Inverse problems for Sturm-Liouville operators on noncompact trees

Gerhard Freiling and Vjacheslav Yurko

Abstract. We study Sturm-Liouville differential operators on noncompact graphs without cycles (i.e. on trees) with standard matching conditions in internal vertices. First we establish properties of the spectral characteristics and then we investigate the inverse problem of recovering the operator from the so-called Weyl vector. For this inverse problem we prove a uniqueness theorem and propose a procedure for constructing the solution using the method of spectral mappings.

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1. Introduction

Analysis on graphs and other similar structures has been developing for quite some time due to various applications in applied sciences. In particular in recent years it has experienced a significant boost in terms of new applications arising and new methods developed and studied.

In our paper we present the solution of an inverse spectral problem for Sturm-Liouville differential operators on noncompact trees. This inverse problem consists in recovering the potential of the Sturm-Liouville operator on a tree from the given spectral characteristics. We recall that differential operators on graphs (networks, trees) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, biology, electronics, nanoscale technology and other branches of natural sciences and engineering (see [1]-[14] and the references therein). Recently there has been increasing interest in spectral theory of Sturm-Liouville or Schrödinger equations on graphs (for a good review of such publications see [15]-[16]). Most of the works in this direction are devoted to the so-called direct problems of studying properties of the spectrum and the root functions. Inverse spectral problems,

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because of their nonlinearity, are more difficult for investigating, and up to now there are only a few papers devoted to inverse problems for differential operators on graphs (see [17]-[26]) although it can be expected that essential parts of the spectral theory for differential operators on intervals remain similarly valid in a more general context on graphs.

In this paper we provide a formulation and the solution of the inverse problem of recovering the potential of the Sturm-Liouville operator on noncompact trees which is a natural generalization of the well-known inverse problems for the classical Sturm-Liouville operators ([27]-[36]). As the main spectral characteristic we introduce and study the so-called Weyl vector which is a generalization of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [37]). We show that the specification of the Weyl vector uniquely determines the potential, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl vector. For studying the inverse problem on noncompact graphs we develop the ideas of the method of spectral mappings [31], [38]-[39]. This method allows one to solve inverse problems for a wide class of graphs. Since different classes of graphs require different techniques, for definiteness we confine ourselves to Sturm-Liouville equations on trees (i.e. on graphs without cycles) with one infinite edge. Note that the obtained results are valid not only for the selfadjoint case but also for the non-selfadjoint one when the potential is a complex-valued function on the tree.

In a recent paper [23], one of the authors studied Sturm-Liouville operators on *compact* trees. In [23] one has the so-called regular case with a pure discrete spectrum. In the present paper we study these operators on *noncompact* trees, where we face with a singular case having more complicated behavior of the spectrum which leads to new qualitative difficulties for studying direct and inverse problems.

The paper is organized as follows: In section 2 we introduce the main notions and give a formulation of the inverse problem. In section 3 properties of the spectrum are studied. In particular, Theorems 1-5 describe the continuous and the discrete spectrum and connections between them. In section 4 an auxiliary inverse problem for the boundary edges is considered, and in section 5 the solution of the global inverse problem is provided. Here we use and develop the method of pseudo-pruning the tree from [23] with necessary modifications. For solving the inverse problem we essentially use the results of section 3.

2. The Weyl vector. Formulation of the inverse problem

Consider a noncompact, connected tree T in $\mathbf{R}^m$ with the set of vertices $V = \{v_1, \dots, v_r\}$, $v_j \in \mathbf{R}^m$, and the set of edges $\mathcal{E} = \{e_1, \dots, e_r\}$, where $e_j = [v_{n_j}, v_j]$, $j = \overline{1, r-1}$, $n_j > j$, are finite segments, and $e_r = (v_0, v_r]$ is an infinite ray, $v_0 := \infty$. We assume for simplicity that the length of each finite edge is equal to 1 (it follows from the proofs that our method also works for arbitrary lengths of the edges). A vertex is called a boundary vertex if it belongs to only one edge. Such an edge is called a boundary edge. All other vertices and edges are called internal. Let

$\Gamma := \{v_1, \ldots, v_p\}$ be the set of finite boundary vertices, and let $E := \{e_1, \ldots, e_p\}$ be the set of compact boundary edges. Notice that we identify here the graph $(V, \mathcal{E}, \varphi)$ (with incidence relation $\varphi \subset V \times \mathcal{E}$) with its corresponding topological graph (network) T.

For two points $a, b \in T$ we will write $a \le b$ if a lies on a unique simple path connecting v_0 with b. We will write $a < b$ if $a \le b$ and $a \ne b$. The relation $<$ defines a partial ordering on T. If $a < b$ we denote $[a, b] := \{z \in T :\ a \le z \le b\}$. In particular, if $e_j = [v_{n_j}, v_j]$, $j = \overline{1, r-1}$, $v_{n_j} < v_j$, is an edge, we call v_{n_j} its initial point, v_j its end point and say that e_j emanates from v_{n_j} and terminates at v_j. For each internal vertex v we denote by $R(v)$ the set of edges emanating from v, and denote by $R^+(v)$ the set of edges incident with v.

Let μ_{kj} be the number of edges between (ignoring partial ordering) the vertices v_j and v_k. Denote $\sigma_k := \max\limits_j \mu_{kj}$. Clearly, $0 \le \mu_{kj} \le \sigma_k$, $\mu_{kk} = 0$, $\mu_{kj} = \mu_{jk}$.

Example 1. Consider the tree T of the fig.1. Then $r = 8$, $p = 5$, $\sigma_8 = 2$, $\sigma_0 = \sigma_1 = \sigma_6 = \sigma_7 = 3$, $\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 4$.

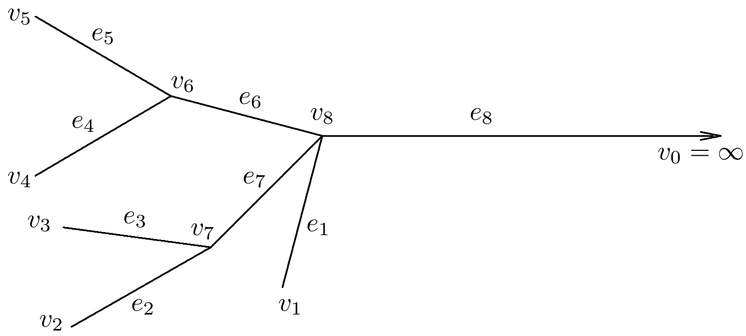

fig.1

Fix v_k. We call μ_{kj} the order of v_j with respect to v_k. Let $V_k^{(\mu)} := \{v_j :\ \mu_{kj} = \mu\}$, be the set of vertices of order μ with respect to v_k.

Fix v_k and e_j. Let ε_{kj} be the maximal order of the vertices of e_j with respect to v_k. The number ε_{kj} is called the order of e_j with respect to v_k. Clearly, $\varepsilon_{kk} = 1$, $1 \le \varepsilon_{kj} \le \sigma_k$. Denote by $\mathcal{E}_k^{(\mu)} := \{e_j :\ \varepsilon_{kj} = \mu\}$, the set of edges of order μ with respect to v_k. Let v_{kj}^+, v_{kj}^- be the vertices of the edge e_j such that the order of v_{kj}^+ with respect to v_k is greater than the order of v_{kj}^-. We call v_{kj}^+ (v_{kj}^-) the distant (near) end of e_j with respect to v_k.

Each compact edge $e_j = [v_{n_j}, v_j] \in \mathcal{E},\ j = \overline{1, r-1}$ is viewed as a segment $[0, 1]$ and is parameterized by the parameter $x_j \in [0, 1]$. It is convenient for us to choose the following orientation: $x_j = 0$ corresponds to the end point v_j, and $x_j = 1$ corresponds to the initial point v_{n_j} of the edge e_j. This means that we identify below the point $\pi_j(x_j) = v_j + x_j(v_{n_j} - v_j) \in e_j \in T$ with parameter value $x_j \in [0,1]$, and for a function $y: T \to \mathbf{C}$ we set $y_j = y \circ \pi_j$ and use the abbreviation $y_j(x_j) = y(\pi_j(x_j))$. The infinite edge $e_r = (v_0, v_r]$ is parameterized similarly by the parameter $x_r \in [0,\infty)$ such that $x_r = 0$ corresponds to the end point v_r.

Hence an integrable function Y on T may be represented as $Y = \{y_j\}_{j\in J}$, where $J := \{j: \ j = \overline{1,r}\}$, and the function $y_j(x_j)$ is defined in the abovementioned sense on the edge e_j. Let $q = \{q_j\}_{j\in J}$ be an integrable real-valued function on T which is called the potential. Consider the Sturm-Liouville equation on T:

$$\ell_j y_j(x_j) := -y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad j \in J, \tag{1}$$

where λ is the spectral parameter, and

$$y_j, y_j' \in AC([0,1]),\ j = \overline{1, r-1};\ y_r, y_r' \in AC_{loc}([0,\infty)). \tag{2}$$

Let the function $Y = \{y_j\}_{j\in J}$ satisfy the following matching conditions in each internal vertex v_k, $k = \overline{p+1, r}$:

$$\left.\begin{aligned} &y_j(1) = y_k(0) \quad \text{for all} \quad e_j \in R(v_k) \quad \text{(continuity condition)},\\ &\sum_{e_j\in R(v_k)} y_j'(1) = y_k'(0) \quad \text{(Kirchhoff's condition)}. \end{aligned}\right\} \tag{3}$$

The matching conditions (3) are called the standard matching conditions. Moreover, we additionally require that the function $Y = \{y_j\}_{j\in J}$ satisfies the following Dirichlet boundary conditions at the boundary vertices:

$$y_j(0) = 0, \quad j = \overline{1,p}. \tag{4}$$

We consider the operator

$$L': \quad D(L') \to L_2(T), \quad Y = \{y_j\}_{j\in J} \to L'Y := \{\ell_j y_j\}_{j\in J},$$

where the domain of definition $D(L')$ consists of functions $Y = \{y_j\}_{j\in J}$ satisfying (2)-(4), and $y_r \in \mathcal{L}_2([0,\infty))$, $\ell_j y_j \in \mathcal{L}_2([0,1])$, $j = \overline{1,r}$. We denote the corresponding boundary value problem (1)-(4) by L.

Let $\lambda = \rho^2$, and let for definiteness $\tau := Im\,\rho \geq 0$. Put $\Omega_0 = \{\rho: \ Im\,\rho > 0\}$, $\Omega = \{\rho: \ Im\,\rho \geq 0,\ \rho \neq 0\}$. Denote by Π the λ-plane with the cut $\lambda \geq 0$, and $\Pi_1 = \overline{\Pi} \setminus \{0\}$; notice that here Π and Π_1 must be considered as subsets of the Riemann surface of the square-root-function. Then, under the map $\rho \to \rho^2 = \lambda$, Π_1 corresponds to the domain Ω.

Let $\Psi_k = \{\psi_{kj}\}_{j\in J}$, $k = \overline{1,p}$, be the solutions of equation (1) satisfying the matching conditions (3) and the boundary conditions

$$\psi_{kj}(0,\lambda) = \delta_{kj}, \quad j = \overline{1,p}, \tag{5}$$

$$\psi_{kr}(x_r,\lambda) = O(\exp(i\rho x_r)),\ x_r \to \infty,\ \rho \in \Omega_0, \tag{6}$$

where δ_{kj} is the Kronecker symbol. The functions Ψ_k are called the Weyl solutions of (1) with respect to the boundary vertex v_k. Denote $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$, where

$M_k(\lambda) := \psi'_{kk}(0, \lambda)$. The functions $M_k(\lambda)$ are called the Weyl functions, and $M(\lambda)$ is called the Weyl vector for equation (1). The inverse problem is formulated as follows:

Inverse Problem 1. Given M, construct the potential q on T.

We mention that the notion of the Weyl vector M is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator ([31], [37]), and Inverse Problem 1 is a generalization of the classical inverse problems for Sturm-Liouville operator from the Weyl function, and (which is equivalent) from the spectral measure (see [31], Ch.1).

3. Properties of the spectrum

Let $T_0 := T \setminus \{e_r\}$ be the compact tree with the edges $e_1, \dots, e_{r-1}$ and with the vertices $v_1, \dots, v_r$. Let $e_{j_1}, \dots, e_{j_s}$ be the edges of T emanating from v_r, i.e. $R(v_r) = \bigcup\limits_{m=1}^{s} e_{j_m}$. Then $T_0 = \bigcup\limits_{m=1}^{s} T_m$, where T_m are the connected compact trees such that $R(v_r) \cap T_m = \{e_{j_m}\}$, and $T_m \cap T_\mu = \{v_r\}$ for $m \neq \mu$. Without loss of generality we assume that $s > 1$.

Let L_0 be the boundary value problem for (1) on the tree T_0 with the standard matching conditions (3) and with the boundary conditions (4). Moreover, denote by $L_{\nu m},\ m = \overline{1, s},\ \nu = 0, 1$, the boundary value problem for (1) on the tree T_m with the standard matching conditions and with the boundary conditions

$$y_j(0) = 0,\ e_j \in E \cap T_m, \quad y_{j_m}^{(\nu)}(1) = 0.$$

Let $C_j(x_j, \lambda),\ S_j(x_j, \lambda),\ j \in J$, be solutions of equation (1) on the edge e_j under the initial conditions $C_j(0, \lambda) = S'_j(0, \lambda) = 1,\ C'_j(0, \lambda) = S_j(0, \lambda) = 0$. For each fixed x_j, the functions $C_j^{(\nu)}(x_j, \lambda)$ and $S_j^{(\nu)}(x_j, \lambda),\ \nu = 0, 1$, are entire in λ of order 1/2. Moreover, $\langle C_j(x_j, \lambda), S_j(x_j, \lambda)\rangle \equiv 1$, where $\langle y, z\rangle := yz' - y'z$ is the Wronskian. Furthermore, let $e(x_r, \rho),\ x_r \geq 0$, be the Jost solution of equation (1) on the edge e_r (see [31, Sec. 2.1]).

Lemma 1. *The function $e(x_r, \rho)$ has the following properties:*

1) *For each fixed $x_r \geq 0$, and $\nu = 0, 1$, the functions $e^{(\nu)}(x_r, \rho)$ are analytic for $\rho \in \Omega_0$, and are continuous for $\rho \in \Omega$.*
2) *For $x_r \to \infty,\ \nu = 0, 1$,*
$$e^{(\nu)}(x_r, \rho) = (i\rho)^\nu \exp(i\rho x_r)(1 + o(1)).$$
For $\rho \in \Omega_0,\ e(x_r, \rho) \in L_2(0, \infty)$. Moreover, $e(x_r, \rho)$ is the unique solution of (1) on e_r (up to a multiplicative constant) having this property.
3) *For $|\rho| \to \infty,\ \rho \in \Omega,\ \nu = 0, 1$,*
$$e^{(\nu)}(x_r, \rho) = (i\rho)^\nu \exp(i\rho x_r)\Big(1 + O(\rho^{-1})\Big),$$
uniformly for $x_r \geq 0$.

4) *For real $\rho \neq 0$, the functions $e(x_r, \rho)$ and $e(x_r, -\rho)$ form a fundamental system of solutions for equation (1) on the edge e_r, and*

$$\langle e(x_r, \rho), e(x_r, -\rho)\rangle = -2i\rho. \tag{7}$$

5) *For real $\rho \neq 0$, $\overline{e^{(\nu)}(x_r, \rho)} = e^{(\nu)}(x_r, -\rho)$.*

The proof of Lemma 1 is given in [31, Sec.2.1].

Consider the Weyl solutions $\Psi_k = \{\psi_{kj}\}_{j\in J}$, $k = \overline{1,p}$, and denote $M^0_{kj}(\lambda) = \psi'_{kj}(0,\lambda)$, $M^1_{kj}(\lambda) = \psi_{kj}(0,\lambda)$. Then

$$\psi_{kj}(x_j, \lambda) = M^1_{kj}(\lambda)C_j(x_j, \lambda) + M^0_{kj}(\lambda)S_j(x_j, \lambda). \tag{8}$$

In particular, $M^0_{kk}(\lambda) = M_k(\lambda)$, $M^1_{kk}(\lambda) = 1$, $M^1_{kj}(\lambda) = 0$ for $k = \overline{1,p}$, $j = \overline{1,p}\setminus k$. Hence

$$\psi_{kk}(x_k, \lambda) = C_k(x_k, \lambda) + M_k(\lambda)S_k(x_k, \lambda). \tag{9}$$

Furthermore, it follows from (6) that

$$\psi_{kr}(x_r, \lambda) = M_{kr}(\lambda)e(x_r, \rho), \tag{10}$$

where $M_{kr}(\lambda)$ does not depend on x_r.

Substituting (8) for $j = \overline{1, r-1}$ and (10) into (3) and (5) we obtain a linear algebraic system s_k with respect to $M^0_{kj}(\lambda), M^1_{kj}(\lambda)$, $j = \overline{1, r-1}$ and $M_{kr}(\lambda)$. The determinant $\Delta(\rho)$ of the system s_k does not depend on k, and has the form

$$\Delta(\rho) = G_0(\lambda)e'(0, \rho) - g_0(\lambda)e(0, \rho), \tag{11}$$

where the functions $g_0(\lambda)$ and $G_0(\lambda)$ are entire in λ of order $1/2$. The function $g_0(\lambda)$ is the characteristic function (see [23]) for the boundary value problem L_0 on the tree T_0, and its zeros coincide with the eigenvalues of L_0. Moreover,

$$G_0(\lambda) = \prod_{m=1}^{s} G_{0m}(\lambda), \quad g_0(\lambda) = G_0(\lambda)\sum_{m=1}^{s} \frac{G_{1m}(\lambda)}{G_{0m}(\lambda)}, \tag{12}$$

where $G_{\nu m}(\lambda)$ are the characteristic functions for the boundary value problems $L_{\nu m}$ on the tree T_m. The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. Solving the system s_k we get by Cramer's rule: $M^\nu_{kj}(\lambda) = \Delta^\nu_{kj}(\rho)/\Delta(\rho)$, $\nu = 0, 1$; $j = \overline{1, r-1}$, where the determinant $\Delta^\nu_{kj}(\rho)$ is obtained from $\Delta(\rho)$ by replacing the column which corresponds to $M^\nu_{kj}(\lambda)$ by the column of free terms. In particular,

$$M_k(\lambda) = -\frac{\Delta_k(\rho)}{\Delta(\rho)}, \quad k = \overline{1,p}, \tag{13}$$

where

$$\Delta_k(\rho) = G_k(\lambda)e'(0, \rho) - g_k(\lambda)e(0, \rho), \quad k = \overline{1,p}. \tag{14}$$

The functions $g_k(\lambda)$ and $G_k(\lambda)$ are obtained from $g_0(\lambda)$ and $G_0(\lambda)$ respectively by replacing $S^{(\xi)}_k(1, \lambda)$, $\xi = 0, 1$, by $C^{(\xi)}_k(1, \lambda)$, $\xi = 0, 1$. The functions $g_k(\lambda)$ and $G_k(\lambda)$ are entire in λ of order $1/2$.

Example 2. *Let $r = p + 1$, i.e. the tree T is a star, and $R^+(v_r) = T$ (see fig.2).*

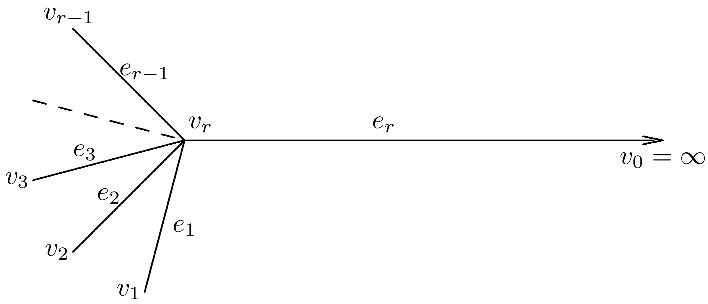

fig.2

Then $s = p$, $T_m = \{e_m\}$, $m = \overline{1, p}$, and

$$G_{\nu m}(\lambda) = S_m^{(\nu)}(1, \lambda), \quad G_0(\lambda) = \prod_{m=1}^{p} S_m(1, \lambda), \quad g_0(\lambda) = G_0(\lambda) \sum_{m=1}^{p} \frac{S'_m(1, \lambda)}{S_m(1, \lambda)}.$$

Example 3. *Consider the tree from Example 1. Then $s = 3$, $j_1 = 1$, $j_2 = 7$, $j_3 = 6$, $T_1 = \{e_1\}$, $T_2 = \{e_1, e_3, e_7\}$, $T_3 = \{e_4, e_5, e_6\}$,*

$$G_{\nu 1}(\lambda) = S_1^{(\nu)}(1, \lambda),$$

$$G_{\nu 2}(\lambda) = S_2(1, \lambda) S'_3(1, \lambda) S_7^{(\nu)}(1, \lambda) + S'_2(1, \lambda) S_3(1, \lambda) S_7^{(\nu)}(1, \lambda) + S_2(1, \lambda) S_3(1, \lambda) C_7^{(\nu)}(1, \lambda),$$

$$G_{\nu 3}(\lambda) = S_4(1, \lambda) S'_5(1, \lambda) S_6^{(\nu)}(1, \lambda) + S'_4(1, \lambda) S_5(1, \lambda) S_6^{(\nu)}(1, \lambda) + S_4(1, \lambda) S_5(1, \lambda) C_6^{(\nu)}(1, \lambda),$$

$\nu = 0, 1$, and $G_0(\lambda)$ and $g_0(\lambda)$ are calculated by (12).

The next assertion follows from (11), (14) and Lemma 1.

Theorem 1. *The functions $\Delta(\rho)$ and $\Delta_k(\rho)$, $k = \overline{1, p}$, are analytic in Ω_0, and continuous in Ω. For real $\rho \neq 0$,*

$$\overline{\Delta(\rho)} = \Delta(-\rho). \tag{15}$$

Denote by $\Lambda := \{\lambda = \rho^2 : \ \rho \in \Omega, \ \Delta(\rho) = 0\}$ the set of zeros of the characteristic function $\Delta(\rho)$ in Ω. Then $\Lambda = \Lambda' \cup \Lambda''$, where

$$\Lambda' := \{\lambda = \rho^2 : \ \rho \in \Omega_0, \ \Delta(\rho) = 0\}, \quad \Lambda'' := \{\lambda = \rho^2 : \ Im\, \rho = 0, \ \rho \neq 0, \ \Delta(\rho) = 0\}.$$

Theorem 2. *The Weyl functions $M_k(\lambda)$, $k = \overline{1, p}$, are analytic in $\Pi \setminus \Lambda'$ and continuous in $\Pi_1 \setminus \Lambda$. The set of singularities of $M(\lambda)$ (as an analytic function) coincides with the set $S := \{\lambda : \ \lambda \geq 0\} \cup \Lambda$.*

Theorem 2 follows from (13) and Theorem 1. Similarly, one gets that the functions $M_{kj}^{\nu}(\lambda)$ and $M_{kr}(\lambda)$ from (8) and (10) are analytic in $\Pi \setminus \Lambda'$ and continuous in $\Pi_1 \setminus \Lambda$. By virtue of (8) and (10), the set of singularities of the Weyl *solutions* $\Psi_k = \{\psi_{kj}\}_{j\in J}$ *coincides with* S *for all* x_j*, since the functions* $C_j(x_j,\lambda)$ *and* $S_j(x_j,\lambda)$ *are entire in* λ *for all* x_j*.*

Definition 1. *The set of singularities of the Weyl vector* $M(\lambda)$ *is called the spectrum of* L*. The value of the parameter* λ *for which (1) has nontrivial solutions satisfying (3)-(4) and* $y_r(\infty)=0$ *(i.e.* $\lim_{x_r\to\infty} y(x_r)=0$*), are called eigenvalues of* L*, and the corresponding solutions are called eigenfunctions.*

Theorem 3. *Let* $\lambda_0=\rho_0^2,\ \rho_0\in\Omega_0$*, i.e.* $\lambda_0\notin[0,\infty)$*. For* λ_0 *to be an eigenvalue of* L*, it is necessary and sufficient that* $\lambda_0\in\Lambda'$*.*

Proof. Let $\lambda_0=\rho_0^2\in\Lambda'$. On the tree T we consider the function $Y=\{y_j\}_{j\in J}$ of the form

$$\left.\begin{aligned} y_j(x_j)=\alpha_j^1 C_j(x_j,\lambda_0)+\alpha_j^0 S_j(x_j,\lambda_0),\quad j=\overline{1,r-1},\\ y_r(x_r)=\alpha_r e(x_r,\rho_0). \end{aligned}\right\} \tag{16}$$

Clearly, Y is a solution of equation (1) for $\lambda=\lambda_0$. Substituting (16) into (3) and (4) we obtain a homogeneous linear algebraic system s_0 with respect to $\alpha_j^1,\alpha_j^0,\ j=\overline{1,r-1}$ and α_r. The determinant of the system s_0 is $\Delta(\rho_0)$. Since $\Delta(\rho_0)=0$, it follows that the system s_0 has a nontrivial solution. This means that $Y=\{y_j\}_{j\in J}$ is an eigenfunction, and λ_0 is an eigenvalue of L.

Conversely, let $\lambda_0=\rho_0^2\notin[0,\infty)$ be an eigenvalue of L, and let $Y=\{y_j\}_{j\in J}$ be a corresponding eigenfunction. Then Y has the form (16), where $\alpha_j^1,\alpha_j^0,\ j=\overline{1,r-1}$ and α_r satisfy system s_0. Since Y is not identically zero, it follows that the system s_0 has a nontrivial solution, and consequently, $\Delta(\rho_0)=0$. □

Since the potential $q=\{q_j\}_{j\in J}$ *is a real-valued integrable function, it is known that the operator* L' *is self-adjoint and bounded from below (see [40]). Together with Theorem 3 this yields that* $\Lambda'\subset(-\infty,0)$ *lies on the negative real half-axis, and* Λ' *is a bounded set of eigenvalues of* L*.*

Denote Λ_0 *the set of common positive zeros of* $g_0(\lambda)$ *and* $G_0(\lambda)$*.*

Theorem 4. $\Lambda''=\Lambda_0$*.*

Proof. Let $\lambda_0\in\Lambda''$. Then $\lambda_0=\rho_0^2>0$ and $\Delta(\rho_0)=0$. It follows from (15) that $\Delta(-\rho_0)=0$. Together with (7) and (11) this yields $g_0(\lambda_0)=G_0(\lambda_0)=0$, i.e. $\lambda_0\in\Lambda_0$.

Conversely, let $\lambda_0\in\Lambda_0$. Then $\lambda_0=\rho_0^2>0$ and $g_0(\lambda_0)=G_0(\lambda_0)=0$. It follows from (11) that $\Delta(\rho_0)=0$, i.e. $\lambda_0\in\Lambda''$. □

Theorem 5. *Let* $\lambda_0=\rho_0^2>0$*. For* λ_0 *to be an eigenvalue of* L*, it is necessary and sufficient that* $\lambda_0\in\Lambda''$*.*

Proof. Let $\lambda_0 = \rho_0^2 > 0$ be an eigenvalue, and let $Y = \{y_j\}_{j\in J}$ be a corresponding eigenfunction. According to (7) the functions $\{e(x_r, \rho_0), e(x_r, -\rho_0)\}$ form a fundamental system of solutions of (1) on e_r, and consequently, $y_r(x_r) = Ae(x_r, \rho_0) + Be(x_r, -\rho_0)$. For $x_r \to \infty$ we have $y_r(x_r) \sim 0$, $e(x_r, \pm\rho_0) \sim \exp(\pm i\rho_0 x_r)$. But this is possible only if $A = B = 0$, i.e. $y_r(x_r) \equiv 0$.

Clearly, $Y^0 := Y \setminus \{y_r\} = \{y_j\}_{j=\overline{1,r-1}}$ is an eigenfunction of the boundary value problem L_0 on the tree T_0, and consequently, $g_0(\lambda_0) = 0$. One has $Y^0 = \bigcup_{m=1}^{s} Y_m^0$, where $Y_m^0 = \{y_j\}_{e_j\in T_m}$ is the restriction of Y^0 to the tree T_m. Since Y^0 is not identically zero, it follows that there exists m such that $Y_m^0 \neq 0$. This means that Y_m^0 is an eigenfunction of L_{0m}, and consequently, $G_{0m}(\lambda_0) = 0$, i.e. $G_0(\lambda_0) = 0$. Thus, $\lambda_0 \in \Lambda_0$. Taking Theorem 4 into account we get $\lambda_0 \in \Lambda''$.

Conversely, let $\lambda_0 \in \Lambda''$. According to Theorem 4, $\lambda_0 \in \Lambda_0$, i.e. $g_0(\lambda_0) = G_0(\lambda_0) = 0$. By virtue of (12), there exist $m_1, \dots, m_l$ such that $G_{0,m_1}(\lambda_0) = \dots = G_{0,m_l}(\lambda_0) = 0$, and $G_{0m}(\lambda_0) \neq 0$ for $m = \overline{1,s}$, $m \neq m_1, \dots, m_l$. This means that λ_0 is a common eigenvalue of the boundary value problems $L_{0,m_1}, \dots, L_{0,m_l}$ and L_0. Let $Y_{m_1}^0, \dots, Y_{m_l}^0$ be eigenfunctions of $L_{0,m_1}^0, \dots, L_{0,m_l}^0$, respectively. Put $y_r \equiv 0$ and $Y_m^0 \equiv 0$ for $m = \overline{1,s}$, $m \neq m_1, \dots, m_l$. Take $Y := \Big(\bigcup_{m=1}^{s} Y_m^0\Big) \cup \{y_r\}$, and choose constants in $Y_{m_j}^0$, $j = \overline{1,l}$, such that Y satisfies Kirchhoff's condition in v_r. Then Y is an eigenfunction of L_0, and λ_0 is an eigenvalue of L_0. $\square$

Thus, the spectrum of L coincides with S, and it consists of the positive half-line $\{\lambda : \ \lambda \geq 0\}$, and the discrete real bounded from below set $\Lambda = \Lambda' \cup \Lambda''$. We note that the set Λ'' of positive eigenvalues can be empty, finite or an infinite unbounded set.

Example 4. *Consider Example 1 with $p = 2$. Then*

$$\Delta(\rho) = S_1(1,\lambda)S_2(1,\lambda)e'(0,\rho) - (S_1(1,\lambda)S_2'(1,\lambda) + S_1'(1,\lambda)S_2(1,\lambda))e(0,\rho),$$

i.e. $G_0(\lambda) = S_1(1,\lambda)S_2(1,\lambda)$, $g_0(\lambda) = S_1(1,\lambda)S_2'(1,\lambda) + S_1'(1,\lambda)S_2(1,\lambda)$. In this case Λ_0 is the set of the common positive eigenvalues of the two scalar problems

$$-y_j'' + q_j(x_j)y_j = \lambda y_j, \ x_j \in (0,1), \ y_j(0) = y_j(1) = 0, \ j = 1,2. \tag{17}$$

It follows from the theory of inverse spectral problems (see, for example, [31, Ch.1]) that for arbitrary sequences of real numbers $\{\lambda_{nj}\}_{n\geq 1}$, $j = 1,2$, of the form

$$\lambda_{nj} = \pi^2 n^2 + c_j + \kappa_{nj}, \quad \{\kappa_{nj}\} \in l_2, \ c_j \in \mathbf{R}, \ j = 1,2,$$

there exist real potentials $q_j \in L_2(0,1)$ for which $\{\lambda_{nj}\}_{n\geq 1}$, $j = 1,2$, are the sequences of eigenvalues of the boundary value problems (17). This means that we can choose q_1 and q_2 such that the set Λ'' will be either empty, finite or an infinite unbounded set. For example, if $q_1 = q_2 = 0$, one has $S_1(1,\lambda) = S_2(1,\lambda) =$

$\dfrac{\sin\rho}{\rho}$, *and consequently,* $\Lambda'' = \{\pi^2 n^2\}_{n\ge 1}$ *is the sequence of eigenvalues, and* $Y_n = \{\sin\pi n x_1, -\sin\pi n x_2, 0\}$ *are the eigenfunctions of* L *corresponding to* $\pi^2 n^2$.

4. An inverse problem for boundary edges

Fix $k = 1,\dots,p$, *and consider the following auxiliary inverse problem on the boundary edge* e_k, *which is called Problem IP(k).*

Problem IP(k). *Given* $M_k(\lambda)$, *construct* q_k.

In order to solve the problem IP(k) we need auxiliary propositions.

It is known (see [40]) that for each fixed $j \in J$ *on the edge* e_j, *there exists a fundamental system of solutions of equation (1)* $\{e_{j1}(x_j,\rho), e_{j2}(x_j,\rho)\}$, $\rho \in \Omega$, $|\rho| \ge \rho^*$ *such that the functions* $e_{j1}^{(\nu)}(x_j,\rho), e_{j2}^{(\nu)}(x_j,\rho)$, $\nu = 0,1$, *are analytic for* $\rho \in \Omega_0$, $|\rho| \ge \rho^*$, *are continuous for* $\rho \in \Omega$, $|\rho| \ge \rho^*$, *and uniformly in* x_j, *the following asymptotical formulae hold*

$$e_{j1}^{(\nu)}(x_j,\rho) = (i\rho)^\nu \exp(i\rho x_j)[1],\ e_{j2}^{(\nu)}(x_j,\rho) = (-i\rho)^\nu \exp(-i\rho x_j)[1],\ \rho\in\Omega,\ |\rho|\to\infty, \tag{18}$$

where $[1] = 1 + O(\rho^{-1}),\ \nu = 0,1.$

Denote $\Omega_\delta := \{\rho :\ \arg\rho \in [\delta, \pi-\delta]\},\ \delta > 0.$

Lemma 2. *Let* $y_j(x_j,\rho),\ j = \overline{1, r-1}$ *be a solution of equation (1) on the edge* e_j, *and let*

$$\frac{y_j'(0,\rho)}{y_j(0,\rho)} = (-i\rho) r_j[1], \quad r_j \ne -1,\ \rho\in\Omega_\delta,\ |\rho|\to\infty. \tag{19}$$

Then for $\nu = 0,1,\ \rho\in\Omega_\delta,\ |\rho|\to\infty$, *uniformly in* $x_j \in [0,1]$,

$$y_j^{(\nu)}(x_j,\rho) = D_j(\rho)\Big((-i\rho)^\nu \exp(-i\rho x_j)[1] - (r_j+1)^{-1}(r_j-1)(i\rho)^\nu \exp(i\rho x_j)[1]\Big), \tag{20}$$

where $D_j(\rho)$ *does not depend on* x.

Proof. Using the fundamental system of solutions $\{e_{j1}(x_j,\rho), e_{j2}(x_j,\rho)\}$, one gets

$$y_j(x_j,\rho) = A_j(\rho) e_{j1}(x_j,\rho) + D_j(\rho) e_{j2}(x_j,\rho). \tag{21}$$

It follows from (18) and (21) that

$$\frac{y_j'(0,\rho)}{y_j(0,\rho)} = (i\rho)\,\frac{A_j(\rho)[1] - D_j(\rho)[1]}{A_j(\rho)[1] + D_j(\rho)[1]}, \quad \rho\in\Omega_\delta,\ |\rho|\to\infty.$$

Taking (19) into account we calculate $A_j(\rho) = D_j(\rho)(r_j+1)^{-1}(r_j-1)[1]$. Substituting this relation into (21) and using (18), we arrive at (20). □

The following lemma is proved by the same way.

Lemma 3. *Let $y_j(x_j,\rho),\ j=\overline{1,r-1}$ be a solution of equation (1) on the edge e_j, and let*

$$\frac{y_j'(1,\rho)}{y_j(1,\rho)}=(i\rho)r_j[1],\quad r_j\ne -1,\ \rho\in\Omega_\delta,\ |\rho|\to\infty.$$

Then for $\nu=0,1,\ \rho\in\Omega_\delta,\ |\rho|\to\infty$, uniformly in $x_j\in[0,1]$,

$$y_j^{(\nu)}(x_j,\rho)=D_j(\rho)\Big((i\rho)^\nu\exp(-i\rho(1-x_j))[1]-(r_j+1)^{-1}(r_j-1)(-i\rho)^\nu\exp(i\rho(1-x_j))[1]\Big),$$

where $D_j(\rho)$ does not depend on x.

Lemma 4. *Fix $k=\overline{1,p}$ and consider the Weyl solution $\Psi_k=\{\psi_{kj}\}_{j=\overline{1,r}}$. Let r_{kj} be the number of edges incident with v_{kj}^+, and let ε_{kj} be the order of e_j with respect to v_k. Denote*

$$d_{kj}=\begin{cases}0 & \text{for}\quad j=r,\\ 1 & \text{for}\quad j=\overline{1,p}\setminus k,\\ (r_{kj}-2)/r_{kj} & \text{for}\quad j=k,\overline{p+1,r-1}.\end{cases}$$

Then for $\nu=0,1,\ \rho\in\Omega_\delta,\ |\rho|\to\infty$, uniformly in x_j,

$$\psi_{kj}^{(\nu)}(x_j,\lambda)=B_{kj}(\rho)\Big((-i\rho)^\nu\exp(-i\rho x_j)[1]-d_{kj}(i\rho)^\nu\exp(i\rho x_j)[1]\Big),\tag{22}$$

if $v_j=v_{kj}^+,\ j=\overline{1,r-1}$, and

$$\psi_{kj}^{(\nu)}(x_j,\lambda)=B_{kj}(\rho)\Big((i\rho)^\nu\exp(-i\rho(1-x_j))[1]-d_{kj}(-i\rho)^\nu\exp(i\rho(1-x_j))[1]\Big),\tag{23}$$

if $v_j=v_{kj}^-,\ j=\overline{1,r-1}$ or $j=r$.

Moreover, for $j=\overline{1,r},\ \rho\in\Omega_\delta,\ |\rho|\to\infty$,

$$B_{kj}(\rho)=b_{kj}\exp(i\rho\varepsilon_{kj})[1],\quad b_{kj}\ne 0,\ b_{kk}=1.\tag{24}$$

Proof. 1) Let $j=\overline{1,p}\setminus k$. Then $v_j=v_{kj}^+$, $\psi_{kj}(0,\lambda)=0$, and in view of (8),

$$\psi_{kj}(x_j,\lambda)=M_{kj}^0(\lambda)S_j(x_j,\lambda).\tag{25}$$

Using (25) and the asymptotics

$$S_j^{(\nu)}(x_j,\lambda)=\frac{1}{2i\rho}\Big((i\rho)^\nu\exp(i\rho x_j)[1]+(-i\rho)^\nu\exp(-i\rho x_j)[1]\Big),\quad |\rho|\to\infty,\tag{26}$$

we arrive at (22) for $j=\overline{1,p}\setminus k$ with a certain $B_{kj}(\rho)$ which does not depend on x_j. It follows from (10) that (23) is valid for $j=r$ where $B_{kr}(\rho)$ does not depend on x_r.

2) We partition all edges $\{e_j\}_{j=\overline{1,r}}$ into the classes $\mathcal{E}_k^{(\mu)},\ \mu=1,\dots\sigma_k$. Let us prove (22)-(23) for all other edges by induction with respect to $\mu=\sigma_k,\sigma_k-1,\dots,1$. If $\mu=\sigma_k$ (i.e. $e_j\in\mathcal{E}_k^{(\sigma_k)}$), then (22)-(23) holds according to the previous arguments.

Fix $\mu < \sigma_k$. Suppose that (22)-(23) have been proved for all $e_j \in \mathcal{E}_k^{(\mu+1)} \cup \ldots \cup \mathcal{E}_k^{(\sigma_k)}$. Let now $e_j \in \mathcal{E}_k^{(\mu)}$. Using the induction assumption and the matching conditions (3) in the vertex v_{kj}^+, we get for $\rho \in \Omega_\delta,\ |\rho| \to \infty$,

$$\frac{\psi'_{kj}(0,\lambda)}{\psi_{kj}(0,\lambda)} = -(i\rho)(r_{kj}-1)[1] \quad \text{if} \quad v_j = v_{kj}^+ \,,$$

$$\frac{\psi'_{kj}(1,\lambda)}{\psi_{kj}(1,\lambda)} = (i\rho)(r_{kj}-1)[1] \quad \text{if} \quad v_j = v_{kj}^- \,.$$

Applying Lemmas 2 and 3 we arrive at (22) and (23) respectively with a certain coefficients $B_{kj}(\rho)$. Thus, (22)-(23) are proved for all $e_j,\ j = \overline{1,r}$. In particular, for $\rho \in \Omega_\delta,\ |\rho| \to \infty$, one has

$$\psi_{kk}^{(\nu)}(x_k,\lambda) = B_{kk}(\rho)\Big((i\rho)^\nu \exp(-i\rho(1-x_k))[1] - d_{kk}(-i\rho)^\nu \exp(i\rho(1-x_k))[1]\Big). \tag{27}$$

Since $\psi_{kk}(0,\lambda) = 1$, it follows from (27) that

$$B_{kk}(\rho) = \exp(i\rho)[1], \quad \rho \in \Omega_\delta,\ |\rho| \to \infty, \tag{28}$$

i.e. (24) is valid for $\mathcal{E}_k^{(1)} = \{e_k\}$.

Fix $\mu > 1$. Suppose that (24) has been proved for all $e_j \in \mathcal{E}_k^{(1)} \cup \ldots \cup \mathcal{E}_k^{(\mu-1)}$. Let now $e_j \in \mathcal{E}_k^{(\mu)}$. Using the induction assumption and the matching conditions (3) in the vertex v_{kj}^- we arrive at (24) for $e_j \in \mathcal{E}_k^{(\mu)}$. Thus, (24) is proved for all $e_j,\ j = \overline{1,r}$. □

Corollary 1. *For $k = \overline{1,p},\ \nu = 0,1,\ x_k \in [0,1),\ \rho \in \Omega_\delta,\ |\rho| \to \infty$,*

$$\psi_{kk}^{(\nu)}(x_k,\lambda) = (i\rho)^\nu \exp(i\rho x_k)[1], \quad M_k(\lambda) = (i\rho)[1]. \tag{29}$$

Indeed, (29) follows from (27), (28) and $M_k(\lambda) = \psi'_{kk}(0,\lambda)$.

Let $\delta > 0$ be sufficiently small and fixed, and let $\Lambda = \{\lambda_l\},\ \lambda_l = \rho_l^2$. Denote $G_\delta := \{\lambda = \rho^2 :\ Im\,\rho \ge 0,\ |\rho| \ge \delta,\ |\rho - \rho_l| \ge \delta, \forall l\}$. Using the standard technique (see [40]) one can show that

$$|\psi_{kk}^{(\nu)}(x_k,\lambda)| \le C|\rho^\nu \exp(i\rho x_k)|,\ |M_k(\lambda)| \le C|\rho|, \quad \lambda \in G_\delta,\ k = \overline{1,p},\ x_k \in [0,1]. \tag{30}$$

Let us prove the uniqueness of the solution of inverse problem IP(k). For this purpose together with L we consider a boundary value problem $\tilde{L}$ of the same form but with different potential $\tilde{q}$. Everywhere below if a symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$.

Theorem 6. *Fix $k = \overline{1,p}$. If $M_k(\lambda) = \tilde{M}_k(\lambda)$, then $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on [0,1]. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k.*

Proof. Consider the functions

$$P_{ks}(x_k,\lambda) = (-1)^{s-1}\Big(\psi_{kk}(x_k,\lambda)\tilde S_k^{(2-s)}(x_k,\lambda) - \tilde\psi_{kk}^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big). \quad (31)$$

Since $\langle C_k(x_k,\lambda), S_k(x_k,\lambda)\rangle \equiv 1$, it follows from (9) that

$$\langle \psi_{kk}(x_k,\lambda), S_k(x_k,\lambda)\rangle \equiv 1. \quad (32)$$

Using (26), (29), (30) and (31) we infer

$$P_{k1}(x_k,\lambda) = 1 + O(\rho^{-1}), \quad P_{k2}(x_k,\lambda) = O(\rho^{-1}), \quad \rho\in\Omega_\delta,\ |\rho|\to\infty,\ x_k\in(0,1], \quad (33)$$

$$|P_{ks}(x_k,\lambda)| \le C|\rho^{1-s}|, \quad \lambda\in G_\delta,\ x_k\in[0,1]. \quad (34)$$

Taking (31) and (32) into account we calculate

$$\left.\begin{aligned} \psi_{kk}(x_k,\lambda) &= P_{k1}(x_k,\lambda)\tilde\psi_{kk}(x_k,\lambda) + P_{k2}(x_k,\lambda)\tilde\psi'_{kk}(x_k,\lambda),\\ S_k(x_k,\lambda) &= P_{k1}(x_k,\lambda)\tilde S_k(x_k,\lambda) + P_{k2}(x_k,\lambda)\tilde S'_k(x_k,\lambda). \end{aligned}\right\} \quad (35)$$

Substituting (9) into (31) and using the assumption of the theorem we conclude that for each fixed x_k, the functions $P_{ks}(x_k,\lambda)$ are entire in λ. Together with (33) and (34) this yields $P_{k1}(x_k,\lambda)\equiv 1$, $P_{k2}(x_k,\lambda)\equiv 0$. Substituting these relations into (35) we get $\psi_{kk}(x_k,\lambda)\equiv\tilde\psi_{kk}(x_k,\lambda)$ and $S_k(x_k,\lambda)\equiv\tilde S_k(x_k,\lambda)$ for all x and λ, and consequently, $q_k(x_k) = \tilde q_k(x_k)$ a.e. on $[0,1]$. □

Using the method of spectral mappings [31] for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for the solution of the inverse problem IP(k). Here we only explain ideas briefly; for details and proofs see [31]. Take the tree $\tilde T$ with the zero potential $\tilde q = 0$. Then $\tilde S_k(x,\lambda) = \frac{\sin\rho x}{\rho}$. Fix $k=\overline{1,p}$. Denote $\lambda' = \min(\inf\Lambda,\ \inf\tilde\Lambda)$ and take a fixed $\delta>0$. In the λ- plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma=\gamma^+\cup\gamma^-\cup\gamma'$, where $\gamma^\pm = \{\lambda:\ \pm Im\,\lambda=\delta;\ Re\,\lambda\ge\lambda'\}$, $\gamma'=\{\lambda:\ \lambda-\lambda'=\delta\exp(i\alpha),\ \alpha\in(\pi/2,3\pi/2)\}$. For each fixed $x_k\in[0,1]$, the function $S_k(x_k,\lambda)$ is the unique solution of the following linear integral equation

$$S_k(x_k,\lambda) = \tilde S_k(x_k,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde D_k(x_k,\lambda,\mu)S_k(x_k,\mu)\,d\mu, \quad (36)$$

where $\tilde D_k(x_k,\lambda,\mu) = \displaystyle\int_0^{x_k}\tilde S_k(t,\lambda)\tilde S_k(t,\mu)\hat M_k(\mu)\,dt,\ \hat M_k(\mu) := M_k(\mu) - \tilde M_k(\mu)$. *The potential q_k on the edge e_k can be constructed from the solution of the integral equation (36) via the formula*

$$q_k(x_k) = \frac{1}{2\pi i}\int_\gamma (S_k(x_k,\lambda)\tilde S_k(x_k,\lambda))'\hat M_k(\lambda)\,d\lambda$$

or by the formula $q_k(x_k) = \lambda + S''_k(x_k,\lambda)/S_k(x_k,\lambda)$.

5. Solution of the inverse problem on the tree

In this section we provide a constructive solution of Inverse Problem 1 of recovering the potential q *on the tree* T *from the given Weyl vector* M*. First we need auxiliary propositions.*

Let us introduce the Weyl solutions for internal vertices. Fix $v_k \in V$*. Denote* $T_k^0 := \{z \in T : \ v_k < z\}$, $T_k^1 := T \setminus T_k^0$*. Clearly,* T_k^1 *is noncompact, and* $e_r \in T_k^1$*. Let* Γ_k *be the set of finite boundary vertices of* T_k^1*, and let* E_k *be the set of compact boundary edges of* T_k^1*. Denote* $J_k := \{j : \ e_j \in T_k^1\}$*. If* $Y = \{y_j\}_{j \in J}$ *is a function on* T*, then* $\{Y\}_k := \{y_j\}_{j \in J_k}$ *is a function on* T_k^1*.*

Fix $v_k \notin \Gamma$ *(i.e.* $k = \overline{p+1, r}$*). Let* $\Psi_k(x,\lambda) = [\psi_{kj}(x,\lambda)]_{j \in J_k}$ *be the solution of equation (1) on* T_k^1 *satisfying (3), (6) and the boundary conditions* $\Psi_{k|v_j} = \delta_{kj}, \quad v_j \in \Gamma_k$*. The function* Ψ_k *is the Weyl solution of (1) on* T_k^1 *with respect to the vertex* v_k*. Denote by* $M_k(\lambda) := \psi'_{kk}(0,\lambda)$, $k = \overline{p+1, r}$ *the Weyl functions for* T_k^1 *with respect to* v_k*.*

Lemma 5. *Fix* $v_m \notin \Gamma$*. Let* $e_k = [v_m, v_k] \in R(v_m)$*. Then*

$$\psi_{mj}(x_j,\lambda) = (\psi_{kk}(1,\lambda))^{-1}\psi_{kj}(x_j,\lambda), \ j \in J_m, \tag{37}$$

$$M_m(\lambda) = (\psi_{kk}(1,\lambda))^{-1} \sum_{e_j \in R(v_m)} \psi'_{kj}(1,\lambda), \tag{38}$$

and Ψ_m, M_m *does not depend on* k*.*

Proof. Denote

$$z_{kj}(x_j,\lambda) := (\psi_{kk}(1,\lambda))^{-1}\psi_{kj}(x_j,\lambda), \ j \in J_m.$$

First we check the behavior of these functions at infinity. It follows from (6) that

$$z_{kr}(x_r,\lambda) = O(\exp(i\rho x_r)), \quad x_r \to \infty, \quad \rho \in \Omega_0. \tag{39}$$

Using the matching conditions (3) we get

$$z_{km}(0,\lambda) = 1. \tag{40}$$

Taking the boundary conditions into account we calculate

$$z_{kj}(0,\lambda) = 0, \quad v_j \in \Gamma_m, \ j \neq m.$$

Together with (39) and (40) this yields $z_{kj}(x_j,\lambda) \equiv \psi_{mj}(x_j,\lambda)$, i.e. (37) holds. We note that $\psi_{mj}(x_j,\lambda)$ does not depend on the choice of k. Furthermore,

$$M_m(\lambda) = \psi'_{mm}(0,\lambda) = \frac{1}{\psi_{kk}(1,\lambda)}\psi'_{km}(0,\lambda).$$

Using the matching conditions (3) again we arrive at (38). □

Denote $M^0_{kj}(\lambda) = \psi'_{kj}(0,\lambda)$, $M^1_{kj}(\lambda) = \psi_{kj}(0,\lambda)$, $k = \overline{p+1,r}$, $j \in J_k$. *Then (8) and (9) are valid for* $k = \overline{1,r}$, $j \in J_k$, *where* $J_k = J$ *for* $k = \overline{1,p}$. *In particular, this yields*

$$\psi^{(\nu)}_{kj}(1,\lambda) = M^1_{kj}(\lambda)C^{(\nu)}_j(1,\lambda) + M^0_{kj}(\lambda)S^{(\nu)}_j(1,\lambda), \quad \nu = 0,1,\ k = \overline{1,r},\ j \in J_k, \tag{41}$$

$$\psi^{(\nu)}_{kk}(1,\lambda) = C^{(\nu)}_k(1,\lambda) + M_k(\lambda)S^{(\nu)}_k(1,\lambda), \quad \nu = 0,1,\ k = \overline{1,r}. \tag{42}$$

Solution of Inverse Problem 1. *Let the Weyl vector* $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$ *for the tree* T *be given. The procedure for the solution of Inverse Problem 1 consists in the realization of the so-called* A_μ*- procedures successively for* $\mu = \sigma, \sigma-1, \dots, 1$, *where* $\sigma := \sigma_0$ *is the height of the tree* T. *These procedures are formally similar to those from [23] but here the Weyl functions have a more complicated structure of their sets of singularities. We use here the method of pseudo-pruning the tree in each internal vertex; this method was suggested in [23] for compact trees, but it works also for noncompact case.*

Let us describe the A_μ*- procedures by induction. Fix* $\mu = \overline{1,\sigma}$, *and suppose that* $A_\sigma, \dots, A_{\mu+1}$*- procedures have been already carried out. Let us carry out* A_μ*-procedure.*

$\mathbf{A}_\mu$**- procedure.** *For each* $v_k \in V^{(\mu)}_0$, *the Weyl functions* $M_k(\lambda)$ *are given. Indeed, if* $v_k \in V^{(\mu)}_0 \cap \Gamma$, *then* $M_k(\lambda)$ *are given a priori, and if* $v_k \in V^{(\mu)}_0 \setminus \Gamma$, *then* $M_k(\lambda)$ *were calculated on the previous steps according to* $A_\sigma, \dots, A_{\mu+1}$*- procedures.*

1) For each edge $e_k \in \mathcal{E}^{(\mu)}_0$, *we solve the inverse problem IP(k) and find* $q_k(x_k)$, $x_k \in [0,1]$ *on the edge* e_k. *If* $\mu = 1$, *then Inverse Problem 1 is solved, and we stop our calculations. If* $\mu > 1$, *we go on to the next step.*

2) For each $e_k \in \mathcal{E}^{(\mu)}_0$, *we construct* $C_k(x_k,\lambda)$, $S_k(x_k,\lambda)$, $x_k \in [0,1]$, *and calculate* $\psi^{(\nu)}_{kk}(1,\lambda)$, $\nu = 0,1$, *by (42).*

3) Returning procedure. For each fixed $v_m \in V^{(\mu-1)}_0 \setminus \Gamma$ *and for any fixed* $e_k, e_i \in R(v_m)$, $i \neq k$, *we consider the compact rooted tree* $T^2_i := T^0_i \cup \{e_i\}$ *with the root* v_m. *Solving the linear algebraic problem* $Z(T^2_i, v_m, \psi_{kk}(1,\lambda))$ *(see [23]), we calculate the transition matrix* $[M^0_{kj}(\lambda), M^1_{kj}(\lambda)]$ *for* $e_j \in T^2_i$.

4) For each fixed $v_m \in V^{(\mu-1)}_0 \setminus \Gamma$ *we calculate the Weyl function* $M_m(\lambda)$ *by (38), where* $\psi'_{kj}(1,\lambda)$ *are constructed via (41) for* $\nu = 1$.

Thus, executing successively $A_\sigma, A_{\sigma-1}, \dots, A_1$*- procedures we obtain the solution of Inverse Problem 1 and prove its uniqueness, i.e. the following assertion holds.*

Theorem 7. *The specification of the Weyl vector* M *uniquely determines the potential* q *on* T. *Solution of Inverse Problem 1 can be obtained by executing successively* $A_\sigma, A_{\sigma-1}, \dots, A_1$*-procedures.*

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Gerhard Freiling
Fachbereich Mathematik Universität Duisburg-Essen D 47057 Duisburg Germany
e-mail: `gerhard.freiling@uni-due.de`

Vjacheslav Yurko
Dep. of Mathematics Saratov State University Astrakhanskaya 83 Saratov 410026 Russia
e-mail: `yurkova@info.sgu.ru`