we consider the first order boundary value problem (2.10)–(2.11) and the $n-$th order boundary value problem

$$\ell_1^n(y) = \lambda y \tag{3.1}$$

$$U_j(y) := \left(\ell_1^{j-1}(y)\right)(-\sigma) - e^{i\phi}\left(\ell_1^{j-1}(y)\right)(\sigma) = 0, \quad j = 1, 2, \ldots, n, \tag{3.2}$$

with the associated differential operator

$$\mathcal{L}_n : D_{\mathcal{L}_n} \longrightarrow L^2(-\sigma, \sigma), \quad y \longmapsto \ell_1^n(y)$$

where $D_{\mathcal{L}_n}$ is defined to be the set of all $y \in C^{n-1}[-\sigma, \sigma]$ such that $y^{(n-1)}$ is absolutely continuous on $[-\sigma, \sigma]$, $\ell^n(y) \in L^2(-\sigma, \sigma)$ and $U_j(y) = 0$, $j = 1, \ldots, n$.
We know from the previous section how to define powers of compact self-adjoint operators; for sake of clarity we shall explain that the way we define powers of differential operators is consistent with the operator theoretic approach. The following lemma proves the desired consistency. If we assume, without loss of generality, that zero is not an eigenvalue of $\mathcal{L}$, then the eigenvalue problem $\mathcal{L}y = \lambda y$ is equivalent to the Fredholm integral operator,

$$y = \lambda \mathcal{G}_1 y, \tag{3.3}$$

$$\mathcal{G}_1 : L^2(-\sigma, \sigma) \longrightarrow L^2(-\sigma, \sigma), \quad \left(\mathcal{G}_1 y\right)(x) := \int_{-\sigma}^{\sigma} G(x, \xi) y(\xi)\, d\xi, \tag{3.4}$$

where $G(x, \xi) = G(x, \xi, 0)$ and $G(x, \xi, \mu)$ is given in (2.15) above.

Lemma 3.1. *If $\lambda = 0$ is not an eigenvalue of $\mathcal{L}$, then the eigenvalue problem* (3.1)–(3.2)*, or $\mathcal{L}_n y = \lambda y$, is equivalent to the Fredholm integral equation*

$$y = \lambda \mathcal{G}_n y, \tag{3.5}$$

$$\mathcal{G}_n : L^2(-\sigma, \sigma) \longrightarrow L^2(-\sigma, \sigma), \quad \left(\mathcal{G}_n y\right)(x) := \int_{-\sigma}^{\sigma} \mathcal{G}_n(x, \xi) y(\xi)\, d\xi, \tag{3.6}$$

where $G_n(x, \xi)$ is defined iteratively by

$$G_m(x, \xi) := \int_{-\sigma}^{\sigma} G(x, s) G_{m-1}(s, \xi)\, ds, \; m = 2, \ldots, n. \tag{3.7}$$

Proof. First, we notice that if $\lambda = 0$ is not an eigenvalue of $\mathcal{L}y = \lambda y$, then zero is not an eigenvalue of $\mathcal{L}_n y = \lambda y$. Since if we let $\mathcal{L}_n y = 0$, then $\mathcal{L}_n y = \mathcal{L}\left(\mathcal{L}_{n-1} y\right) = 0$. Thus $\mathcal{L}_{n-1} y = 0$ since zero is not an eigenvalue of $\mathcal{L}$. Therefore by mathematical induction $y = 0$, i.e. $\lambda = 0$ is not an eigenvalue of $\mathcal{L}_n$. We will prove the lemma using mathematical induction. The required is obviously true when $n = 1$. Now, assume that the lemma is correct when $n = m - 1$, $m = 2, 3, \ldots$. We prove the lemma is true when $n = m$. First assume that y satisfies (3.5). Then $\mathcal{G}_n y \in D_{\mathcal{L}_n}$ and

$$\mathcal{L}_m y = \lambda \mathcal{L}_m \mathcal{G}_m y = \lambda \mathcal{L}_{m-1} \left(\mathcal{L}\mathcal{G} \left(\mathcal{G}_{m-1} y \right) \right) = \lambda \mathcal{L}_{m-1} \left(\mathcal{G}_{m-1} y \right) = \lambda y.$$

where $\mathcal{G}_{n-1}$ is the Fredholm operator whose kernel is $G_{n-1}(x, \xi)$.
Conversely let $\mathcal{L}_n y = \lambda y$. Then by the definition of $\mathcal{L}_n$, $\mathcal{L}_n y \in L^2(-\sigma, \sigma)$ and

$$\lambda \mathcal{G}_n y = \mathcal{G}_n \mathcal{L}_n y = \mathcal{G} \left(\mathcal{G}_{n-1} \mathcal{L}_{n-1} \left(\mathcal{L} y \right) \right) = \mathcal{G} \left(\mathcal{L} y \right) = y.$$

□

Now we derive sampling theorems associated with $\mathcal{L}_n$ from those associated with $\mathcal{L}$. We notice that the fundamental set of solutions of (2.10) consists of one solution, namely (2.12), while the n-th order one has n linearly independent solutions. The following lemma indicates how to construct a fundamental set of solutions of (3.1) from (2.10). Let $V := V(\omega_1, \ldots, \omega_n)$ denote the Vandermonde matrix

$$V := V(\omega_1, \ldots, \omega_n) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \omega_1 & \omega_2 & \ldots & \omega_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \omega_2^{n-1} & \ldots & \omega_n^{n-1} \end{pmatrix} \tag{3.8}$$

and let V_{jk} denote the $(n-1) \times (n-1)$ matrix that arises from V by deleting the $j-$th row and the $k-$th column.

Lemma 3.2. *Let $\rho := \sqrt[n]{\lambda}$. There are functions (of λ), a_{jk}, $j, k = 1, \ldots, n$ such that*

$$y_j(x, \lambda) = \sum_{k=1}^{n} a_{jk}\, \varphi(x, \omega_k \rho) = \sum_{k=1}^{n} a_{jk}\, e^{i\omega_k \rho (x+\sigma)}, \; j = 1, \ldots, n, \tag{3.9}$$

is a fundamental set of solutions of (3.1)*, which satisfies,*

$$y_j^{(k-1)}(-\sigma, \lambda) = \delta_{jk}, \; j, k = 1, 2, \ldots, n. \tag{3.10}$$

Proof. By direct computations we can see that

$$\varphi_k(x, \lambda) := \varphi(x, \omega_k \rho) = e^{i\omega_k \rho (x+\sigma)}, \; k = 1, 2, \ldots, n, \tag{3.11}$$

are linearly independent solutions of (3.1). Now, let $y_j(x, \lambda)$ be a desired f.s. of (3.1), which satisfies (3.10). Then there are unique functions a_{jk} such that

$$y_j(x, \lambda) = \sum_{k=1}^{n} a_{jk}\, \varphi_k(x, \lambda), \quad 1 \leq j, k \leq n. \tag{3.12}$$

To find these functions we apply the initial conditions (3.10) to (3.12) to get the system of the form

$$M(\rho) \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix} = \begin{pmatrix} \delta_{j1} \\ \delta_{j2} \\ \vdots \\ \delta_{jn} \end{pmatrix}, \tag{3.13}$$

with

$$M(\rho) := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \varphi'(-\sigma, \omega_1\rho) & \varphi'(-\sigma, \omega_2\rho) & \dots & \varphi'(-\sigma, \omega_n\rho) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}(-\sigma, \omega_1\rho) & \varphi^{(n-1)}(-\sigma, \omega_2\rho) & \dots & \varphi^{(n-1)}(-\sigma, \omega_n\rho) \end{pmatrix}.$$

Equivalently we obtain the system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ i\rho\omega_1 & i\rho\omega_2 & \dots & i\rho\omega_n \\ \vdots & \vdots & \ddots & \vdots \\ (i\rho\omega_1)^{(n-1)} & (i\rho\omega_2)^{(n-1)} & \dots & (i\rho\omega_n)^{(n-1)} \end{pmatrix} \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix} = \begin{pmatrix} \delta_{j1} \\ \delta_{j2} \\ \vdots \\ \delta_{jn} \end{pmatrix}. \tag{3.14}$$

Since the determinant of the coefficient matrix does not vanish for $\lambda \in \mathbb{C}$, then the constants a_{jk} are unique and using Cramer's method we obtain

$$a_{jk} = (-1)^{j+k}(i\rho)^{-j+1}\frac{\det(V_{jk})}{\det(V)}, \ 1 \le j, k \le n. \tag{3.15}$$

□

The eigenvalues of the operator $\mathcal{L}_n$ are the zeros of the CD

$$\Delta_n(\lambda) := \det(U_k(y_j))_{1\le k,j\le n} = \begin{vmatrix} U_1(y_1(x,\lambda)) & \dots & U_1(y_n(x,\lambda)) \\ U_2(y_1(x,\lambda)) & \dots & U_2(y_n(x,\lambda)) \\ \vdots & \ddots & \vdots \\ U_n(y_1(x,\lambda)) & \dots & U_n(y_n(x,\lambda)) \end{vmatrix}. \tag{3.16}$$

Lemma 3.3. *The characteristic determinant of* (3.1)–(3.2), $\Delta_n(\lambda)$, *satisfies*

$$\Delta_n(\lambda) = \frac{\widetilde{V}}{i^{n(n-1)/2}\left(\det(V)\right)^n}\prod_{j=1}^{n}\Delta(\omega_j\rho), \tag{3.17}$$

where

$$\widetilde{V} = \sum_{\substack{j_1, j_2, \dots, j_n = 1, \\ j_k \ne j_l,\ for\ l \ne k}}^{n} \begin{vmatrix} (-1)^{j_1+1}\det(V_{1j_1}) & \dots & (-1)^{j_1+n}\det(V_{nj_1}) \\ (-1)^{j_2+1}\omega_{j_2}\det(V_{1j_2}) & \dots & (-1)^{j_2+n}\omega_{j_2}\det(V_{nj_2}) \\ \vdots & \ddots & \vdots \\ (-1)^{j_n+1}\omega_{j_n}^{n-1}\det(V_{1j_n}) & \dots & (-1)^{j_n+n}\omega_{j_n}^{n-1}\det(V_{nj_n}) \end{vmatrix}.$$

Proof. From (3.9) and (3.2), we have

$$\begin{aligned}
\Delta_n(\lambda) &= \begin{vmatrix} \sum_{k=1}^{n} a_{1k}U_1(\varphi(x,\omega_k\rho)) & \dots & \sum_{k=1}^{n} a_{nk}U_1(\varphi(x,\omega_n\rho)) \\ \sum_{k=1}^{n} a_{1k}\omega_k\rho U_1(\varphi_1(x,\omega_k\rho)) & \dots & \sum_{k=1}^{n} a_{nk}\omega_k\rho U_1(\varphi(x,\omega_k\rho)) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{1k}(\omega_k\rho)^{n-1}U_1(\varphi_1(x,\omega_k\rho)) & \dots & \sum_{k=1}^{n} a_{nk}(\omega_k\rho)^{n-1}U_1(\varphi(x,\omega_k\rho)) \end{vmatrix} \\
&= \lambda^{\frac{n-1}{2}} \sum_{j_1,j_2,\dots,j_n=1}^{n} \begin{vmatrix} a_{1j_1}\Delta(\omega_{j_1}\rho)) & \dots & a_{nj_1}\Delta(\omega_{j_1}\rho)) \\ \omega_{j_2}a_{1j_2}\Delta(\omega_{j_2}\rho)) & \dots & \omega_{j_2}a_{nj_2}\Delta(\omega_{j_2}\rho)) \\ \vdots & \ddots & \vdots \\ \omega_{j_n}^{n-1}a_{1j_n}\Delta(\omega_{j_n}\rho)) & \dots & \omega_{j_n}^{n-1}a_{nj_n}\Delta(\omega_{j_n}\rho)) \end{vmatrix} \\
&= \lambda^{\frac{n-1}{2}} C(\lambda) \prod_{j=1}^{n} \Delta(\omega_j\rho),
\end{aligned}$$

where,

$$\begin{aligned}
C(\lambda) &= \sum_{\substack{j_1,j_2,\dots,j_n=1,\\ j_k\neq j_l \ for\ l\neq k;}}^{n} \begin{vmatrix} a_{1j_1} & \dots & a_{nj_1} \\ \omega_{j_2}a_{1j_2} & \dots & \omega_{j_2}a_{nj_2} \\ \vdots & \ddots & \vdots \\ \omega_{j_n}^{n-1}a_{1j_n} & \dots & \omega_{j_n}^{n-1}a_{nj_n} \end{vmatrix} \\
&= \frac{\displaystyle\sum_{\substack{j_1,j_2,\dots,j_n=1,\\ j_k\neq j_l \ for\ l\neq k}}^{n} \begin{vmatrix} (-1)^{j_1+1}\det(V_{1j_1}) & \dots & (-1)^{j_1+n}\det(V_{nj_1}) \\ (-1)^{j_2+1}\omega_{j_2}\det(V_{1j_2}) & \dots & (-1)^{j_2+n}\omega_{j_2}\det(V_{nj_2}) \\ \vdots & \ddots & \vdots \\ (-1)^{j_n+1}\omega_{j_n}^{n-1}\det(V_{1j_n}) & \dots & (-1)^{j_n+n}\omega_{j_n}^{n-1}\det(V_{nj_n}) \end{vmatrix}}{\lambda^{\frac{n-1}{2}} i^{n(n-1)/2} (\det(V))^n}.
\end{aligned}$$

Therefore $\lambda^{\frac{n-1}{2}}C(\lambda)$ is a complex constant which is independent of λ and doesn't equal zero since if it does, then $\Delta_n(\lambda)$ is identically zero, implying that all the complex numbers are eigenvalues of $\mathcal{L}_n$ which is a contradiction. ☐

According to the previous lemma, the eigenvalues of the boundary value problem (3.1)–(3.2) are

$$\Lambda_k := \lambda_k^n = \left(\frac{k\pi}{\sigma} - \frac{\theta}{2\sigma}\right)^n, \ k \in \mathbb{Z}. \tag{3.18}$$

If n is odd, then all eigenvalues are simple and the sequence of the eigenfunctions is generated by,

$$\phi_1(x,\lambda) := \varphi(x,\rho). \tag{3.19}$$

If n is even, then the multiplicity of the eigenvalues depends on whether $\frac{\theta}{\pi} \in \mathbb{Z}$ or not. We will have two cases:

(1) If $\frac{\theta}{\pi} \notin \mathbb{Z}$, then all eigenvalues are simple and the sequence of the eigenfunctions is generated by (3.19).
(2) If $\frac{\theta}{\pi} = m \in \mathbb{Z}$, then we distinguish between two subcases. If m is even, then the eigenvalues $\{\Lambda_{-\frac{m}{2}+k}\}_{k=1}^{\infty}$ are double and $\Lambda_{-\frac{m}{2}} = 0$ is a simple eigenvalue if $\lambda = 0$ is an eigenvalue of $\mathcal{L}$, while if m is odd, then the eigenvalues $\{\Lambda_{-\frac{m+1}{2}+k}\}_{k=1}^{\infty}$ are double. The sequence of the eigenfunctions, when Λ_k is double, is generated by

$$\phi_1(x,\lambda) := \varphi(x,\rho), \qquad \phi_2(x,\lambda) := \varphi(x,-\rho). \tag{3.20}$$

Summarizing we obtain a sampling result associated with eigenvalue problem (3.1)–(3.2).

Theorem 3.4. *If $\rho := \sqrt[n]{\lambda}$ and $f(\cdot) \in L^2(-\sigma,\sigma)$, then the transform*

$$F_1(\lambda) = \int_{-\sigma}^{\sigma} f(x)\,\varphi(x,\rho)\,dx, \quad \lambda \in \mathbb{C} \tag{3.21}$$

has the sampling representation

$$F_1(\lambda) = \sum_{k\in\mathbb{Z}} F\left(\frac{k\pi}{\sigma} - \frac{\theta}{2\sigma}\right) \frac{e^{i(\sigma\rho+\theta/2)}\sin(\sigma\rho+\theta/2)}{\sigma\left(\rho - \left(\frac{k\pi}{\sigma} - \frac{\theta}{2\sigma}\right)\right)}, \tag{3.22}$$

where the convergence is uniform on $\mathbb{R}$ and on compact subsets of $\mathbb{C}$.

Proof. From (2.16) and (3.21) we notice that,

$$F_1(\lambda) = \int_{-\sigma}^{\sigma} f(x)\varphi(x,\rho)\,dx = F(\rho).$$

Hence the assertion of the theorem follows from Theorem A, this can be accomplished by direct substitution. □

Similar to the proof of this theorem, we can derive the sampling expansion

$$F_2(\lambda) = -\sum_{k\in\mathbb{Z}} F\left(\frac{k\pi}{\sigma} - \frac{\theta}{2\sigma}\right) \frac{e^{i(-\sigma\rho+\theta)}\sin(-\sigma\rho+\theta)}{\sigma\left(\rho + \left(\frac{k\pi}{\sigma} - \frac{\theta}{2\sigma}\right)\right)}, \tag{3.23}$$

for the transform

$$F_2(\lambda) = \int_{-\sigma}^{\sigma} f(x)\varphi(x,-\rho)\,dx, \quad \lambda \in \mathbb{C}. \tag{3.24}$$

Next, we derive a sampling theorem associated with Green's function of $\mathcal{L}_n y = \lambda y$ iteratively from that of Theorem A above. Below are two results concerning this task. The first gives Green's function of (3.1)–(3.2) in terms of $G(x,\xi,\lambda)$ given in (2.15) above. The second is the desired sampling theorem.

Lemma 3.5. *Green's function of* (3.1)–(3.2) *is given by,*

$$G_n(x,\xi,\lambda) = \sum_{j=1}^{n} \frac{(-1)^{j-1}\det(V_{1j})}{\rho^{n-1}\det(V)}\, G(x,\xi,\omega_j\rho). \tag{3.25}$$

Proof. As is mentioned before, Green's function is the unique solution of (2.3). To seek this solution, let $G_n(x,\xi,\lambda)=\sum_{j=1}^{n} c_j G(x,\xi,\omega_j\rho)$ for some constants c_j, $j=1,\ldots,n$. Then, by mathematical induction,

(3.26)
$$\lambda\int_{-\sigma}^{\sigma} G_n(x,t,\lambda)G_n(t,\xi)\,dt = G_n(x,\xi,\lambda)-\sum_{j=1}^{n-1}\sum_{k=1}^{n} c_k\left(\rho\omega_k\right)^{n-j} G_{n-j+1}(x,\xi),$$

where $G_j(x,\xi)$, $j=1,\ldots,n$ are given in (3.7) above. Since $G_n(x,\xi,\lambda)$ is the resolvent kernel of $G_n(x,\xi)$, then it satisfies (2.3), and the right-hand-side of (3.26) should be $G_n(x,\xi,\lambda)-G_n(x,\xi)$. We seek suitable constants c_j; such c_j's are solutions of the system

$$\begin{pmatrix} \omega_1^{n-1} & \ldots & \omega_n^{n-1} \\ \omega_1^{n-2} & \ldots & \omega_n^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}=\begin{pmatrix} \frac{1}{\rho^{n-1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.27}$$

Solving this system and substituting in $G_n(x,\xi,\lambda)$ we obtain (3.25). These c_j's are unique hence $G_n(x,\xi,\lambda)$ is unique. □

The following theorem gives the sampling associated with Green's function of the boundary value problem (3.1)–(3.2).

Theorem 3.6. *If* $\rho:=\sqrt[n]{\lambda}$ *and* $\xi_0\in[-\sigma,\sigma]$ *is fixed, then the transform,*

(3.28)
$$\begin{aligned} F_n^*(\lambda) &= \int_{-\sigma}^{\sigma} f(x)\Delta_n(\lambda)G_n(x,\xi_0,\lambda)\,dx \\ &= \sum_{j=1}^{n}\frac{(-1)^{j-1}\widetilde{V}\det(V_{1j})\prod_{\substack{l=1\\ l\neq j}}^{n}\Delta(\omega_l\rho)}{\rho^{n-1}i^{n(n-1)/2}\left(\det(V)\right)^{n+1}}\int_{-\sigma}^{\sigma} f(x)\Delta(\omega_j\rho)G(x,\xi_0,\omega_j\rho)\,dx,\ \lambda\in\mathbb{C} \end{aligned}$$

has the sampling representation,

(3.29)
$$F_n^*(\lambda)=\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}} F^*\left(\frac{j\pi}{\sigma}-\frac{\theta}{2\sigma}\right)\frac{(-1)^{j-1}\widetilde{V}\det(V_{1j})\prod_{l=1}^{n}e^{i(\sigma\omega_l\rho+\theta/2)}\sin(\sigma\omega_l\rho+\theta/2)}{\rho^{n-1}i^{n(n-1)/2}\left(\det(V)\right)^{n+1}\sigma\left(\omega_k\rho-\left(\frac{j\pi}{\sigma}-\frac{\theta}{2\sigma}\right)\right)}.$$

Proof. From (3.28) and (2.16) we notice that,

$$\begin{aligned} F_n^*(\lambda) &= \sum_{j=1}^{n}\frac{(-1)^{j-1}\widetilde{V}\det(V_{1j})\prod_{\substack{l=1\\ l\neq j}}^{n}\Delta(\omega_l\rho)}{\rho^{n-1}i^{n(n-1)/2}\left(\det(V)\right)^{n+1}}\int_{-\sigma}^{\sigma} f(x)\Delta(\omega_j\rho)G(x,\xi_0,\omega_j\rho)\,dx \\ &= \sum_{j=1}^{n}\frac{(-1)^{j-1}\widetilde{V}\det(V_{1j})\prod_{\substack{l=1\\ l\neq j}}^{n}\Delta(\omega_l\rho)}{\rho^{n-1}i^{n(n-1)/2}\left(\det(V)\right)^{n+1}}F^*(\omega_j\sqrt{\lambda}). \end{aligned}$$

Hence we get from Theorem A the assertion of the theorem . □

4. Iterations from regular second order operators

This section includes sampling theorems associated with the operator $\mathcal{D}_n := \mathcal{D}^n$, where $n \in \mathbb{Z}^+$, iterated from the second order operator $\mathcal{D}$. We will derive sampling theorems for integral transforms whose kernels are either the solutions or Green's functions of $\mathcal{D}_n$. For this aim we investigate the relationship between $\mathcal{D}$, $\mathcal{D}_n$ and the associated sampling from many aspects. We compare the fundamental solutions, characteristic determinants and sampling results.

Consider the second order boundary value problem (2.18)–(2.20) and the $2n-$th order boundary value problem

$$\ell_2^n(y) = \mu y, \tag{4.1}$$

$$V_{j1}y := \alpha_{11}(\ell_2^{j-1}y)(a) + \alpha_{12}(\ell^{j-1}y)'(a) + \beta_{11}(\ell^{j-1}y)(b) + \beta_{12}(\ell^{j-1}y)'(b) = 0, \tag{4.2}$$

$$V_{j2}y := \alpha_{21}(\ell^{j-1}y)(a) + \alpha_{22}(\ell^{j-1}y)'(a) + \beta_{21}(\ell^{j-1}y)(b) + \beta_{22}(\ell^{j-1}y)'(b) = 0, \tag{4.3}$$

with the associated differential operator

$$\mathcal{D}_n : D_{\mathcal{D}_n} \longrightarrow L^2(a,b), \quad y \longmapsto \ell_2^n(y),$$

where $D_{\mathcal{D}_n}$ is defined to be the set of all $y \in C^{(2n-1)}[a,b]$ such that $y^{(2n-1)}$ is absolutely continuous on $[a,b]$, $\ell^n(y) \in L^2(a,b)$ and $V_{jk}(y) = 0$, $j = 1,\dots,n$, $k = 1,2$. As in the previous section the following lemma proves in an operator theoretic language that $\mathcal{D}_n$ is exactly $\mathcal{D}^n$. Assume that zero is not an eigenvalue of $\mathcal{D}$. Then the eigenvalue problem $\mathcal{D}y = \mu y$ is equivalent to the Fredholm integral operator,

$$y = \mu \mathcal{H}_1 y, \tag{4.4}$$

$$\mathcal{H}_1 : L^2(a,b) \longrightarrow L^2(a,b), \quad (\mathcal{H}_1 y)(x) := \int_a^b H(x,\xi) y(\xi)\, d\xi, \tag{4.5}$$

where $H(x,\xi) = H(x,\xi,0)$ and $H(x,\xi,\mu)$ is given in (2.26) above.

Lemma 4.1. *If $\mu = 0$ is not an eigenvalue of $\mathcal{D}$, then the eigenvalue problem* (4.1)–(4.3) *is equivalent to the Fredholm integral equation*

$$y = \mu \mathcal{H}_n y, \tag{4.6}$$

$$\mathcal{H}_n : L^2(a,b) \longrightarrow L^2(a,b), \quad (\mathcal{H}_n y)(x) := \int_a^b H_n(x,\xi) y(\xi)\, d\xi, \tag{4.7}$$

where $H_n(x,\xi)$ is defined iteratively by

$$H_m(x,\xi) := \int_a^b H(x,s) H_{m-1}(s,\xi)\, ds, \; m = 2,\dots,n, \; H_1(x,\xi) = H(x,\xi). \tag{4.8}$$

Proof. Similar to that of Lemma 3.1. □

Now we derive sampling theorems associated with $\mathcal{D}_n$ from those associated with $\mathcal{D}$. We notice that the fundamental set of solutions of (2.18) consists of two linearly independent solutions, namely (2.22), while the n-th iterative one has $2n$ linearly independent solutions. The following lemma indicates how to construct a fundamental set of solutions of (4.1) from (2.18). For simplicity, let $c_{jk} := (-1)^{j+k}(\alpha_{jk}+\beta_{jk})$ and let $W := W(\omega_1,\ldots,\omega_n)$ denote the matrix

$$W := W(\omega_1,\ldots,\omega_n) = \begin{pmatrix} c_{22} & \cdots & c_{22} & c_{12} & \cdots & c_{12} \\ c_{21} & \cdots & c_{21} & c_{11} & \cdots & c_{11} \\ \omega_1 c_{22} & \cdots & \omega_n c_{22} & \omega_1 c_{12} & \cdots & \omega_n c_{12} \\ \omega_1 c_{21} & \cdots & \omega_n c_{21} & \omega_1 c_{11} & \cdots & \omega_n c_{11} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} c_{22} & \cdots & \omega_n^{n-1} c_{22} & \omega_1^{n-1} c_{12} & \cdots & \omega_n^{n-1} c_{12} \\ \omega_1^{n-1} c_{21} & \cdots & \omega_n^{n-1} c_{21} & \omega_1^{n-1} c_{11} & \cdots & \omega_n^{n-1} c_{11} \end{pmatrix}. \tag{4.9}$$

Let also $W_{j,k}$ denote the $(n-1)\times(n-1)$ matrix that arises from W by deleting the $j-$th row and the $k-$th column.

Lemma 4.2. *Let $\eta := \sqrt[n]{\mu}$. There are functions of μ, a_{jk}, b_{jk}, $j = 1,\ldots,2n$, $k = 1,\ldots,n$, such that*

$$y_j(x,\mu) = \sum_{k=1}^{n} a_{jk}\,\phi(x,\omega_k\eta) + b_{jk}\,\chi(x,\omega_k\eta),\ j, = 1,2,\ldots,2n, \tag{4.10}$$

is a fundamental set of solutions of (4.1)*, which satisfies,*

$$y_j^{(k-1)}(a,\mu) = \delta_{jk},\ j,k = 1,2,\ldots,2n, \tag{4.11}$$

Proof. By direct computations we can see that

$$\phi_{1k}(x,\mu) := \phi(x,\omega_k\eta), \quad \phi_{2k}(x,\mu) := \chi(x,\omega_k\eta), \quad k = 1,2,\ldots,n, \tag{4.12}$$

are linearly independent solutions of (4.1). Now, let $y_j(x,\mu)$ be the desired f.s. of (4.1), which satisfies (4.11). Then there are unique functions a_{jk}, and b_{jk} such that

$$y_j(x,\mu) = \sum_{k=1}^{n} a_{jk}\,\phi_{1k}(x,\mu) + b_{jk}\,\phi_{2k}(x,\mu), \quad 1 \le j \le 2n,\ 1 \le k \le n. \tag{4.13}$$

To find these constants we apply the initial conditions (4.11) to (4.13) to get the system

$$M_1 \begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \\ b_{j1} \\ \vdots \\ b_{jn} \end{pmatrix} = \begin{pmatrix} \delta_{j,1} \\ \vdots \\ \delta_{j,n} \\ \delta_{j,n+1} \\ \vdots \\ \delta_{j,2n} \end{pmatrix} \tag{4.14}$$

with

$$M_1 = \begin{pmatrix} \phi_{11}(a,\mu) & \dots & \phi_{1n}(a,\mu) & \phi_{21}(a,\mu) & \dots & \phi_{2n}(a,\mu) \\ \phi'_{11}(a,\mu) & \dots & \phi'_{1n}(a,\mu) & \phi'_{21}(a,\mu) & \dots & \phi'_{2n}(a,\mu) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{11}^{(2n-1)}(a,\mu) & \dots & \phi_{1n}^{(2n-1)}(a,\mu) & \phi_{21}^{(2n-1)}(a,\mu) & \dots & \phi_{2n}^{(2n-1)}(a,\mu) \end{pmatrix}.$$

Obviously, this system has a unique solution, since the determinant of the coefficients does not vanish for any $\mu \in \mathbb{C}$. However, the computation of the solution of this system will depend on whether μ is a simple or a double eigenvalue. If all eigenvalues of (2.18)–(2.20) are simple, then $\Omega'(\mu) \neq 0$ for $\mu \in \mathbb{C}$. In this case system (4.14) is nothing but

$$M_2 \begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \\ b_{j1} \\ \vdots \\ b_{jn} \end{pmatrix} = \begin{pmatrix} \delta_{j,1} \\ \vdots \\ \delta_{j,n} \\ \delta_{j,n+1} \\ \vdots \\ \delta_{j,2n} \end{pmatrix} \tag{4.15}$$

with

$$M_2 = \begin{pmatrix} \frac{c_{22}}{\Omega'(\mu)} & \dots & \frac{c_{22}}{\Omega'(\mu)} & \frac{c_{12}}{\Omega'(\mu)} & \dots & \frac{c_{12}}{\Omega'(\mu)} \\ \frac{c_{21}}{\Omega'(\mu)} & \dots & \frac{c_{21}}{\Omega'(\mu)} & \frac{c_{11}}{\Omega'(\mu)} & \dots & \frac{c_{11}}{\Omega'(\mu)} \\ \frac{\eta\omega_1 c_{22}}{\Omega'(\mu)} & \dots & \frac{\eta\omega_n c_{22}}{\Omega'(\mu)} & \frac{\eta\omega_1 c_{12}}{\Omega'(\mu)} & \dots & \frac{\eta\omega_n c_{12}}{\Omega'(\mu)} \\ \frac{\eta\omega_1 c_{21}}{\Omega'(\mu)} & \dots & \frac{\eta\omega_n c_{21}}{\Omega'(\mu)} & \frac{\eta\omega_1 c_{11}}{\Omega'(\mu)} & \dots & \frac{\eta\omega_n c_{11}}{\Omega'(\mu)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{(\eta\omega_1)^{n-1} c_{22}}{\Omega'(\mu)} & \dots & \frac{(\eta\omega_n)^{n-1} c_{22}}{\Omega'(\mu)} & \frac{(\eta\omega_1))^{n-1} c_{12}}{\Omega'(\mu)} & \dots & \frac{(\eta\omega_n))^{n-1} c_{12}}{\Omega'(\mu)} \\ \frac{(\eta\omega_1)^{n-1} c_{21}}{\Omega'(\mu)} & \dots & \frac{(\eta\omega_n)^{n-1} c_{21}}{\Omega'(\mu)} & \frac{(\eta\omega_1)^{n-1} c_{11}}{\Omega'(\mu)} & \dots & \frac{(\eta\omega_n)^{n-1} c_{11}}{\Omega'(\mu)} \end{pmatrix}.$$

Solving this system we obtain

$$\begin{aligned} a_{jk} &= (-1)^{j+k}(\eta/\Omega'(\mu))^{-j+1}\tfrac{\det(W_{j,k})}{\det(W)}, \\ b_{jk} &= (-1)^{j+k+2n}(\eta/\Omega'(\mu))^{-j+1}\tfrac{\det(W_{j+n,k+n})}{\det(W)}, \end{aligned} \tag{4.16}$$

where $1 \leq j,k \leq n$, $\mu \in \mathbb{C}$ and $W_{j,k}$ is defined above. If μ_0 is a double eigenvalue, i.e. $\Omega'(\mu_0) = 0$, then the system (4.14) is equivalent to

$$M_3 \begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \\ b_{j1} \\ \vdots \\ b_{jn} \end{pmatrix} = \begin{pmatrix} \delta_{j,1} \\ \vdots \\ \delta_{j,n} \\ \delta_{j,n+1} \\ \vdots \\ \delta_{j,2n} \end{pmatrix} \tag{4.17}$$

with

$$M_3 = \begin{pmatrix} \frac{c_{22}}{\Omega''(\mu)} & \cdots & \frac{c_{22}}{\Omega''(\mu)} & \frac{c_{12}}{\Omega''(\mu)} & \cdots & \frac{c_{12}}{\Omega''(\mu)} \\ \frac{c_{21}}{\Omega''(\mu)} & \cdots & \frac{c_{21}}{\Omega''(\mu)} & \frac{c_{11}}{\Omega''(\mu)} & \cdots & \frac{c_{11}}{\Omega''(\mu)} \\ \frac{\eta\omega_1 c_{22}}{\Omega''(\mu)} & \cdots & \frac{\eta\omega_n c_{22}}{\Omega''(\mu)} & \frac{\eta\omega_1 c_{12}}{\Omega''(\mu)} & \cdots & \frac{\eta\omega_n c_{12}}{\Omega''(\mu)} \\ \frac{\eta\omega_1 c_{21}}{\Omega''(\mu)} & \cdots & \frac{\eta\omega_n c_{21}}{\Omega''(\mu)} & \frac{\eta\omega_1 c_{11}}{\Omega''(\mu)} & \cdots & \frac{\eta\omega_n c_{11}}{\Omega''(\mu)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{(\eta\omega_1)^{n-1} c_{22}}{\Omega''(\mu)} & \cdots & \frac{(\eta\omega_n)^{n-1} c_{22}}{\Omega''(\mu)} & \frac{(\eta\omega_1))^{n-1} c_{12}}{\Omega''(\mu)} & \cdots & \frac{(\eta\omega_n))^{n-1} c_{12}}{\Omega''(\mu)} \\ \frac{(\eta\omega_1)^{n-1} c_{21}}{\Omega''(\mu)} & \cdots & \frac{(\eta\omega_n)^{n-1} c_{21}}{\Omega''(\mu)} & \frac{(\eta\omega_1)^{n-1} c_{11}}{\Omega''(\mu)} & \cdots & \frac{(\eta\omega_n)^{n-1} c_{11}}{\Omega''(\mu)} \end{pmatrix}.$$

Solving this system we obtain

$$\begin{aligned} a_{jk} &= (-1)^{j+k}(\eta/\Omega''(\mu))^{-j+1}\tfrac{\det(W_{j,k})}{\det(W)}, \\ b_{jk} &= (-1)^{j+k+2n}(\eta/\Omega''(\mu))^{-j+1}\tfrac{\det(W_{j+n,k+n})}{\det(W)}, \end{aligned} \tag{4.18}$$

where $1 \leq j, k \leq n$, $\mu \in \mathbb{C}$ □

The eigenvalues of the operator $\mathcal{D}_n$ are the zeros of CD

$$\Omega_n(\mu) := \begin{vmatrix} V_1(y_1(x,\mu)) & \dots & V_1(y_{2n}(x,\mu)) \\ V_2(y_1(x,\mu)) & \dots & V_2(y_{2n}(x,\mu)) \\ \vdots & \ddots & \vdots \\ V_n(y_1(x,\mu)) & \dots & V_{2n}(y_{2n}(x,\mu)) \end{vmatrix}. \tag{4.19}$$

Lemma 4.3. *The characteristic determinant of* (4.1)–(4.3), $\Omega_n(\mu)$, *satisfies*

$$\Omega_n(\mu) = \frac{\widetilde{W}}{(\det(W))^n} \prod_{j=1}^{n} \Omega^2(\omega_j \eta), \tag{4.20}$$

where

$$\widetilde{W} := \sum_{\substack{l_1,l_2,\dots,l_n=1, \\ j_1,j_2,\dots,j_n=1}}^{n} \begin{vmatrix} (-1)^{l_1+1}\det(W_{1,l_1}) & \dots & (-1)^{l_1+2n}\det(W_{2n,l_1}) \\ (-1)^{l_2+1}\omega_{l_2}\det(W_{1,l_2}) & \dots & (-1)^{l_2+2n}\omega_{l_2}\det(W_{2n,l_2}) \\ \vdots & \ddots & \vdots \\ (-1)^{l_n+1}\omega_{l_n}^{n-1}\det(W_{1,l_n}) & \dots & (-1)^{l_n+2n}\omega_{l_n}^{n-1}\det(W_{2n,l_n}) \\ (-1)^{j_1+1+n}\det(W_{1,j_1+n}) & \dots & (-1)^{j_1+3n}\det(W_{2n,j_1+n}) \\ \vdots & \ddots & \vdots \\ (-1)^{j_1+1+n}\omega_{j_n}^{n-1}\det(W_{1,j_n+n}) & \dots & (-1)^{j_n+3n}\omega_{j_n}^{n-1}\det(W_{2n,j_n+n}) \end{vmatrix}.$$

Notice that $\widetilde{W} \neq 0$ *if* $\Omega'(\mu) \neq 0$. *If* $\Omega'(\mu_0) = 0$, *then* $\Omega_n(\mu_0^n) = 0$.

Proof. From (4.10) and (4.2)–(4.3), we have, where $\Omega'(\mu) \neq 0$,

$$\Omega_n(\mu) =$$

$$\left|\begin{matrix}
\sum_{k=1}^{n} a_{1,k}V_1(\phi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} a_{2n,k}V_1(\phi(x,\omega_k\eta)) \\
\sum_{k=1}^{n} a_{1,k}\omega_k\eta V_1(\phi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} a_{2n,k}\omega_k\eta V_1(\phi(x,\omega_k\eta)) \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{n} a_{1,k}(\omega_k\eta)^{n-1}V_1(\phi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} a_{2n,k}(\omega_k\eta)^{n-1}V_1(\phi(x,\omega_k\eta)) \\
\sum_{k=1}^{n} b_{1,k}V_2(\chi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} b_{2n,k}V_2(\chi(x,\omega_k\eta)) \\
\sum_{k=1}^{n} b_{1,k}\omega_k\eta V_2(\chi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} b_{2n,k}\omega_k\eta V_2(\chi(x,\omega_k\eta)) \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{n} b_{1,k}(\omega_k\eta)^{n-1}V_2(\chi(x,\omega_k\eta)) & \cdots & \sum_{k=1}^{n} b_{2n,k}(\omega_k\eta)^{n-1}V_2(\chi(x,\omega_k\eta))
\end{matrix}\right|$$

$$= \sum_{\substack{l_1,\ldots,l_n=1 \\ j_1,\ldots,j_n=1}}^{n} \mu^{n-1}
\left|\begin{matrix}
a_{1,l_1}\frac{\Omega(\omega_{l_1}\eta)}{\Omega'(\omega_{l_1}\eta)} & \cdots & a_{2n,l_1}\frac{\Omega(\omega_{l_1}\eta)}{\Omega'(\omega_{l_1}\eta)} \\
a_{1,l_2}\omega_{l_2}\frac{\Omega(\omega_{l_2}\eta)}{\Omega'(\omega_{l_2}\eta)} & \cdots & a_{2n,l_2}\omega_{l_2}\frac{\Omega(\omega_{l_2}\eta)}{\Omega'(\omega_{l_2}\eta)} \\
\vdots & \ddots & \vdots \\
a_{1,l_n}\omega_{l_n}^{n-1}\frac{\Omega(\omega_{l_n}\eta)}{\Omega'(\omega_{l_n}\eta)} & \cdots & a_{2n,l_n}\omega_{l_n}^{n-1}\frac{\Omega(\omega_{l_n}\eta)}{\Omega'(\omega_{l_n}\eta)} \\
b_{1,j_1}\frac{\Omega(\omega_{j_1}\eta)}{\Omega'(\omega_{j_1}\eta)} & \cdots & b_{2n,j_1}\frac{\Omega(\omega_{j_1}\eta)}{\Omega'(\omega_{j_1}\eta)} \\
b_{1,j_2}\omega_{j_2}\frac{\Omega(\omega_{j_2}\eta)}{\Omega'(\omega_{j_2}\eta)} & \cdots & b_{2n,j_2}\omega_{j_2}\frac{\Omega(\omega_{j_2}\eta)}{\Omega'(\omega_{j_2}\eta)} \\
\vdots & \ddots & \vdots \\
b_{1,j_n}\omega_{j_n}^{n-1}\frac{\Omega(\omega_{j_n}\eta)}{\Omega'(\omega_{j_n}\eta)} & \cdots & b_{2n,j_n}\omega_{j_n}^{n-1}\frac{\Omega(\omega_{j_n}\eta)}{\Omega'(\omega_{j_n}\eta)}
\end{matrix}\right|.$$

Substituting from (4.16) we obtain (4.20) where $\widetilde{W}$ doesn't equal to zero since if it does, then $\Omega_n(\mu)$ is identically zero implying that all the complex numbers are eigenvalues of $\mathcal{D}_n$ which is a contradiction. If $\Omega'(\mu_0) = 0$, i.e. μ_0 is a double eigenvalue, then we can see that also $\Omega_n(\mu_0^n) = 0$. □

According to the previous lemma, the eigenvalues of the boundary value problem (4.1)–(4.3) are

$$\tilde{\mu}_k = \mu_k^n,\ k = 1,\ldots,\infty. \tag{4.21}$$

Since the eigenvalues μ_k of the boundary value problem (2.18)–(2.20) are positive and have at most double multiplicity, then each eigenvalue $\tilde{\mu}_k$ has the same multiplicity as μ_k, thus all the eigenvalues of $\mathcal{D}_n$ are at most of double multiplicity and the sequence of the eigenfunctions could be shown to be generated by the two functions

$$\Phi_1(x,\mu)=\phi(x,\eta),\qquad \Phi_2(x,\mu)=\chi(x,\eta). \tag{4.22}$$

Indeed, $\Phi_1(x,\mu)$, $\Phi_2(x,\mu)$ satisfy (4.1) and (4.2)–(4.3).

Now we derive the sampling result associated with the eigenvalue problem (4.1)–(4.3).

Theorem 4.4. *If $f(\cdot)\in L^2(a,b)$, then the transforms,*

$$\begin{pmatrix}\mathcal{F}_1(\mu)\\ \mathcal{F}_2(\mu)\end{pmatrix}=\int_a^b f(x)\begin{pmatrix}\Phi_1(x,\mu)\\ \Phi_2(x,\mu)\end{pmatrix}dx,\quad \mu\in\mathbb{C} \tag{4.23}$$

have the sampling representations,

$$\begin{pmatrix}\mathcal{F}_1(\mu)\\ \mathcal{F}_2(\mu)\end{pmatrix}=\sum_{j=1}^{\infty}\begin{pmatrix}\mathcal{F}(\mu_j)\\ \mathcal{F}^*(\mu_j)\end{pmatrix}\frac{D(\eta)}{(\eta-\mu_j))\,D'(\mu_j)}. \tag{4.24}$$

where the convergence is uniform on $\mathbb{R}$ and on compact subsets of $\mathbb{C}$.

Proof. From (4.23) and (2.27) we notice that,

$$\begin{pmatrix}\mathcal{F}_1(\mu)\\ \mathcal{F}_2(\mu)\end{pmatrix}=\int_a^b f(x)\begin{pmatrix}\phi(x,\eta)\\ \chi(x,\eta)\end{pmatrix}dx=\begin{pmatrix}\mathcal{F}(\eta)\\ \mathcal{F}^*(\eta)\end{pmatrix}.$$

Hence, using Theorem B, the theorem is proved by direct substitution. □

Now, we derive a sampling theorem associated with Green's function of $\mathcal{D}_n y=\mu y$ iteratively from that of Theorem *B* above. Below are two results concerning this task. The first gives Green's function of (4.1)–(4.3) in terms of $H(x,\xi,\mu)$ given in (2.26) above. The second is the desired sampling theorem. The proofs are omitted since they are similar to those of the previous section.

Lemma 4.5. *Green's function of* (4.1)–(4.3) *is given by,*

$$H_n(x,\xi,\mu)=\sum_{j=1}^{n}\frac{(-1)^{j-1}\det(V_{1j})}{\eta^{n-1}\det(V)}\,H(x,\xi,\omega_j\eta). \tag{4.25}$$

Theorem 4.6. *If $\eta:=\sqrt[n]{\mu}$ and $\xi_0\in[a,b]$ is fixed, then the transform,*

$$\begin{aligned}\mathcal{F}_n^{**}(\mu)&=\int_a^b f(x)\Omega_n(\mu)H_n(x,\xi_0,\mu)\,dx\\ &=\sum_{j=1}^{n}\frac{(-1)^{j-1}\widetilde{W}\prod_{\substack{l=1\\ l\neq j}}^{n}\Omega(\omega_l\eta)}{\eta^{n-1}\det(V)\left(\det(W)\right)^n}\int_a^b f(x)\Omega(\omega_j\eta)H(x,\xi_0,\omega_j\eta)\,dx,\ \mu\in\mathbb{C}\end{aligned} \tag{4.26}$$

has the sampling representation,

$$\mathcal{F}_n^{**}(\mu)=\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\mathcal{F}^{**}(\mu_n)\frac{(-1)^{j-1}\widetilde{W}\prod_{l=1}^{n}\Omega(\omega_l\eta)}{\eta^{n-1}\det(V)\left(\det(W)\right)^n(\omega_j\eta-\mu_n)\,\Omega'(\mu_n)}. \tag{4.27}$$

The theory developed above could be extended to

- Non self adjoint problems with strongly regular boundary conditions.
- Singular Sturm-Liouville problems with limit-circle and limit-points singularities.
- The derivation of a sampling theorem associated with a differential operator which is a product of two differential operators from the sampling theorems associated with each of them. The situation here will be more complicated because the product of the operators does not always preserve self adjointness.

References

1. M.H. Annaby, Interpolation expansions associated with nonselfadjoint boundary value problems, *Comm. Appl. Anal.* **3** (1999), 545–561.
2. M.H. Annaby and G. Freiling, Sampling expansions associated with Kamke problems, *Math. Z.* **234** (2000), 163–189.
3. M.H. Annaby and G. Freiling, Sampling integro-differential transforms arising from second order differential operators, *Math. Nachr.* **216** (2000), 25–43.
4. M.H. Annaby, G. Freiling and A.I. Zayed, Discontinuous boundary-value problems expansion, *J. Integral Equations and Appl.* **16** (2004), 1–23.
5. M.H. Annaby, H.A. Hassan, A sampling theorem associated with boundary-value problems with not necessary simple eigenvalues, *Internat. J. Math. Sci.* **21** (1998), 571–580.
6. M. H. Annaby and A. I. Zayed, On the use of Green's function in sampling theory, *J. Integral. Equations. and Appl.* **10** 1998, 117–139.
7. P. L. Butzer and G. Schöttler, Sampling theorems associated with fourth and higher order self-adjoint eigenvalue problems, *J. Comput. Appl. Math.* **51** 1994, 159–177.
8. P. L. Butzer, G. Schmeisser and R. C. Stens, An introduction to sampling analysis, *In: Non Uniform Sampling; Theory and Practice, F. Marvasti (ed), Kluwer, New York* (2001), 17–121.
9. L. L. Campbell, A comparison of the sampling theorems of Kramer and Whittaker, *SIAM J. Appl. Math* **12** (1964), 117–130.
10. J.A. Cochran, *The analysis of linear integral equations*, McGraw-Hill, New York, 1972.
11. E.A. Coddington and N. Levinson, *Theory of ordinary differential equations,* McGraw-Hill, New York, 1955.
12. M.S.P. Estham, *Theory of ordinary differential equations,* Van Nostrand Reinhold, 1970.
13. W.N. Everitt, The Sturm-Liouville problem for fourth-order differential equations. *Quart. J. Math.* Oxford(**2**), **8** (1957), 146–160.
14. W. N. Everitt and G. Nasri-Roudsari, Interpolation and sampling theories, and linear ordinary boundary value problems, In: *Sampling theory in Fourier and signal analysis: advanced topics.* Oxford University Press, Oxford. Edited by J. R. Higgins and R. L. Stens, (1999), 96–129.
15. W. N. Everitt and G. Nasri-Roudsari, Sturm-Liouville problems with coupled boundary conditions and Lagrange interpolation series, *J. Comput. Appl. Math.* , **1**, (1999), 319–347.
16. W. N. Everitt and G. Nasri-Roudsari, Sturm-Liouville problems with coupled boundary conditions and Lagrange interpolation series: II, *Rend. di Mat., Roma (7)* **20** (2000), 199–238.
17. W. N. Everitt and A. Poulkou, Kramer analytic kernels and first-order boundary value problems, *J. Comput. Appl. Math.* **148** (2002), 29–47.
18. W. N. Everitt, G. Schöttler and P. L. Butzer, Sturm-Liouville boundary value problems and Lagrange interpolation series, *Rend. di Mat., Roma (7)* **14** (1994), 87–126.
19. I. Gohberg and S. Goldberg, *Basic operator theory,* Boston, Basel, Stuttgart: Birkhäuser, 1980.
20. A. Haddad, K. Yao and J. Thomas, Genalized methods for the derivation of the sampling theorems, *IEEE Trans. Inf. Theory* **IT-13** (1967), 227–230.
21. J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*, Oxford University Press, Oxford, 1996.

22. J. R. Higgins , Sampling theorms from the iteration of low order differetial operators, Morder sampling theory, *Appl. Numer. Harmon. Anal.*, Birkhäuser Boston, Boston, MA, (2001), 219–227.
23. V. Kotel'nikov, On the carrying capacity of the "ether" and wire in telecommunications, *material for the first All-Union Conference on Questions of Communications, Izd. Red. Upr. Svyazi RKKA, Moscow, Russian* **55** (1933), 55–64. ,
24. H. P. Kramer, A Generalized sampling theorem, *J. Math. Phys.* **38** (1959), 68–72.
25. M.A. Naimark, *Linear differential operators I,* Gerge Harrap, London, 1967.
26. C. E. Shannon, Communications in the presence of noise, *Proc. IRE* **37**, (1949), 10–21.
27. P. Weiss, Sampling theorems associated with Sturm-Liouville systems, *Bull. Amer. Math. Soc.* **163** (1956), 242.
28. E. Whittaker, On the functions which are represented by the expansion of the interpolation theory, *Proc. Roy. Soc. Edin. sec.* **A 35** (1915), 181–194.
29. A. I. Zayed , On Kramer's sampling theorem associated with general Sturm-Liouville problems and Lagrange interpolation, *SIAM J. Appl. Math* **51** (1991), 575–604.
30. A. I. Zayed, A new role of Green's function interpolation and sampling theory, *J. Math. Anal. Appl.* **175** (1993), 222–238.
31. A.l. Zayed, *Advances in Shannon's sampling theory*, CRC Press, Boca Raton, 1993.
32. A. I. Zayed, M. A. El-Sayed and M. H. Annaby, On Lagrange interpolation and Kramer sampling theorems associated with self-adjoint boundary-value problems, *J. Math. Anal. Appl.* **158** (1991), 269–284.
33. A. I. Zayed, G. Hinsen and P. L. Butzer, On Lagrange interpolation and Kramer-type sampling theorems associated with Sturm-Liouville problems, *SIAM J. Appl. Math.* **50** (1990), 893–909.

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