SAMPLING THEOREMS ASSOCIATED WITH BIRKHOFF-REGULAR PROBLEMS

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Abstract. Sampling expansions are derived associated with arbitrary Birkhoff-regular boundary value problems including problems with multiple eigenvalues. As a kernel of the sampled transforms, the Green's function multiplied by the characteristic determinant, is used. The sampling expansions obtained are Hermite-type interpolation series. An example illustrating the obtained results is given.

1. Introduction

One of the approaches for deriving sampling theorems is connected with the application of Kramer's lemma [17] or its biorthogonal generalization given by Higgins in [14]. The later deals with the derivation of sampling representations for two dual transforms whose kernels generate biorthogonal bases. In other words, let $I \subset \mathbb{R}$ be a bounded interval and $K(\cdot,\lambda),\ K^*(\cdot,\lambda) : I \times \mathbb{C} \longrightarrow \mathbb{C}$ be $L^2(I)-$functions for all $\lambda \in \mathbb{C}$. Suppose that there are sequences $\{\lambda_k\}_{k\in\mathbb{N}},\ \{\lambda_k^*\}_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $\{K(\cdot,\lambda_k)\}_{k\in\mathbb{N}}$ is a basis of $L^2(I)$ and $\{K^*(\cdot,\lambda_k^*)\}_{k\in\mathbb{N}}$ is the biorthogonal one. Set

$$\nu_k := \int_I K(x,\lambda_k)\,\overline{K^*(x,\lambda_k^*)}\,dx.$$

Then the integral transform

$$f(\lambda) = \int_I g(x)K(x,\lambda)\,dx, \quad \lambda \in \mathbb{C},\ g \in L^2(I)$$

has the following sampling representation

$$f(\lambda) = \sum_{k=1}^{\infty} f(\lambda_k)\,S_k(\lambda), \quad S_k(\lambda) = \frac{1}{\nu_k}\int_I K(x,\lambda)\,\overline{K^*(x,\lambda_k^*)}\,dx. \tag{1.1}$$

Dually the transform

$$f^*(\lambda) = \int_I g(x)K^*(x,\lambda)\,dx, \quad \lambda \in \mathbb{C},\ g \in L^2(I)$$

has the expansion

$$f^*(\lambda) = \sum_{k=1}^{\infty} f^*(\lambda_k^*)\,S_k^*(\lambda), \quad S_k^*(\lambda) = \frac{1}{\overline{\nu_k}}\int_I K^*(x,\lambda)\,\overline{K(x,\lambda_k)}\,dx. \tag{1.2}$$

Moreover, the series in (1.1) and (1.2) converge uniformly on any subset of $\mathbb{C}$ on which $\|K(\cdot,\lambda)\|_{L^2(I)}$ and $\|K^*(\cdot,\lambda)\|_{L^2(I)}$ are bounded, respectively.

It is known that the examples of such kernels, $K(\cdot,\lambda),\ K^*(\cdot,\lambda)$, can sometimes be derived as solutions of an ordinary differential equation depending on a spectral parameter λ and of its adjoint one. In this case the points $\lambda_k,\ k \in \mathbb{N}$, are chosen to be the eigenvalues of the corresponding boundary value problem (BVP) and $\lambda_k^* = \overline{\lambda_k}$ are those of the adjoint problem, and

$\{K(\cdot, \lambda_k)\}_{k\in\mathbb{N}}$, $\{K^*(\cdot, \overline{\lambda_k})\}_{k\in\mathbb{N}}$ are biorthogonal bases of the corresponding eigenfunctions (see e.g. [2, 3, 4, 5, 8, 10, 12, 15, 24, 25, 26]). Moreover, there is also a possibility to construct these kernels using Green's function of the BVP. In both cases the sampling representations (1.1), (1.2) turn out to be Lagrange interpolation series. However, this approach imposes rather strong requirements on the BVP's used. In particular, even for a selfadjoint BVP there may be no solution $K(\cdot, \lambda)$ of the corresponding differential equation that gives a complete system of eigenfunctions while λ varies on the spectrum. Furthermore, the system of eigenfunctions of a non-selfadjoint BVP may, generally speaking, form no basis even in its closed linear hull. For such cases Kramer's lemma cannot give any sampling expansions.

In the present paper we use another approach of deriving sampling representations associated with BVP's. This approach is connected with the analytical nature of solutions of differential equations with a spectral parameter and appears to be more natural for BVP's. It does not require neither basisness of eigenfunctions nor even their completeness. The single requirement on the BVP considered is that it is Birkhoff-regular. As a kernel of the sampled transforms we use the Green's function multiplied by the characteristic determinant, which gives a more rich variety of such kernels. We note that the use of the Green's function in the sampling theory can also be found in [1, 3, 5, 6, 13] and other papers. In the following section we remind necessary properties of the Green's function and of Birkhoff-regular BVP's. In Section 3 we obtain sampling expansions of Hermite-type and give an example illustrating the obtained results.

2. Green's function

In the following we consider the BVP consisting of the differential equation

$$\ell(y) := i^n y^{(n)} + \sum_{j=0}^{n-2} p_j(x) y^{(j)} = \lambda y =: \rho^n y, \quad 0 \le x \le 1, \tag{2.1}$$

and the boundary conditions

$$U_\nu(y) := U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \quad \nu = \overline{1, n}, \tag{2.2}$$

where $p_j(\cdot) \in L(0,1)$ and

$$U_{\nu 0}(y) = \alpha_\nu y^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu - 1} \alpha_{\nu j} y^{(j)}(0), \qquad U_{\nu 1}(y) = \beta_\nu y^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu - 1} \beta_{\nu j} y^{(j)}(1).$$

Without any loss of generality we assume that the boundary conditions are normalized, i.e.

$$n - 1 \ge k_1 \ge k_2 \ge \ldots \ge k_n, \quad k_\nu > k_{\nu+2}, \quad |\alpha_j| + |\beta_j| > 0, \quad j = \overline{1, n}.$$

We remind that boundary conditions (2.2) are called *regular* (or, more precisely, *Birkhoff-regular*) [7, 20, 22, 23], if certain determinants θ_0, θ_1 for odd n and θ_{-1}, θ_1 for even n do not vanish. We note, that θ_0, θ_1, θ_{-1} and consequently the regularity of (2.2) depend only on the coefficients α_ν, β_ν of the leading derivatives in the conditions (2.2) and not on $\alpha_{\nu j}$, $\beta_{\nu j}$. The conditions (2.2) are called *strongly regular*, if they are regular and if additionally $\theta_0^2 \ne 4\theta_1\theta_{-1}$ for even n. The BVP (2.1), (2.2) is called *regular* (*strongly regular*), if its boundary conditions (2.2) are regular (strongly regular). We mention that in particular according to [11, 19, 21] all selfadjoint BVP's of the form (2.1), (2.2) are regular. The adjoint BVP of a (strongly) regular one is also (strongly) regular (see e.g. [18]). It is known, that the system of eigen- and associated functions (e.a.f.'s) of a regular BVP is complete in $L_2(0,1)$ (see e.g. [20]). Moreover, it has been proved by Kesel'man [16] and Mikhailov [18] that e.a.f.'s of a strongly regular BVP form an unconditional basis of $L_2(0,1)$.

It is assumed in the sequel that the BVP (2.1), (2.2) is regular. Let $y_1(\cdot,\lambda),\dots,y_n(\cdot,\lambda)$ be the fundamental system of solutions of the differential equation (2.1) satisfying the following initial conditions

$$y_k^{(j-1)}(0,\lambda)=\delta_{jk},\quad j,k=\overline{1,n}.$$

Then for every fixed $x\in[0,1]$ the functions $y_1(x,\lambda),\dots,y_n(x,\lambda)$ are entire with respect to λ and the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ ($|\lambda_k|\le|\lambda_{k+1}|$, $\lambda_k\ne\lambda_j$ for $k\ne j$) of (2.1), (2.2) are the zeros of the characteristic determinant

$$\Delta(\lambda)=\det\|U_j(y_k)\|_{j,k=\overline{1,n}},$$

which is also an entire function. Moreover, one can split all eigenvalues (counted with multiplicities) into two sequences $\{\lambda'_k\}_{k\in\mathbb{N}}$, $\{\lambda''_k\}_{k\in\mathbb{N}}$ with the following asymptotics (see [20]):

$$\lambda'_k=(2k\pi)^n\Big(1+\frac{A'}{k}+O\Big(\frac{1}{k^\alpha}\Big)\Big),\quad \lambda''_k=(-2k\pi)^n\Big(1+\frac{A''}{k}+O\Big(\frac{1}{k^\alpha}\Big)\Big),$$

where for the strongly regular case $\alpha=2$, otherwise generally speaking $\alpha=\frac{3}{2}$. We also note that for the strongly regular case almost all (i.e. all except some finite number) eigenvalues are algebraically simple, otherwise almost all of them have algebraic multiplicity $m_a(\lambda_k)\le 2$. The geometric multiplicity of λ_k (i.e. the number of linearly independent eigenfunctions corresponding to λ_k) is denoted by $m_g(\lambda_k)$.

If λ is not an eigenvalue of (2.1), (2.2), then for any function $f\in L_2(0,1)$ the solution of the BVP $\ell(y)=\lambda y+f,\ U_\nu(y)=0,\ \nu=\overline{1,n}$, has the form

$$y(x)=\int_0^1 G(x,\xi,\lambda)f(\xi)d\xi,\quad 0\le x\le 1,$$

where $G(x,\xi,\lambda)$ is Green's function of (2.1), (2.2).

Let us fix an arbitrary $\xi_0\in[0,1]$ and consider the function

$$\varphi(x,\lambda):=\Delta(\lambda)G(x,\xi_0,\lambda). \tag{2.3}$$

After removing the singularities the function $\varphi(x,\lambda)$ is entire with respect to λ for every fixed $x\in[0,1]$. Consider the set $\mathbb{F}$ of linear transformations $F(\lambda)$ of the form

$$F(\lambda)=\int_0^1 f(x)\varphi(x,\lambda)\,dx,\quad f(x)\in L_2(0,1). \tag{2.4}$$

Our goal is to derive sampling representations for functions from $\mathbb{F}$. The classical sampling theorem corresponds to the case $n=1$ (see e.g. [13, 14]). For definiteness let $n>1$.

Let $\rho_k^n=\lambda_k$. The following assertion is valid (see [20]).

Lemma 2.1. *Fix $\delta>0$. In the domain $\mathbb{C}_\delta:=\{\lambda:|\rho-\rho_k|\ge\delta,k\in\mathbb{N}\}$ the estimate*

$$|G(x,\xi,\lambda)|\le\frac{M}{|\rho|^{n-1}}$$

holds.

Using the contour integral method and taking into account Lemma 2.1 together with the analytical nature of Green's function [20] one can prove the subsequent known result.

Lemma 2.2. *The following representation holds (for $\lambda\ne\lambda_k$)*

$$G(x,\xi,\lambda)=\sum_{k=1}^{\infty}\sum_{j=1}^{m_g(\lambda_k)}\sum_{\nu=1}^{m_{kj}}\frac{1}{(\lambda-\lambda_k)^\nu}\sum_{l=1}^{m_{kj}+1-\nu}\overline{z_{k,j,l-1}(\xi)}y_{k,j,m_{kj}+1-\nu-l}(x), \tag{2.5}$$

where

$$y_{k,j,0},\, y_{k,j,1},\, \dots,\, y_{k,j,m_{kj}-1}, \quad j = \overline{1, m_g(\lambda_k)},$$

is an arbitrary canonical system of e.a.f.'s corresponding to the eigenvalue λ_k*, and*

$$z_{k,j,0},\, z_{k,j,1},\, \dots,\, z_{k,j,m_{kj}-1}, \quad j = \overline{1, m_g(\lambda_k)},$$

is an appropriately normalized one of the adjoint BVP corresponding to the eigenvalue $\overline{\lambda_k}$*. The series in* (2.5) *and all its termwise derivatives with respect to* λ *converge uniformly for* $x, \xi \in [0,1]$ *and for* λ *on bounded subsets* $B \subset \mathbb{C}$*, i.e. for all* $p \in \mathbb{N} \cup \{0\}$

$$\sup_{x,\xi\in[0,1],\lambda\in B} \Big| \sum_{k=N}^{\infty} \frac{\partial^p}{\partial \lambda^p} \sum_{j=1}^{m_g(\lambda_k)} \sum_{\nu=1}^{m_{kj}} \frac{1}{(\lambda-\lambda_k)^\nu} \sum_{l=1}^{m_{kj}+1-\nu} \overline{z_{k,j,l-1}(\xi)} y_{k,j,m_{kj}+1-\nu-l}(x) \Big| \to 0,$$

as $N \to \infty$.

Remark 2.3. The value m_{kj}, appearing in Lemma 2.2, is called the multiplicity of the eigenfunction $y_{k,j,0}(x)$ (or $z_{k,j,0}(x)$), moreover,

$$\sum_{j=1}^{m_g(\lambda_k)} m_{kj} = m_a(\lambda_k).$$

If the BVP (2.1), (2.2) is selfadjoint, then $m_g(\lambda_k) = m_a(\lambda_k)$ for all $k \in \mathbb{N}$, i.e. $m_{kj} = 1$ for all k, j and $G(x, \xi, \lambda)$ has only simple poles.

Remark 2.4. Denote by $\mathbb{S}$ the set of all zeros of the eigenfunctions $z_{k,j,0}(x)$ of the adjoint BVP. Obviously $\mathbb{S}$ is at most countable. If $m_g(\lambda_k) = 1$ for all k and $\xi_0 \in [0,1] \setminus \mathbb{S}$, then the transformation (2.4) is a bijection from $L_2(0,1)$ to $\mathbb{F}$. For the strongly regular case and $\xi_0 \in [0,1] \setminus \mathbb{S}$ this transformation has an at most finite-dimensional kernel in $L_2(0,1)$.

3. The Main Results

This section contains two sampling theorems associated with the problem (2.1)–(2.2), which we assume to be regular. The sampling series obtained are, generally speaking, of Hermite interpolation type. At the end of the section we give an illustrative example. Let us first show the idea by assuming that $m_a(\lambda_k) \le 2$ and consequently $m_g(\lambda_k) \le 2$ for all $k \in \mathbb{N}$. Denote

$$\begin{aligned} \mathbb{N}_1 &:= \{k :\ k \in \mathbb{N},\ m_a(\lambda_k) = 1\}, \\ \mathbb{N}_2 &:= \{k :\ k \in \mathbb{N},\ m_a(\lambda_k) = 2,\ m_g(\lambda_k) = 1\}, \\ \mathbb{N}_3 &:= \{k :\ k \in \mathbb{N},\ m_a(\lambda_k) = 2,\ m_g(\lambda_k) = 2\}. \end{aligned}$$

Then our standing assumption means $\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 = \mathbb{N}$. Moreover, (2.5) takes the form

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \sum_{j=1}^{2} \frac{R_{k,j}(x,\xi)}{(\lambda-\lambda_k)^j}, \tag{3.1}$$

where

$$R_{k,1}(x,\xi) = \overline{z_{k,1,0}(\xi)} y_{k,1,0}(x), \quad R_{k,2}(x,\xi) = 0, \qquad k \in \mathbb{N}_1,$$

$$\left.\begin{aligned} R_{k,1}(x,\xi) &= \overline{z_{k,1,0}(\xi)} y_{k,1,1}(x) + \overline{z_{k,1,1}(\xi)} y_{k,1,0}(x), \\ R_{k,2}(x,\xi) &= \overline{z_{k,1,0}(\xi)} y_{k,1,0}(x), \end{aligned}\right\} \quad k \in \mathbb{N}_2,$$

$$\left.\begin{aligned} R_{k,1}(x,\xi) &= \overline{z_{k,1,0}(\xi)} y_{k,1,0}(x) + \overline{z_{k,2,0}(\xi)} y_{k,2,0}(x), \\ R_{k,2}(x,\xi) &= 0, \end{aligned}\right\} \quad k \in \mathbb{N}_3.$$

Theorem 3.1. *For any function $F(\lambda)$ of the form (2.4) the following representation holds*

$$F(\lambda)=\sum_{k=1}^{\infty}\Big(F(\lambda_k)\Big[\frac{A_k\Delta(\lambda)}{(\lambda-\lambda_k)^2}+\frac{B_k\Delta(\lambda)}{\lambda-\lambda_k}\Big]+F'(\lambda_k)\frac{C_k\Delta(\lambda)}{\lambda-\lambda_k}\Big), \tag{3.2}$$

where

$$A_k=0,\qquad B_k=\frac{1}{\Delta'(\lambda_k)},\qquad C_k=0,\qquad k\in\mathbb{N}_1;$$

$$A_k=\frac{2}{\Delta''(\lambda_k)},\quad B_k=-\frac{2\Delta'''(\lambda_k)}{3(\Delta''(\lambda_k))^2},\quad C_k=\frac{2}{\Delta''(\lambda_k)},\quad k\in\mathbb{N}_2;$$

$$A_k=0,\qquad B_k=0,\qquad C_k=\frac{2}{\Delta''(\lambda_k)},\quad k\in\mathbb{N}_3.$$

The series in the right-hand side of (3.2) *and all its termwise derivatives converge uniformly on every bounded subset of* $\mathbb{C}$.

Proof. Substituting (2.3) and then (3.1) with $\xi=\xi_0$ into (2.4) we have

$$F(\lambda)=\sum_{k=1}^{\infty}\sum_{j=1}^{2}\frac{\Delta(\lambda)}{(\lambda-\lambda_k)^j}\int_0^1 f(x)R_{k,j}(x,\xi_0)\,dx. \tag{3.3}$$

The series in the right-hand side of (3.3) and all its termwise derivatives converge uniformly on every bounded subset of $\mathbb{C}$. Let $k\in\mathbb{N}_1$, then taking in (3.3) $\lambda=\lambda_k$ we arrive at

$$\int_0^1 f(x)R_{k,1}(x,\xi_0)\,dx=\frac{F(\lambda_k)}{\Delta'(\lambda_k)},\qquad \int_0^1 f(x)R_{k,2}(x,\xi_0)\,dx=0,\quad k\in\mathbb{N}_1. \tag{3.4}$$

Differentiation in (3.3) with respect to λ gives

$$F'(\lambda)=\sum_{k=1}^{\infty}\sum_{j=1}^{2}\frac{\Delta'(\lambda)(\lambda-\lambda_k)^j-j(\lambda-\lambda_k)^{j-1}\Delta(\lambda)}{(\lambda-\lambda_k)^{2j}}\int_0^1 f(x)R_{k,j}(x,\xi_0)\,dx. \tag{3.5}$$

Further, let $k\in\mathbb{N}_2$. For $\lambda=\lambda_k$ formula (3.3) takes the form

$$F(\lambda_k)=\frac{\Delta''(\lambda_k)}{2}\int_0^1 f(x)R_{k,2}(x,\xi_0)\,dx,\quad k\in\mathbb{N}_2. \tag{3.6}$$

Taking $\lambda=\lambda_k$ in (3.5) we obtain

$$F'(\lambda_k)=\frac{\Delta''(\lambda_k)}{2}\int_0^1 f(x)R_{k,1}(x,\xi_0)\,dx+\frac{\Delta'''(\lambda_k)}{6}\int_0^1 f(x)R_{k,2}(x,\xi_0)\,dx,\quad k\in\mathbb{N}_2. \tag{3.7}$$

Solving the system of linear algebraic equations (3.6), (3.7) we find

$$\left.\begin{aligned}\int_0^1 f(x)R_{k,1}(x,\xi_0)\,dx&=\frac{2F'(\lambda_k)}{\Delta''(\lambda_k)}-\frac{2\Delta'''(\lambda_k)F(\lambda_k)}{3(\Delta''(\lambda_k))^2},\\ \int_0^1 f(x)R_{k,2}(x,\xi_0)\,dx&=\frac{2F(\lambda_k)}{\Delta''(\lambda_k)},\end{aligned}\right\}\quad k\in\mathbb{N}_2. \tag{3.8}$$

For $k\in\mathbb{N}_3$ it is obvious that $F(\lambda_k)=0$. Moreover, formula (3.5) gives

$$\int_0^1 f(x)R_{k,1}(x,\xi_0)\,dx=\frac{2F'(\lambda_k)}{\Delta''(\lambda_k)},\qquad \int_0^1 f(x)R_{k,2}(x,\xi_0)\,dx=0,\quad k\in\mathbb{N}_3. \tag{3.9}$$

Substituting (3.4), (3.8), (3.9) into (3.3) we arrive at (3.2). □

Now let us consider the general case. Denote

$$m_k := \max_{j=\overline{1,m_g(\lambda_k)}} \{m_{kj}\}.$$

Thus, m_k is the maximal multiplicity of an eigenfunction corresponding to the eigenvalue λ_k.

Theorem 3.2. *For any function $F(\lambda)$ of the form (2.4) the following representation holds*

$$F(\lambda) = \sum_{k=1}^{\infty} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda), \tag{3.10}$$

where

$$S_{k,\nu}(\lambda) = \frac{1}{\nu!} \sum_{j=1}^{m_a(\lambda_k)-\nu} C_{k,j} \frac{\Delta(\lambda)}{(\lambda-\lambda_k)^{m_a(\lambda_k)-\nu+1-j}}\,,$$

and the numbers $C_{k,j}$, $j = \overline{1, m_k}$, can be found from the triangular non-singular system of linear algebraic equations

$$\sum_{j=1}^{s} C_{k,j} \frac{\Delta^{(m_a(\lambda_k)+s-j)}(\lambda_k)}{(m_a(\lambda_k)+s-j)!} = \delta_{s,1}, \quad s = \overline{1, m_k}. \tag{3.11}$$

The series in (3.10) *and all its termwise derivatives converge uniformly on every bounded subset of* $\mathbb{C}$.

Proof. We note that the formula (3.2) is a particular case of (3.10). Since $m_a(\lambda_k) \le 2$ for almost all k the formula (3.10) differs from (3.2) in an at most finite number of terms. Hence, the convergence of the series in (3.10) follows from the proof of Theorem 3.1. Thus, the function

$$\widetilde{F}(\lambda) := \sum_{k=1}^{\infty} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda)$$

is entire and it remains to prove the equality $F(\lambda) \equiv \widetilde{F}(\lambda)$. We note that

$$F^{(\nu)}(\lambda_k) = 0, \quad \nu = \overline{0, m_a(\lambda_k) - m_k - 1}, \tag{3.12}$$

for those k with $m_k < m_a(\lambda_k)$. Let us show that

$$S_{k,\nu}^{(l)}(\lambda_p) = \delta_{l,\nu}\delta_{k,p}\,, \quad l = \overline{0, m_a(\lambda_p)-1}, \ \nu = \overline{m_a(\lambda_k)-m_k, m_a(\lambda_k)-1}, \ k,p \in \mathbb{N}. \tag{3.13}$$

Clearly, if $p \neq k$, then $S_{k,\nu}^{(l)}(\lambda_p) = 0$, $l = \overline{0, m_a(\lambda_p)-1}$. Hence, it is only to prove (3.13) for $p = k$. The lowest power of $(\lambda - \lambda_k)$ in $S_{k,\nu}(\lambda)$ is ν. Thus, for $l = \overline{0, \nu-1}$ (3.13) is valid. For $l = \overline{\nu, m_a(\lambda_k)-1}$ by differentiation we get

$$S_{k,\nu}^{(l)}(\lambda_k) = \frac{l!}{\nu!} \sum_{j=1}^{l-\nu+1} C_{k,j} \frac{\Delta^{(m_a(\lambda_k)+l-\nu+1-j)}(\lambda_k)}{(m_a(\lambda_k)+l-\nu+1-j)!}, \quad 1 \le l-\nu+1 \le m_k.$$

Hence, using (3.11), we obtain $S_{k,\nu}^{(l)}(\lambda_k) = \delta_{l-\nu+1,1} = \delta_{l,\nu}$, and (3.13) is proved. According to (3.12), (3.13) the function

$$\gamma(\lambda) := \frac{F(\lambda) - \widetilde{F}(\lambda)}{\Delta(\lambda)}$$

is, after removing the singularities, entire with respect to λ. Moreover, $\gamma(\lambda)$ is bounded in $\mathbb{C}_\delta$ and consequently in $\mathbb{C}$. Therefore Liouville's theorem implies that the function $\gamma(\lambda)$ is constant. On the other hand, $\gamma(\lambda) \to 0$ as $\lambda \to \infty$. Thus, $F(\lambda) \equiv \widetilde{F}(\lambda)$ and the theorem is proved. □

Finally we provide an example illustrating the theorems given above.

Example 5.1. Consider the following regular BVP

$$-y'' = \lambda y, \quad 0 < x < 1, \quad y(0) = y'(0) + y'(1) = 0. \tag{3.14}$$

Then we have

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x, \quad y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$\Delta(\lambda) = -1 - \cos\sqrt{\lambda}x = -2\prod_{k=1}^{\infty}\Big(1 - \frac{\lambda}{\lambda_k}\Big)^2, \quad \lambda_k = \pi^2(2k-1)^2.$$

Obviously, $m_a(\lambda_k) = 2$, $m_g(\lambda_k) = 1$ for all $k \in \mathbb{N}$. Hence, $\mathbb{N}_2 = \mathbb{N}$. It is easy to show (see also [9]), that the Green's function of (3.14) has the form

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\sin\sqrt{\lambda}x\cos\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}(1+\cos\sqrt{\lambda})} - \dfrac{\sin\sqrt{\lambda}(x-\xi)}{\sqrt{\lambda}}, & \xi \le x, \\[2ex] \dfrac{\sin\sqrt{\lambda}x\cos\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}(1+\cos\sqrt{\lambda})}, & \xi \ge x. \end{cases}$$

Let us for simplicity choose $\xi_0 = 1$, then

$$\varphi(x,\lambda) = \Delta(\lambda)G(x,1,\lambda) = -\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Besides, it is easy to calculate:

$$\Delta''(\lambda_k) = -\frac{1}{4\lambda_k}, \quad \Delta'''(\lambda_k) = \frac{3}{8\lambda_k^2}.$$

Thus, according to Theorem 3.1, for any function $F(\lambda)$ of the form

$$F(\lambda) = -\int_0^1 f(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\,dx, \quad f(x) \in L_2(0,1),$$

the following representation holds

$$F(\lambda) = \sum_{k=1}^{\infty}\Big(F(\lambda_k)\frac{4(\lambda+\lambda_k)(1+\cos\sqrt{\lambda})}{(\lambda-\lambda_k)^2} + F'(\lambda_k)\frac{8\lambda_k(1+\cos\sqrt{\lambda})}{\lambda-\lambda_k}\Big),$$

where $\lambda_k = \pi^2(2k-1)^2$.

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