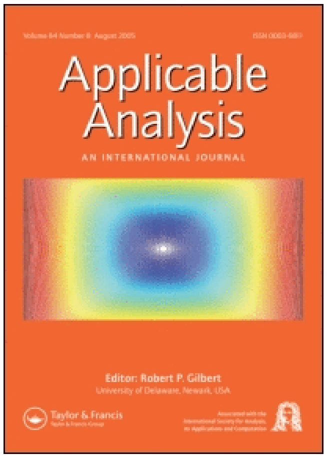

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Inverse problems for differential operators on trees with general matching conditions

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Inverse problems for differential operators on trees with general matching conditions

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Sturm–Liouville differential operators on compact trees with general matching conditions in internal vertices are studied. We establish properties of the spectral characteristics and investigate three inverse problems of recovering the operator either from the so-called Weyl functions, or from discrete spectral data or from a system of spectra. For these inverse problems, we prove the corresponding uniqueness theorems and obtain procedures for constructing their solutions by the method of spectral mappings.

Keywords: Sturm–Liouville equations on graphs; Inverse spectral problems; Method of spectral mappings

AMS Classifications: 34A55; 34B45; 34L05; 47E05

1. Introduction

We study inverse spectral problems for Sturm–Liouville differential operators on compact graphs without circles (i.e., on trees) with general matching conditions in internal vertices. The inverse problem consists in recovering the potential of the Sturm–Liouville operator on a graph from the given spectral characteristics. Differential operators on graphs (networks, trees) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, biology, electronics, nanoscale technology, and other branches of natural sciences and engineering (see [1–15] and the references therein). In recent years, there has been considerable interest in spectral theory of Sturm–Liouville equations on graphs (see a good review of such

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publications in [16,17]). Most of the works in this direction are devoted to the so-called direct problems of studying properties of the spectrum and the root functions. Inverse spectral problems, because of their nonlinearity, are more difficult for investigating, and nowadays there are only isolated fragments, not constituting a general picture, in the inverse problem theory for differential operators on graphs. Some aspects of the inverse problem theory on graphs were studied in [18–22] and other works, but mostly only very particular questions are considered there. We mark the important paper [22] where there was a first attempt to suggest a global formulation of the inverse problem on compact trees and to give an approach to its solution. But unfortunately the formulation of the inverse problem in [22] is over-determined (even in the simplest case of the classical Sturm–Liouville operator on an interval), and the question how to formulate the inverse problem correctly remains open.

In this article, we provide formulations and the solutions of the inverse problems for Sturm–Liouville operators on compact trees which are not over-determined and which are natural generalizations of the well-known inverse problems for the classical Sturm–Liouville operators on an interval ([23–32]). We introduce spectral characteristics which uniquely determine the potential on the tree, study their properties, prove the corresponding uniqueness theorems and provide a constructive procedure for the solution. First, we study the inverse problem of recovering the potential on the tree from the so-called Weyl vector which is a generalization of the Weyl function (m-function) for the classical Sturm–Liouville operator [33]. Then we consider the inverse problem of recovering the potential from a system of spectra which is a generalization of the classical Borg's inverse problem for the Sturm–Liouville operator on an interval. We also consider the inverse problem on trees from the so-called spectral data which is a generalization of the classical Marchenko's inverse problem. For studying these inverse problems on trees, we develop the ideas of the method of spectral mappings [34]. This method allows one to solve inverse problems for a wide class of graphs (not only on trees). Since different classes of graphs require different techniques, for definiteness we confine ourselves to Sturm–Liouville equations on trees (i.e., on graphs without cycles). Note that the obtained results are valid not only for the selfadjoint case but also for the non-selfadjoint one when the potential is a complex-valued function on the tree. We also note that in the recent paper [35] an inverse problem on a tree is considered for the particular case of the so-called standard matching conditions.

2. Main notions

Consider a compact, connected tree T in $\mathbf{R^m}$ with the root v_0, the set of vertices $V = \{v_0, \ldots, v_r\}$ and the set of edges $\mathcal{E} = \{e_1, \ldots, e_r\}$. We suppose that the length of each edge is equal to 1. A vertex is called a boundary vertex if it belongs to only one edge. Such an edge is called a boundary edge. All other vertices and edges are called internal. Without loss of generality we assume that v_0 is a boundary vertex.

For two points $a, b \in T$ we will write $a \leq b$ if a lies on a unique simple path connecting the root v_0 with b; let $|b|$ stand for the length of this path. We will write $a < b$ if $a \leq b$ and $a \neq b$. The relation $<$ defines a partial ordering on T. If $a < b$, we denote $[a, b] := \{z \in T\colon\ a \leq z \leq b\}$. In particular, if $e = [v, w]$ is an edge, we call v its initial point, w its end point and say that e emanates from v and terminates at w. For each

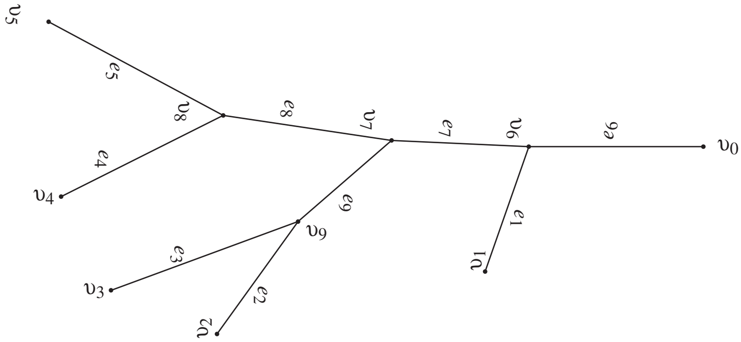

Figure 1.

internal vertex v, we denote by $R(v) := \{e \in \mathcal{E}\colon\ e = [v, w], w \in V\}$ the set of edges emanating from v. For any $v \in V$ the number $|v|$ is a nonnegative integer, which is called the order of v. For $e \in \mathcal{E}$ its order is defined as the order of its end point. The number $\sigma := \max_{j=\overline{1,r}} |v_j|$ is called the height of the tree T. Let $V^{(\mu)} := \{v \in V\colon |v| = \mu\}$, $\mu = \overline{0, \sigma}$ be the set of vertices of order μ, and let $\mathcal{E}^{(\mu)} := \{e \in \mathcal{E}\colon e = [v, w], v \in V^{(\mu-1)}, w \in V^{(\mu)}\}$, $\mu = \overline{1, \sigma}$ be the set of edges of order μ.

Each edge $e \in \mathcal{E}$ is now parametrized by the parameter $x \in [0, 1]$. It is convenient for us to choose the following parametrization on each edge $e = [v, w] \in \mathcal{E}$: we put $z(x) = w + x(v - w) \in e$; in particular $z(0) = w, z(1) = v$, i.e., $x = 0$ corresponds to the end point w, and $x = 1$ corresponds to the initial point v of the edge e. This means that the parametrization of each edge is made in opposite direction to its orientation. Thus, for each edge $e \in \mathcal{E}$ there is a bijection from $[0, 1]$ to e. For definiteness, we enumerate the vertices v_j as follows: $\Gamma := \{v_0, v_1, \ldots, v_p\}$ are the boundary vertices, $v_{p+1} \in V^{(1)}$, and $v_j, j > p + 1$ are enumerated in order of increasing $|v_j|$. We enumerate the edges similarly, namely: $e_j = [v_{j_k}, v_j]$, $j = \overline{1, r}$, $j_k < j$. In particular, $E := \{e_1, \ldots, e_{p+1}\}$ is the set of boundary edges, $e_{p+1} = [v_0, v_{p+1}]$. Clearly, $e_j \in \mathcal{E}^{(\mu)}$ if and only if $v_j \in V^{(\mu)}$. As an example see figure 1 where $r = 9,\ p = 5,\ \sigma = 4$.

3. Sturm–Liouville equation with general matching conditions

An integrable function Y on T may be represented as a vector $Y(x) = [\,y_j(x)]_{j \in J}$, $x \in [0, 1]$, where $J := \{j\colon\ j = \overline{1, r}\}$, and the function $y_j(x)$ is defined on the edge e_j. Let $q = [q_j]_{j \in J}$ be an integrable complex-valued function on T which is called the potential. Consider the Sturm–Liouville equation on T:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad x \in [0, 1], \tag{1}$$

where $j \in J$, λ is the spectral parameter, the functions $y_j(x), y_j'(x)$ are absolutely continuous on $[0, 1]$ and satisfy the following matching conditions in each internal vertex v_k, $k = \overline{p+1, r}$:

$$\left.\begin{aligned} y_k(0) &= a_{kj} y_j(1) \quad \text{for all } e_j \in R(v_k), \\ y_k'(0) &= \sum_{e_j \in R(v_k)} \left(a_{kj}^1 y_j'(1) + a_{kj}^0 y_j(1) \right) \quad \text{(generalized Kirchhoff's condition)}, \end{aligned}\right\} \tag{2}$$

where a_{kj}, a_{kj}^0, a_{kj}^1 are complex numbers, and $a_{kj} a_{kj}^1 \neq 0$ for these k, j. Assume that

$$r_k := \sum_{e_j \in R(v_k)} \frac{a_{kj}^1}{a_{kj}} \neq -1, \quad k = \overline{p+1, r}. \tag{3}$$

Condition (3) is called the regularity condition for matching. Sturm–Liouville operators on T which do not satisfy the regularity condition for matching (3), possess qualitatively different properties for the formulation and the investigation of direct and inverse problems, and are not considered in this article; they require a separate investigation. We note that if $a_{kj} = a_{kj}^1 = 1, a_{kj}^0 = 0$ for all k, j, then the conditions (2) are called the standard conditions. For standard matching conditions, (3) is obviously satisfied. Note that in (2) we have $2r - p - 1$ conditions. In order to define a boundary value problem for (1) we need additionally $p+1$ conditions at the boundary vertices v_j, $j = \overline{0, p}$. For this purpose we introduce the following linear forms in the boundary vertices v_j, $j \in \Gamma$:

$$U_{js}(Y) := \sum_{\nu=0}^{1} h_{js}^{\nu} Y_{|v_j}^{(\nu)}, \quad s = 0, 1, \ j = \overline{0, p},$$

where h_{js}^{ν} are complex numbers such that $\det[h_{js}^{\nu}]_{s,\,\nu=0,1} \neq 0$. Denote by L the boundary value problem for equation (1) with the matching conditions (2) and with the boundary conditions $U_{j0}(Y) = 0$, $j = \overline{0, p}$. We also will consider the boundary value problems L_k, $k = \overline{0, p}$, for equation (1) with the matching conditions (2) and with the boundary conditions $U_{k1}(Y) = 0$, $U_{j0}(Y) = 0$, $j = \overline{0, p} \backslash k$.

4. Formulations of the inverse problems

Let $\Psi_k(x, \lambda) = [\psi_{kj}(x, \lambda)]_{j \in J}$, $k = \overline{0, p}$, be solutions of equation (1) satisfying (2) and the boundary conditions

$$U_{j0}(\Psi_k) = \delta_{jk}, \quad j = \overline{0, p}, \tag{4}$$

where δ_{jk} is the Kronecker symbol. The functions Ψ_k are called the Weyl solutions of (1) with respect to the boundary vertex v_k. Denote $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$, where $M_k(\lambda) := U_{k1}(\Psi_k)$. The functions $M_k(\lambda)$ are called the Weyl functions, and $M(\lambda)$ is called the Weyl vector for equation (1).

For definiteness, we will consider the case when $U_{j0}(Y) = Y'_{|v_j} + h_j Y_{|v_j}$, $U_{j1}(Y) = Y_{|v_j}$, i.e., $h_{j0}^1 = h_{j1}^0 = 1, h_{j1}^1 = 0$, $h_{j0}^0 = h_j$. Other cases are treated similarly.

Let $\varphi_j(x,\lambda), S_j(x,\lambda),\ j\in J, x\in[0,1]$ be solutions of equation (1) on the edge e_j under the initial conditions $\varphi_j(0,\lambda)=S_j'(0,\lambda)=1,\ \ \varphi_j'(0,\lambda)=-h_j, S_j(0,\lambda)=0$. For each fixed x, the functions $\varphi_j^{(\nu)}(x,\lambda)$ and $S_j^{(\nu)}(x,\lambda), \nu=0,1$, are entire in λ of order 1/2. Moreover, $\langle\varphi_j(x,\lambda), S_j(x,\lambda)\rangle\equiv 1$, where $\langle y,z\rangle := yz'-y'z$ is the Wronskian of y and z. Denote

$$M_{kj}^1(\lambda)=\psi_{kj}(0,\lambda),\quad M_{kj}^0(\lambda)=\psi_{kj}'(0,\lambda)+h_j\psi_{kj}(0,\lambda).$$

It is easy to check that

$$\psi_{kj}(x,\lambda)=M_{kj}^0(\lambda)S_j(x,\lambda)+M_{kj}^1(\lambda)\varphi_j(x,\lambda). \tag{5}$$

In particular, for $k=\overline{1,p}$, we have $M_{kk}^1(\lambda)=M_k(\lambda)$, $M_{kk}^0(\lambda)=1$, $M_{kj}^0(\lambda)=0$ for $j=\overline{1,p}\backslash k$, and consequently,

$$\psi_{kk}(x,\lambda)=S_k(x,\lambda)+M_k(\lambda)\varphi_k(x,\lambda). \tag{6}$$

Substituting (5) into (2) and (4) we obtain a linear algebraic system s_k with respect to $M_{kj}^0(\lambda), M_{kj}^1(\lambda)$. By the well-known method (see, for example, [27,36]), one can show that the determinant of this system $\Delta(\lambda)$ is an entire function of order 1/2, and that $\Delta(\lambda)$ is the characteristic function of the boundary value problem L, i.e., the zeros of $\Delta(\lambda)$ coincide with the eigenvalues $\{\lambda_l\}_{l\ge 0}$ of the boundary value problem L. Solving the system s_k we get by Cramer's rule: $M_{kj}^s(\lambda)=\Delta_{kj}^s(\lambda)/\Delta(\lambda)$, $s=0,1$, $j=\overline{1,r}$, where the determinant $\Delta_{kj}^s(\lambda)$ is obtained from $\Delta(\lambda)$ by the replacement of the column which corresponds to $M_{kj}^s(\lambda)$ by the column of the free terms. In particular,

$$M_k(\lambda)=\frac{\Delta_k(\lambda)}{\Delta(\lambda)},\quad k=\overline{1,p}, \tag{7}$$

where $\Delta_k(\lambda):=\Delta_{kk}^0(\lambda)$ is the characteristic function for the boundary value problem L_k. The function $\Delta_k(\lambda)$ is also entire in λ of order 1/2, and its zeros coincide with the eigenvalues $\{\lambda_{lk}\}_{l\ge 0}$ of the boundary value problem L_k.

We note that similarly to the classical Sturm–Liouville operators on an interval, it can be shown that λ_l and λ_{lk} lie in the strip $\{\lambda\colon |\mathrm{Im}\,\lambda|\le c_1, \mathrm{Re}\,\lambda\ge c_0\}$ for some c_0, c_1. It follows from (7) that the Weyl functions $M_k(\lambda)$ are meromorphic in λ with the poles $\{\lambda_l\}_{l\ge 0}$. If all poles are simple, we introduce also the data $S:=\{\lambda_l, \alpha_{lk}\}_{l\ge 0, k=\overline{1,p}}$, where α_{lk} are the residues of $M_k(\lambda)$ at λ_l; the data S are called the spectral data for L.

In this article, we study three inverse problems of recovering the potential $q=[q_j]_{j\in J}$ and the coefficients $h=[h_j]_{j\in J}$ from the following spectral characteristics:

(1) from the Weyl vector $M=[M_k]_{k=\overline{1,p}}$;
(2) from the system of $p+1$ spectra $\Sigma:=\{\lambda_l, \lambda_{lk}\colon l\ge 0, k=\overline{1,p}\}$;
(3) from the spectral data S.

For each of these inverse problems we provide a constructive procedure for the solution and prove its uniqueness.

We note that the notion of the Weyl vector M is a generalization of the notion of the Weyl function (m-function) for the classical Sturm–Liouville operator [27,33]. If $r=1$ (i.e., the tree T is the interval $[0,1]$), then $p=1$ and the Weyl vector M coincides with the classical Weyl function. Thus, Inverse problem 1 is a generalization of the classical inverse problem for Sturm–Liouville operators on an interval from the Weyl function, and (which is equivalent) from the spectral measure. Inverse problem 2 is a generalization of the classical Borg's inverse problem for the Sturm–Liouville operator on an interval from two spectra. If $r=1$, then $p=1$ and Inverse problem 2 coincides with the classical Borg's inverse problem from two spectra. Inverse problem 3 is a generalization of the classical Marchenko's inverse problem for the Sturm–Liouville operator on an interval (see [23,27] for details).

5. Properties of the Weyl solutions

Let $\lambda=\rho^2, \operatorname{Im}\rho\geq 0$. Denote $\Lambda:=\{\rho\colon \operatorname{Im}\rho\geq 0\}$. It is known [36] that for each fixed $j\in J$ on the edge e_j, there exists a fundamental system of solutions of equation (1) $\{e_{j1}(x,\rho), e_{j2}(x,\rho)\}$, $x\in[0,1], \rho\in\Lambda, |\rho|\geq\rho^*$ with the following properties:

(1) the functions $e_{js}^{(\nu)}(x,\rho), \nu=0,1,$ are continuous for $x\in[0,1], \rho\in\Lambda, |\rho|\geq\rho^*$;
(2) for each $x\in[0,1]$, the functions $e_{js}^{(\nu)}(x,\rho), \nu=0,1,$ are analytic with respect to $\rho\in\Lambda, |\rho|\geq\rho^*$;
(3) uniformly for $x\in[0,1]$, the following asymptotical formulae hold

$$e_{j1}^{(\nu)}(x,\rho)=(i\rho)^\nu\exp(i\rho x)[1],\quad e_{j2}^{(\nu)}(x,\rho)=(-i\rho)^\nu\exp(-i\rho x)[1],\ \rho\in\Lambda,\ |\rho|\to\infty, \tag{8}$$

where $[1]=1+O(\rho^{-1}),\ \nu=0,1.$

Denote $\Lambda_\delta:=\{\rho\colon\ \arg\rho\in[\delta,\pi-\delta]\}$, $\delta>0$.

LEMMA 1 *Let $y_j(x,\rho)$ be a solution of equation* (1) *on the edge e_j, and let*

$$\frac{y_j'(0,\rho)}{y_j(0,\rho)}=(-i\rho)r_j[1],\quad r_j\neq-1,\ \rho\in\Lambda_\delta,\ |\rho|\to\infty. \tag{9}$$

Then for $\nu=0,1,\ \rho\in\Lambda_\delta,\ |\rho|\to\infty,$ uniformly in $x\in[0,1]$,

$$y_j^{(\nu)}(x,\rho)=D_j(\rho)\big((-i\rho)^\nu\exp(-i\rho x)[1]-(r_j+1)^{-1}(r_j-1)(i\rho)^\nu\exp(i\rho x)[1]\big), \tag{10}$$

where $D_j(\rho)$ does not depend on x.

Proof Using the fundamental system of solutions $\{e_{j1}(x,\rho), e_{j2}(x,\rho)\}$, one gets

$$y_j(x,\rho)=A_j(\rho)e_{j1}(x,\rho)+D_j(\rho)e_{j2}(x,\rho). \tag{11}$$

It follows from (8) and (11) that

$$\frac{y_j'(0,\rho)}{y_j(0,\rho)} = (i\rho)\frac{A_j(\rho)\,[1] - D_j(\rho)\,[1]}{A_j(\rho)\,[1] + D_j(\rho)\,[1]}, \quad \rho \in \Lambda_\delta,\ |\rho| \to \infty.$$

Taking (9) into account, we calculate $A_j(\rho) = D_j(\rho)\,(r_j+1)^{-1}(r_j-1)[1]$. Substituting this relation into (11) and using (8), we arrive at (10). ■

In the next lemmas, we formulate asymptotic estimates for the Weyl solutions.

LEMMA 2 *Let* $e_j \in \mathcal{E}^{(\mu)}$. *Then for* $\nu = 0, 1$, $\rho \in \Lambda_\delta$, $|\rho| \to \infty$, *uniformly in* $x \in [0, 1]$, *one has*

$$\psi_{0j}^{(\nu)}(x,\lambda) = B_j(\rho)\exp(i\rho\mu)\big((-i\rho)^{\nu-1}\exp(-i\rho x)[1] - (i\rho)^{\nu-1}d_j\exp(i\rho x)[1]\big), \tag{12}$$

where $d_j = 1$ *for* $j = \overline{1,p}$, *and* $d_j = (1+r_j)^{-1}(1-r_j)$ *for* $j = \overline{p+1,r}$. *Moreover, for* $\rho \in \Lambda_\delta$, $|\rho| \to \infty$,

$$B_j(\rho) = b_j[1], \quad b_j \neq 0,\ b_{p+1} = 1. \tag{13}$$

In particular, for $\rho \in \Lambda_\delta$, $|\rho| \to \infty$,

$$\psi_{0j}^{(\nu)}(x,\lambda) = (-i\rho)^{\nu-1}b_j\exp(i\rho(\mu - x))[1], \quad \nu = 0, 1,\ x \in (0, 1]. \tag{14}$$

Proof (1) Let $j = \overline{1,p}$. Then $\psi_{0j}'(0,\lambda) + h_j\psi_{0j}(0,\lambda) = 0$, and in view of (5),

$$\psi_{0j}(x,\lambda) = M_{0j}^1(\lambda)\varphi_j(x,\lambda). \tag{15}$$

Using (15) and the asymptotics

$$\varphi_j^{(\nu)}(x,\lambda) = \frac{1}{2}((i\rho)^\nu\exp(i\rho x)[1] + (-i\rho)^\nu\exp(-i\rho x)[1]), \quad |\rho| \to \infty, \tag{16}$$

we arrive at (12) for $j = \overline{1,p}$.

(2) Let us prove (12) for all other edges by induction with respect to $\mu = \sigma, \sigma-1, \ldots, 1$, where σ is the height of the tree T. If $\mu = \sigma$ (i.e., $e_j \in \mathcal{E}^{(\sigma)}$), then $1 \le j \le p$, and (12) holds according to the previous arguments.

Fix $\mu < \sigma$. Suppose that (12) has been proved for all $e_k \in \mathcal{E}^{(\mu+1)} \cup \cdots \cup \mathcal{E}^{(\sigma)}$. Let $e_j \in \mathcal{E}^{(\mu)}$. Clearly, if $e_k \in R(v_j)$, then $e_k \in \mathcal{E}^{(\mu+1)}$. Therefore, for each $e_k \in R(v_j)$, (12) holds by the induction assumption. In particular, for $\rho \in \Lambda_\delta, |\rho| \to \infty, e_k \in R(v_j)$,

$$\psi_{0k}(1,\lambda) = B_k(\rho)(-i\rho)^{-1}\exp(i\rho\mu)[1], \quad \psi_{0k}'(1,\lambda) = B_k(\rho)\exp(i\rho\mu)[1]. \tag{17}$$

Using the matching conditions (2), we calculate

$$\frac{\psi'_{0j}(0,\lambda)}{\psi_{0j}(0,\lambda)} = \sum_{e_k \in R(v_j)} \frac{a^1_{jk}\psi'_{0k}(1,\lambda) + a^0_{jk}\psi_{0k}(1,\lambda)}{a_{jk}\psi_{0k}(1,\lambda)}.$$

Together with (17) this yields

$$\frac{\psi'_{0j}(0,\lambda)}{\psi_{0j}(0,\lambda)} = (-i\rho)r_j[1].$$

By the regularity condition for matching, $r_j \neq -1$. Applying Lemma 1, we arrive at (13) with a certain coefficient $B_j(\rho)$. Thus, (12) is proved for all edges $e_j \in \mathcal{E}$. Furthermore, it follows from (2) that $\psi_{0j}(0,\lambda) = a_{jk}\psi_{0k}(1,\lambda)$ for all $e_k \in R(v_j)$, and consequently, $a_{jk}B_k(\rho) = 2(r_j+1)^{-1}B_j(\rho)[1]$. Since $\psi'_{0,p+1}(1,\lambda) + h_0\psi_{0,p+1}(1,\lambda) = 1$, we obtain (13). Hence (14) is also valid. ■

Symmetrically to (12)–(14), one can get the asymptotics for all other Weyl solutions $\Psi_k, k = \overline{1,p}$. In particular, the following assertion is a corollary of Lemma 2.

LEMMA 3 *For* $k = \overline{1,p}$, $\nu = 0, 1$, *one has*

$$\psi^{(\nu)}_{kk}(x,\lambda) = (i\rho)^{\nu-1}\exp(i\rho x)[1],\ M_k(\lambda) = (i\rho)^{-1}[1], \quad \rho \in \Lambda_\delta,\ |\rho| \to \infty,\ x \in [0,1). \tag{18}$$

Let $\delta > 0$ be sufficiently small and fixed. Denote $G_\delta := \{\rho\colon\ |\rho - \rho_l| \geq \delta, \forall l \geq 0\}$, where $\lambda_l = \rho_l^2$ are the eigenvalues of the boundary value problem L. Using the standard technique [35] one can show that

$$|\psi^{(\nu)}_{kk}(x,\lambda)| \leq C|\rho^{\nu-1}\exp(i\rho x)|,\ |M_k(\lambda)| \leq C|\rho|^{-1}, \quad \rho \in G_\delta \cap \Lambda,\ x \in [0,1]. \tag{19}$$

6. Auxiliary inverse problem

Fix $k = \overline{1,p}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k). Given $M_k(\lambda)$, construct $q_k(x), x \in [0,1]$ and h_k.

Let us prove the uniqueness of the solution of the inverse problem IP(k), this will be done as we described in [27]. For this purpose together with T, we consider a tree $\tilde{T}$ of the same form but with different $\tilde{q}$ and $\tilde{h}$. Everywhere below if a symbol α denotes an object related to T, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{T}$.

LEMMA 4 *If* $M_k(\lambda) = \tilde{M}_k(\lambda)$, *then* $q_k(x) = \tilde{q}_k(x)$ *a.e. on* [0,1], *and* $h_k = \tilde{h}_k$. *Thus, the specification of the Weyl function* M_k *uniquely determines the potential* q_k *on the edge* e_k *and the coefficient* h_k.

Proof We introduce the functions

$$P^k_{1s}(x,\lambda) = (-1)^{s-1}\Big(\varphi_k(x,\lambda)\tilde{\psi}^{(2-s)}_{kk}(x,\lambda) - \tilde{\varphi}^{(2-s)}_k(x,\lambda)\psi_{kk}(x,\lambda)\Big), \quad s = 1,2. \tag{20}$$

It follows from (6) that $\langle \varphi_k(x,\lambda), \psi_{kk}(x,\lambda)\rangle \equiv 1$. Then, by direct calculations we get

$$\varphi_k(x,\lambda) = P^k_{11}(x,\lambda)\tilde{\varphi}_k(x,\lambda) + P^k_{12}(x,\lambda)\tilde{\varphi}'_k(x,\lambda). \tag{21}$$

Using (16) and (18)–(20) we obtain

$$P^k_{1s}(x,\lambda) = \delta_{1s} + O(\rho^{-1}), \quad \rho \in \Lambda_\delta,\ |\rho| \to \infty,\ x \in (0,1], \tag{22}$$

$$|P^k_{1s}(x,\lambda)| \le C|\rho|^{1-s}, \quad \rho \in G_\delta \cap \Lambda,\ x \in [0,1]. \tag{23}$$

According to (6) and (20),

$$P^k_{1s}(x,\lambda) = (-1)^{s-1}\Big(\Big(\varphi_k(x,\lambda)\tilde{S}^{(2-s)}_k(x,\lambda) - S_k(x,\lambda)\tilde{\varphi}^{(2-s)}_k(x,\lambda)\Big) + (\tilde{M}_k(\lambda) - M_k(\lambda))\varphi_k(x,\lambda)\tilde{\varphi}^{(2-s)}_k(x,\lambda)\Big).$$

Since $M_k(\lambda) = \tilde{M}_k(\lambda)$, it follows that for each fixed x, the functions $P^k_{1s}(x,\lambda)$ are entire in λ. Together with (22) and (23) this yields $P^k_{11}(x,\lambda) \equiv 1$, $P^k_{12}(x,\lambda) \equiv 0$. Substituting these relations into (21) we get $\varphi_k(x,\lambda) \equiv \tilde{\varphi}_k(x,\lambda)$ for all x and λ, and consequently, $q_k(x) = \tilde{q}_k(x)$ a.e. on $[0,1]$, and $h_k = \tilde{h}_k$. ■

Using the method of spectral mappings [34] for the Sturm–Liouville operator on the edge e_k one can get a constructive procedure for the solution of the local inverse problem IP(k). Here we only explain ideas briefly; for details and proofs see [27] or [34]. Take the tree $\tilde{T}$ with $\tilde{q} = 0$ and $\tilde{h} = 0$. Then $\tilde{\varphi}_k(x,\lambda) = \cos\rho x$. Fix $k = \overline{1,p}$. Denote $\lambda' = \min_{l \ge 0}(\operatorname{Re}\lambda_l, \operatorname{Re}\tilde{\lambda}_l)$, and take a fixed $c > 0$ such that $|\operatorname{Im}\lambda_l|, |\operatorname{Im}\tilde{\lambda}_l| < c$. In the λ-plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma = \gamma^+ \cup \gamma^- \cup \gamma'$, where $\gamma^\pm = \{\lambda:\ \pm\operatorname{Im}\lambda = c;\ \operatorname{Re}\lambda \ge \lambda'\}$, $\gamma' = \{\lambda:\ \lambda - \lambda' = c\exp(i\alpha),\ \alpha \in (\pi/2, 3\pi/2)\}$. For each fixed $x \in [0,1]$, the function $\varphi_k(x,\lambda)$ is the unique solution of the following linear integral equation

$$\tilde{\varphi}_k(x,\lambda) = \varphi_k(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\varphi_k(x,\mu)\,\mathrm{d}\mu, \tag{24}$$

where $\tilde{D}_k(x,\lambda,\mu) = \int_0^x \tilde{\varphi}_k(t,\lambda)\,\tilde{\varphi}_k(t,\mu)\,\mathrm{d}t$, $\hat{M}_k(\mu) := M_k(\mu) - \tilde{M}_k(\mu)$. The potential q_k on the edge e_k can be constructed from the solution of the integral equation (24) via the formula

$$q_k(x) = \frac{1}{2\pi i}\int_\gamma (\varphi_k(x,\lambda)\,\tilde{\varphi}_k(x,\lambda))'\hat{M}_k(\lambda)\,\mathrm{d}\lambda$$

or by the formula $q_k(x) = \lambda + \varphi_k''(x,\lambda)/\varphi_k(x,\lambda)$. Moreover $h_k = -\varphi'(0,\lambda)$. It is also possible to construct the potential from the discrete spectral data $\{\lambda_l, \alpha_{lk}\}_{l\geq 0}$. For this purpose, we can calculate the contour integral in (24) by the residue theorem and transform the integral equation (24) to the following linear equation in a space of bounded sequences (for each fixed x):

$$\tilde{\varphi}_{kns}(x) = \varphi_{kns}(x) + \sum_{l,j} \tilde{P}_{kns}^{lj}(x)\varphi_{klj}(x), \quad l,n \geq 0,\ s,j = 0,1,$$

where $\varphi_{kns}(x) = \varphi_k(x, \lambda_n^s)$, $\tilde{\varphi}_{kns}(x) = \tilde{\varphi}_k(x, \lambda_n^s)$, $\tilde{P}_{kns}^{lj}(x) = (-1)^j \tilde{D}_k(x, \lambda_n^s, \lambda_l^j)\alpha_{lk}^j$, $\lambda_l^0 = \lambda_l$, $\lambda_l^1 = \tilde{\lambda}_l$, $\alpha_{lk}^0 = \alpha_{lk}$, $\alpha_{lk}^1 = \tilde{\alpha}_{lk}$; for details see [27].

7. Problem $Z(T, v_0, a)$

Let $\Psi = [\psi_j]_{j\in J}$ be the solution of equation (1) on T satisfying (2) and the boundary conditions

$$\Psi_{|v_0} = a, \quad U_{j0}(\Psi) = 0,\ j = \overline{1,p}, \tag{25}$$

where a is a complex number. Denote $m_j^1(\lambda) = \psi_j(0,\lambda)$, $m_j^0(\lambda) = \psi_j'(0,\lambda) + h_j\psi_j(0,\lambda)$, $j\in J$. Then

$$\psi_j(x,\lambda) = m_j^0(\lambda)S_j(x,\lambda) + m_j^1(\lambda)\varphi_j(x,\lambda). \tag{26}$$

Substituting (26) into (2) and (25) we obtain a linear algebraic system with respect to $m_j^0(\lambda)$, $m_j^1(\lambda)$, $j\in J$. The determinant of this system is $\Delta_0(\lambda)$. Solving this system by Cramer's rule we find the transition matrix $[m_j^0(\lambda), m_j^1(\lambda)]_{j\in J}$ for T with respect to v_0 and a. The problem of calculating the transition matrix $[m_j^0(\lambda), m_j^1(\lambda)]_{j\in J}$ by Cramer's rule is called Problem $Z(T, v_0, a)$. This problem will be used below for describing the procedure for the solution of the inverse problems.

8. Weyl solutions for internal vertices

Fix $v_k \in V$. Denote $T_k^0 := \{z\in T\colon\ v_k < z\}$, $T_k := T\setminus T_k^0$. Clearly, T_k is a tree with the root v_0. Let Γ_k be the set of boundary vertices of T_k, and let E_k be the set of boundary edges of T_k. Denote $J_k := \{j\colon\ e_j\in T_k\}$. If $Y = [y_j]_{j\in J}$ is a function on T, then $\{Y\}_k := [y_j]_{j\in J_k}$ is a function on T_k.

Fix $v_k \notin \Gamma$ (i.e., $k\in\{p+1,\ldots,r\}$). Let $\Psi_k(x,\lambda) = [\psi_{kj}(x,\lambda)]_{j\in J_k}$ be the solution of equation (1) on T_k satisfying (2) and the boundary conditions $U_{j0}(\Psi_k) = \delta_{kj}$, $v_j\in\Gamma_k$, where $U_{k0}(Y) = Y'_{|v_k} + h_k Y_{|v_k}$, and h_k is a complex number. The vector Ψ_k is the Weyl solution of (1) on T_k with respect to the vertex v_k. Denote by $M_k(\lambda) := \psi_{kk}(0,\lambda)$, $k = \overline{p+1,r}$ the Weyl functions for T_k with respect to v_k.

LEMMA 5 *Fix $v_m \notin \Gamma$. Let $e_k = [v_m, v_k] \in R(v_m)$. Then*

$$\Psi_m(x,\lambda) = \left\{ \frac{1}{A_{mk}(\lambda)} \Psi_k(x,\lambda) \right\}_m, \quad i.e., \quad \psi_{mj}(x,\lambda) = \frac{1}{A_{mk}(\lambda)} \psi_{kj}(x,\lambda),\ j \in J_m, \tag{27}$$

$$M_m(\lambda) = \frac{a_{mk}}{A_{mk}(\lambda)} \psi_{kk}(1,\lambda), \tag{28}$$

where

$$A_{mk}(\lambda) = \sum_{e_j \in R(v_m)} a^1_{mj} \psi'_{kj}(1,\lambda) + a_{mk} \left(h_m + \sum_{e_j \in R(v_m)} \frac{a^0_{mj}}{a_{mj}} \right) \psi_{kk}(1,\lambda), \tag{29}$$

and Ψ_m, M_m does not depend on k.

Proof Since $U_{j0}(\Psi_k) = U_{j0}(\Psi_m) = 0$ for $j \in J_m \backslash m$, we get (27) for some $A_{mk}(\lambda)$. Using the condition $U_{m0}(\Psi_m) = 1$, we calculate $A_{mk}(\lambda) = \psi'_{km}(0,\lambda) + h_m \psi_{km}(0,\lambda)$. Taking (2) into account we arrive at (29). Furthermore,

$$M_m(\lambda) = \psi_{mm}(0,\lambda) = \frac{1}{A_{mk}(\lambda)} \psi_{km}(0,\lambda).$$

Using the matching conditions (2) again, we obtain (28). ■

Denote $M^1_{kj}(\lambda) = \psi_{kj}(0,\lambda)$, $M^0_{kj}(\lambda) = \psi'_{kj}(0,\lambda) + h_j \psi_{kj}(0,\lambda)$ for $k = \overline{p+1, r}$, $j \in J_k$. Then (5) and (6) are valid for $k = \overline{1, r}$, $j \in J_k$, where $J_k = J$ for $k = \overline{1, p}$. In particular, this yields

$$\psi^{(\nu)}_{kj}(1,\lambda) = M^0_{kj}(\lambda) S^{(\nu)}_j(1,\lambda) + M^1_{kj}(\lambda) \varphi^{(\nu)}_j(1,\lambda), \quad \nu = 0, 1,\ k = \overline{1, r},\ j \in J_k, \tag{30}$$

$$\psi^{(\nu)}_{kk}(1,\lambda) = S^{(\nu)}_k(1,\lambda) + M_k(\lambda) \varphi^{(\nu)}_k(1,\lambda), \quad \nu = 0, 1,\ k = \overline{1, r}. \tag{31}$$

9. Solution of Inverse problem 1

Let the Weyl vector $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$ for the tree T be given. The procedure for the solution of Inverse problem 1 consists in the realization of the so-called A_μ-procedures successively for $\mu = \sigma, \sigma - 1, \ldots, 1$, where σ is the height of the tree T. Let us describe these A_μ-procedures.

A_σ-procedure

(1) For each edge $e_k \in \mathcal{E}^{(\sigma)}$, we solve the local inverse problem IP(k) and find $q_k(x), x \in [0, 1]$ on the edge e_k and h_k.
(2) For each $e_k \in \mathcal{E}^{(\sigma)}$, we construct $\varphi_k(x,\lambda), S_k(x,\lambda), x \in [0, 1]$, and calculate $\psi^{(\nu)}_{kk}(1,\lambda)$, $\nu = 0, 1$, by (31).

(3) *Returning procedure*. For each fixed $v_m \in V^{(\sigma-1)}\setminus\Gamma$ and for all $e_j, e_k \in R(v_m)$, $j \neq k$, we construct $M^s_{kj}(\lambda), s = 0, 1,$ by the formulae

$$M^0_{kj}(\lambda) = 0,\ M^1_{kj}(\lambda) = \frac{a_{mk}}{a_{mj}} \frac{\psi_{kk}(1, \lambda)}{\varphi_j(1, \lambda)}, \quad e_j, e_k \in R(v_m),\ j \neq k.$$

(4) For each fixed $v_m \in V^{(\sigma-1)}\setminus\Gamma$ we calculate the Weyl function $M_m(\lambda)$ by (28), where $A_{mk}(\lambda)$ and $\psi'_{kj}(1, \lambda)$ are constructed via (29) and (30).

Next we carry out A_μ-procedures for $\mu = \overline{1, \sigma - 1}$ by induction. Fix $\mu \in \{1, \ldots, \sigma - 1\}$, and suppose that $A_\sigma, \ldots, A_{\mu+1}$-procedures have been already carried out. Let us now carry out the A_μ-procedure.

$\mathbf{A_\mu}$-procedure. For each $v_k \in V^{(\mu)}$, the Weyl functions $M_k(\lambda)$ are given. Indeed, if $v_k \in V^{(\mu)} \cap \Gamma$, then $M_k(\lambda)$ are given *a priori*, and if $v_k \in V^{(\mu)}\setminus\Gamma$, then $M_k(\lambda)$ were calculated on the previous steps according to $A_\sigma, \ldots, A_{\mu+1}$-procedures.

(1) For each edge $e_k \in \mathcal{E}^{(\mu)}$, we solve the local inverse problem IP(k) and find $q_k(x), x \in [0, 1]$ on the edge e_k and h_k. If $\mu = 1$, then Inverse problem 1 is solved, and we stop our calculations. If $\mu > 1$, we go on to the next step.
(2) For each $e_k \in \mathcal{E}^{(\mu)}$, we construct $\varphi_k(x, \lambda), S_k(x, \lambda), x \in [0, 1]$, and calculate $\psi^{(\nu)}_{kk}(1, \lambda)$, $\nu = 0, 1$, by (31).
(3) *Returning procedure*. For each fixed $v_m \in V^{(\mu-1)}\setminus\Gamma$ and for any fixed $e_k, e_i \in R(v_m), i \neq k$, we consider the tree $T^1_i := T^0_i \cup \{e_i\}$ with the root v_m. Solving the problem $Z(T^1_i, v_m, \psi_{kk}(1, \lambda))$, we calculate the transition matrix $[M^0_{kj}(\lambda), M^1_{kj}(\lambda)]$ for $e_j \in T^1_i$.
(4) For each fixed $v_m \in V^{(\mu-1)}\setminus\Gamma$, we calculate the Weyl function $M_m(\lambda)$ by (28), where $A_{mk}(\lambda)$ and $\psi'_{kj}(1, \lambda)$ are constructed via (29) and (30).

Thus, we have obtained the solution of Inverse problem 1 and proved its uniqueness, i.e., the following assertion holds.

THEOREM 1 *The specification of the Weyl vector M uniquely determines the potential q on T and h. The solution of Inverse problem 1 can be obtained by executing successively* $A_\sigma, A_{\sigma-1}, \ldots, A_1$*-procedures.*

10. Solution of Inverse problem 2

Let the system of spectra $\Sigma := \{\lambda_l, \lambda_{lk}; l \geq 0; k = \overline{1, p}\}$ be given. The numbers $\{\lambda_l\}_{l\geq 0}$ and $\{\lambda_{lk}\}_{l\geq 0}$ coincide with the zeros of the characteristic functions $\Delta(\lambda)$ and $\Delta_k(\lambda)$, respectively. These functions are entire in λ of order 1/2. By Hadamard's factorization theorem [37], the functions $\Delta(\lambda)$ and $\Delta_k(\lambda)$ are uniquely determined up to multiplicative constants by their zeros:

$$\Delta(\lambda) = C \prod_{l=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_l}\right), \quad \Delta_k(\lambda) = C_k \prod_{l=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{lk}}\right)$$

(the case when $\Delta(0)=0$ or/and $\Delta_k(0)=0$ requires minor modifications). Then, by virtue of (7),

$$M_k(\lambda)=m_k\prod_{l=0}^{\infty}\left(1-\frac{\lambda}{\lambda_{lk}}\right)\left(1-\frac{\lambda}{\lambda_l}\right)^{-1},\quad k=\overline{1,p},\ m_k-const. \tag{32}$$

Using (18), we obtain

$$m_k=\lim_{|\rho|\to\infty}(i\rho)^{-1}\prod_{l=0}^{\infty}\left(1-\frac{\lambda}{\lambda_l}\right)\left(1-\frac{\lambda}{\lambda_{lk}}\right)^{-1},\quad \rho\in\Lambda_\delta,\ k=\overline{1,p}. \tag{33}$$

Thus, using the given spectra Σ, one can construct uniquely the Weyl vector $M(\lambda)=[M_k(\lambda)]_{k=\overline{1,p}}$ by (32) and (33). In other words, the solution of Inverse problem 2 is reduced to the solution of Inverse problem 1, and the following assertion holds.

THEOREM 2 *The specification of the system of spectra Σ uniquely determines the potential q on T and h. For constructing the solution of Inverse problem* 2, *we calculate the Weyl vector M by* (32) *and* (33), *and then construct q and h by solving Inverse problem* 1.

11. Solution of Inverse problem 3

Let all poles of $M(\lambda)$ be simple (for example, this is always true in the selfadjoint case), and let the spectral data S be given. Take positive numbers $R_N\to\infty$ such that for sufficiently small $\delta>0$, the circles $|\rho|=R_N$ lie in G_δ for all N. Let us show that

$$M_k(\lambda)=\sum_{l=0}^{\infty}\frac{\alpha_{lk}}{\lambda-\lambda_l}, \tag{34}$$

where the series in (34) converges "with brackets": $\sum_{l=0}^{\infty}:=\lim_{N\to\infty}\sum_{|\lambda_l|<R_N^2}$.

Indeed, consider the contour integral

$$J_{N,k}(\lambda):=\frac{1}{2\pi i}\int_{\gamma_N}\frac{M_k(\mu)}{\lambda-\mu}\,\mathrm{d}\mu,\quad \lambda\in\operatorname{int}\gamma_N, \tag{35}$$

where $\gamma_N=\{\mu:\ |\mu|=R_N^2\}$. It follows from (18) and (19) that $\lim_{N\to\infty}J_{N,k}(\lambda)=0$. On the other hand, calculating the contour integral in (35) by the residue theorem we obtain

$$J_{N,k}(\lambda)=-M_k(\lambda)+\sum_{|\lambda_l|<R_N^2}\frac{\alpha_{lk}}{\lambda-\lambda_l},$$

and consequently, (34) is valid.

Thus, using the given spectral data S, we can construct uniquely the Weyl vector $M(\lambda)=[M_k(\lambda)]_{k=\overline{1,p}}$ by (34). In other words, the solution of Inverse problem 3 is reduced to the solution of Inverse problem 1, and the following assertion holds.

Theorem 3 *The specification of the spectral data S uniquely determines the potential q on T and h. For constructing the solution of Inverse problem* 3, *we calculate the Weyl vector M by* (34), *and then we construct q and h by solving Inverse problem* 1.

Remark In the paper [37], the uniqueness theorem is proved for the inverse problem of recovering Sturm–Liouville equation (1) from a matrix $\Lambda(\lambda)=[\Lambda_{kj}(\lambda)]_{k,j=\overline{0,p}}$, which is called in [38] the Dirichlet-to-Neumann map. This inverse problem is overdetermined: for its unique solvability it is enough to take only a part of the diagonal of $\Lambda(\lambda)$, namely Λ_{kk}, $k=\overline{1,p}$. Note that in our notations $\Lambda_{kk}=M_k(\lambda)$ for $k=\overline{1,p}$.

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