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An inverse problem for the non-selfadjoint matrix Sturm–Liouville equation on the half-line

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Abstract. An inverse spectral problem is studied for the non-selfadjoint matrix Sturm–Liouville differential equation on the half-line. We give a formulation of the inverse problem, prove the corresponding uniqueness theorem and provide a constructive procedure for the solution of the inverse problem by the method of spectral mappings. The obtained results are natural generalizations of the classical results in inverse problem theory for scalar Sturm-Liouville operators.

Key words. Matrix Sturm-Liouville operators, inverse spectral problems, Weyl matrix, method of spectral mappings.

AMS classification. 34A55 34B24 34L40 47E05.

1. Introduction

Consider the matrix Sturm–Liouville equation

$$\ell Y := -Y'' + Q(x)Y = \lambda Y, \qquad x > 0, \tag{1.1}$$

with the boundary condition

$$U(Y) := HY'(0) - hY(0) = 0. \tag{1.2}$$

Here $Y = [y_k]_{k=\overline{1,m}}$ is a column-vector, λ is the spectral parameter, $h = [h_{k\nu}]_{k,\nu=\overline{1,m}}$, $H = [H_{k\nu}]_{k,\nu=\overline{1,m}}$, and $Q(x) = [Q_{k\nu}(x)]_{k,\nu=\overline{1,m}}$ are matrices, where $Q_{k\nu}(x)$ are integrable complex-valued functions, $H_{k\nu}$, $h_{k\nu}$ are complex numbers, and rank $[H, h] = m$. The matrix $Q(x)$ is called the potential. We study the inverse problem of spectral analysis for the non-selfadjoint matrix Sturm-Liouville boundary value problem of the form (1.1), (1.2). Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems often appear in many branches of natural sciences and engineering.

The scalar case ($m = 1$) has been studied fairly completely (see the monographs [9, 12–14] and the references therein). The investigation of the matrix case is more difficult, and nowadays there are only isolated fragments, not constituting a general picture, in the inverse problem theory for matrix Sturm–Liouville differential operators. Some aspects of the inverse problem theory for the matrix case were studied in [1, 2, 4–7, 10, 11, 15, 18, 23] and other works, but only particular questions and mostly for the selfadjoint case are considered there. We note that inverse problems for various classes of first order differential systems were studied in [3, 16, 17, 19, 22, 24] and other works.

In this paper we investigate an inverse problem for matrix Sturm–Liouville operators on the half-line which is a natural generalization of the well-known inverse problem for the classical scalar Sturm-Liouville operators. Note that we consider not only selfadjoint case but also the non-selfadjoint one. As a main spectral characteristic we use the so-called Weyl matrix, which is a generalization of the Weyl function (m-function) for the scalar case (see [9, 13]). The Weyl functions and their generalizations often appear in applications and in pure mathematical problems, and they are the most natural spectral characteristics in the inverse problem theory for various classes of differential operators. In this paper for studying the inverse problem for matrix Sturm–Liouville operators we develop the ideas of the method of spectral mappings [21]. In Section 2 the properties of the Weyl matrix are studied. In Section 3 it is proved that the specification of the Weyl matrix uniquely determines the matrix Sturm–Liouville operator. In Section 4 we provide a constructive procedure for the solution of the inverse problem from the Weyl matrix. The central role there is played by the so-called main equation which is a linear integral equation in a corresponding Banach space. We give a derivation of the main equation and prove its unique solvability. Using the solution of the main equation we construct the solution of the inverse problem. We note that the obtained results are nontrivial generalizations of the classical results in the inverse problem theory for scalar Sturm–Liouville operators.

2. The Weyl matrix

Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$, and let for definiteness $\tau := \text{Im}\ \rho \geq 0$. Denote by Π the λ-plane with the cut $\lambda \geq 0$, and $\Pi_1 = \overline{\Pi} \setminus \{0\}$; notice that here Π and Π_1 must be considered as subsets of the Riemann surface of the square-root-function. Then, under the map $\rho \to \rho^2 = \lambda$, Π_1 corresponds to the domain $\Omega = \{\rho\colon \text{Im}\ \rho \geq 0,\ \rho \neq 0\}$. Put $\Omega_\delta = \{\rho\colon \text{Im}\ \rho \geq 0,\ |\rho| \geq \delta\}$.

First we construct a special fundamental system of solutions for equation (1.1) in Ω having asymptotic behavior at infinity like $\exp(\pm i\rho x)$. The following assertions are proved analogously to the scalar case (see [9, Ch. 2]).

Lemma 2.1. *Equation* (1.1) *has a unique matrix solution* $e(x,\rho) = [e_{k\nu}(x,\rho)]_{k,\nu=\overline{1,m}}$, $\rho \in \Omega$, $x \geq 0$, *satisfying the integral equation*

$$e(x,\rho) = \exp(i\rho x)I_m - \frac{1}{2i\rho}\int_x^\infty \big(\exp(i\rho(x-t)) - \exp(i\rho(t-x))\big)Q(t)e(t,\rho)\,dt,$$

where $I_m = [\delta_{k\nu}]_{k,\nu=\overline{1,m}}$ *is the identity matrix, and* $\delta_{k\nu}$ *is the Kronecker symbol. The matrix-function* $e(x,\rho)$ *has the following properties:*

(i_1) *For* $x \to \infty$, $\nu = 0, 1$, *and each fixed* $\delta > 0$,

$$e^{(\nu)}(x,\rho) = (i\rho)^\nu \exp(i\rho x)\big(I_m + o(1)\big), \tag{2.1}$$

uniformly in Ω_δ.

(i_2) *For* $|\rho|\to\infty$, $\rho\in\mathbf{\Omega}$, $\nu=0,1$,

$$e^{(\nu)}(x,\rho)=(i\rho)^{\nu}\exp{(i\rho x)}\Big(I_m-\frac{1}{2i\rho}\int_x^{\infty}Q(t)\,dt+o\Big(\frac{1}{\rho}\Big)\Big), \tag{2.2}$$

uniformly for $x\ge 0$.

(i_3) *For each fixed* $x\ge 0$, *and* $\nu=0,1$, *the matrix-functions* $e^{(\nu)}(x,\rho)$ *are analytic for* $\mathrm{Im}\ \rho>0$, *and are continuous for* $\rho\in\mathbf{\Omega}$.

The matrix-function $e(x,\rho)$ *is called the Jost solution for* (1.1).

Lemma 2.2. *For each* $\delta>0$, *there exists* $a=a_\delta\ge 0$ *such that equation (1) has a unique matrix solution* $E(x,\rho)=[E_{k\nu}(x,\rho)]_{k,\nu=\overline{1,m}}$, $\rho\in\mathbf{\Omega}_\delta$, *satisfying the integral equation*

$$E(x,\rho)=\exp{(-i\rho x)}I_m+\frac{1}{2i\rho}\int_a^x\exp{(i\rho(x-t))}Q(t)E(t,\rho)\,dt$$
$$+\frac{1}{2i\rho}\int_x^{\infty}\exp{(i\rho(t-x))}Q(t)E(t,\rho)\,dt.$$

The matrix-function $E(x,\rho)$, *called the Birkhoff solution for* (1.1), *has the following properties:*

(i_1) $E^{(\nu)}(x,\rho)=(-i\rho)^{\nu}\exp{(-i\rho x)}(I_m+o(1))$, $x\to\infty$, $\nu=0,1$, *uniformly for* $|\rho|\ge\delta$, $\mathrm{Im}\ \rho\ge\alpha$, *for each fixed* $\alpha>0$*;*

(i_2) $E^{(\nu)}(x,\rho)=(-i\rho)^{\nu}\exp{(-i\rho x)}(I_m+O(\rho^{-1}))$, $|\rho|\to\infty$, $\rho\in\mathbf{\Omega}$, *uniformly for* $x\ge a$*;*

(i_3) *for each fixed* $x\ge 0$, *the matrix-functions* $E^{(\nu)}(x,\rho)$ *are analytic for* $\mathrm{Im}\ \rho>0$, $|\rho|\ge\delta$, *and are continuous for* $\rho\in\mathbf{\Omega}_\delta$*;*

(i_4) *the matrix-functions* $e(x,\rho)$ *and* $E(x,\rho)$ *form a fundamental system of solutions for equation* (1.1)*;*

(i_5) *if* $\delta\ge\int_0^{\infty}Q(t)\,dt$, *then one can take above* $a=0$.

Everywhere below let for definiteness $H=I_m$, i. e. (1.2) takes the form

$$U(Y):=Y'(0)-hY(0)=0. \tag{2.3}$$

Other cases can be treated similarly. Denote by $L=L(Q,h)$ the boundary value problem of the form (1.1), (2.3). Let $\varphi(x,\lambda)=[\varphi_{jk}(x,\lambda)]_{j,k=\overline{1,m}}$ and $S(x,\lambda)=[S_{jk}(x,\lambda)]_{j,k=\overline{1,m}}$ be the solutions of (1.1) under the initial conditions $\varphi(0,\lambda)=I_m$, $\varphi'(0,\lambda)=h$, $S(0,\lambda)=0$, $S'(0,\lambda)=I_m$, where 0 is the zero $m\times m$ matrix. Clearly, $U(\varphi)=0$. For each fixed x, the matrix-functions $\varphi^{(\nu)}(x,\lambda)$ and $S^{(\nu)}(x,\lambda)$, $\nu=0,1$, are entire in λ. The matrix-functions $\varphi(x,\lambda)$ and $S(x,\lambda)$ form a fundamental system of solutions for equation (1.1), and $\det[\varphi^{(\nu)}(x,\lambda),S^{(\nu)}(x,\lambda)]_{\nu=0,1}\equiv 1$.

Denote

$$u(\rho):=U(e(x,\rho))=e'(0,\rho)-he(0,\rho),\qquad \Delta(\rho):=\det u(\rho).$$

By virtue of Lemma 2.2, the functions $u(\rho)$ and $\Delta(\rho)$ are analytic for Im $\rho > 0$, and continuous for $\rho \in \Omega$. It follows from (2.2) that for $|\rho| \to \infty$, $\rho \in \Omega$,

$$e(0,\rho) = I_m - \frac{1}{2i\rho}\int_0^\infty Q(t)\,dt + o\Big(\frac{1}{\rho}\Big),$$

$$u(\rho) = (i\rho)\Big(I_m - \frac{1}{2i\rho}\int_0^\infty Q(t)\,dt - \frac{h}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big), \tag{2.4}$$

$$\Delta(\rho) = (i\rho)^m\big(1 + O(\rho^{-1})\big).$$

Denote

$$\Lambda = \{\lambda = \rho^2\colon \rho \in \Omega,\ \Delta(\rho) = 0\},$$

$$\Lambda' = \{\lambda = \rho^2\colon \text{Im } \rho > 0,\ \Delta(\rho) = 0\},$$

$$\Lambda'' = \{\lambda = \rho^2\colon \text{Im } \rho = 0,\ \rho \neq 0,\ \Delta(\rho) = 0\}.$$

Obviously, $\Lambda = \Lambda' \cup \Lambda''$ is a bounded set, and Λ' is a bounded and at most countable set.

Denote

$$\Phi(x,\lambda) = e(x,\rho)(u(\rho))^{-1}. \tag{2.5}$$

The matrix-function $\Phi(x,\lambda) = [\Phi_{jk}(x,\lambda)]_{j,k=\overline{1,m}}$, $x \geq 0$, satisfies (1.1) and on account of Lemma 2.1 also the conditions

$$U(\Phi) = I_m, \quad \Phi(x,\lambda) = O(\exp(i\rho x)), \quad x \to \infty, \quad \rho \in \Omega \setminus \Lambda.$$

Denote $M(\lambda) := \Phi(0,\lambda)$. The matrix $M(\lambda)$ is called the *Weyl matrix* for L. It follows from (2.5) that

$$M(\lambda) = e(0,\rho)(u(\rho))^{-1}. \tag{2.6}$$

Moreover,

$$\Phi(x,\lambda) = S(x,\lambda) + \varphi(x,\lambda)M(\lambda). \tag{2.7}$$

The matrix-functions $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$ form a fundamental system of solutions for equation (1), and det $[\varphi^{(\nu)}(x,\lambda), \Phi^{(\nu)}(x,\lambda)]_{\nu=0,1} \equiv 1$.

Theorem 2.3. *The Weyl matrix $M(\lambda)$ is analytic in $\Pi \setminus \Lambda'$ and continuous in $\Pi_1 \setminus \Lambda$. The set of singularities of $M(\lambda)$ (as an analytic function) coincides with the set $\Lambda_0 := \{\lambda\colon \lambda \geq 0\} \cup \Lambda$. For $|\rho| \to \infty$, $\rho \in \Omega$,*

$$M(\lambda) = (i\rho)^{-1}\big(I_m + O(\rho^{-1})\big).$$

Theorem 2.3 follows from (2.4), (2.6) and Lemma 2.1.

Theorem 2.4. *The boundary value problem L has no eigenvalues $\lambda > 0$.*

Proof. Suppose that $\lambda_0 = \rho_0^2 > 0$ is an eigenvalue, and let $Y_0(x)$ be a corresponding eigenvector. Since the matrix-functions $\{e(x,\rho_0), e(x,-\rho_0)\}$ form a fundamental system of solutions of (1), we have $Y_0(x) = e(x,\rho_0)A + e(x,-\rho_0)B$, where A and B are constant column-vectors. For $x \to \infty$, $Y_0(x) \sim 0$, $e(x,\pm\rho_0) \sim \exp(\pm i\rho_0 x)$. But this is possible only if $A = B = 0$. □

Theorem 2.5. *Let* $\lambda_0 \notin [0,\infty)$. *For* λ_0 *to be an eigenvalue, it is necessary and sufficient that* $\Delta(\rho_0) = 0$. *In other words, the set of nonzero eigenvalues coincides with* Λ'.

Proof. Let $\lambda_0 \in \Lambda'$, i.e. $\Delta(\rho_0) = 0$. By virtue of (2.1), $\lim\limits_{x\to\infty} e(x,\rho_0) = 0$. Consider the following linear algebraic system $u(\rho_0)\alpha = 0$ with respect to the column-vector $\alpha = [\alpha_k]_{k=\overline{1,m}}$. Since $\det u(\rho_0) = \Delta(\rho_0) = 0$, this system has a nonzero solution α^0. Put $Y_0(x) = e(x,\rho_0)\alpha^0$. Then $U(Y_0) = 0$, $\lim_{x\to\infty} Y_0(x) = 0$, and consequently, $Y_0(x)$ is an eigenfunction, and $\lambda_0 = \rho_0^2$ is an eigenvalue.

Conversely, let $\lambda_0 = \rho_0^2$, $\operatorname{Im}\rho_0 > 0$ be an eigenvalue, and let $Y_0(x)$ be a corresponding eigenfunction. Since the matrix-functions $e(x,\rho_0)$ and $E(x,\rho_0)$ form a fundamental system of solutions of equation (1.1), we get $Y_0(x) = e(x,\rho_0)a + E(x,\rho_0)b$, where a and b are constant column-vectors. As $x \to \infty$, we calculate $b = 0$, i.e. $Y_0(x) = e(x,\rho_0)a$, $a \neq 0$. Since $U(Y_0) = 0$, it follows that $u(\rho_0)a = 0$. Consequently, $\det u(\rho_0) = 0$, i.e. $\Delta(\rho_0) = 0$. Theorem 2.5 is proved. □

The set Λ_0 of singularities of the Weyl matrix $M(\lambda)$ coincides with the spectrum of L. Thus, the spectrum of L consists of the positive half-line $\{\lambda : \lambda \geq 0\}$, and the discrete set $\Lambda = \Lambda' \cup \Lambda''$. Each element of Λ' is an eigenvalue of L. According to Theorem 2.4, the points of Λ'' are not eigenvalues, they are called *spectral singularities* of L.

3. The uniqueness theorem

The inverse problem is formulated as follows: *Given the Weyl matrix* M, *construct* Q *and* h.

In this section we prove the uniqueness theorem for the solution of this inverse problem. For this purpose we agree that together with L we consider a boundary value problem $\tilde{L}$ of the same form but with different $\tilde{Q}$ and $\tilde{h}$. Everywhere below if a symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$, and $\hat{\alpha} = \alpha - \tilde{\alpha}$.

Theorem 3.1. *If* $M = \tilde{M}$, *then* $Q = \tilde{Q}$ *and* $h = \tilde{h}$. *Thus, the specification of the Weyl matrix* M *uniquely determines the potential* Q *and the coefficients of the boundary conditions* (2.3).

Proof. Denote

$$\ell^* Z := -Z'' + ZQ(x), \qquad \langle Z, Y\rangle := Z'Y - ZY',$$

$$U^*(Z) := Z'(0) - Z(0)h,$$

where $Z = [Z_k]^t_{k=\overline{1,m}}$ is a row vector (t is the sign for the transposition). Then

$$\langle Z, Y\rangle_{|x=0} = U^*(Z)Y(0) - Z(0)U(Y). \tag{3.1}$$

Moreover, if $Y(x,\lambda)$ and $Z(x,\mu)$ satisfy the equations $\ell Y(x,\lambda) = \lambda Y(x,\lambda)$ and $\ell^* Z(x,\mu) = \mu Z(x,\mu)$, respectively, then

$$\frac{d}{dx}\langle Z(x,\mu), Y(x,\lambda)\rangle = (\lambda-\mu)Z(x,\mu)Y(x,\lambda). \tag{3.2}$$

Let $\varphi^*(x,\lambda)$, $S^*(x,\lambda)$ and $\mathbf{\Phi}^*(x,\lambda)$ be the matrices satisfying the equation $\ell^* Z = \lambda Z$ and the conditions

$$\varphi^*(0,\lambda) = I_m, \quad \varphi^{*\prime}(0,\lambda) = h, \quad S^*(0,\lambda) = 0, \quad S^{*\prime}(0,\lambda) = I_m,$$

$$U^*(\mathbf{\Phi}^*) = I_m, \quad \mathbf{\Phi}^*(x,\lambda) = O(\exp(i\rho x)), \qquad x\to\infty, \quad \rho\in\mathbf{\Omega}.$$

Denote $M^*(\lambda) := \mathbf{\Phi}^*(0,\lambda)$. Then

$$\mathbf{\Phi}^*(x,\lambda) = S^*(x,\lambda) + M^*(\lambda)\varphi^*(x,\lambda). \tag{3.3}$$

According to (3.2), $\langle \mathbf{\Phi}^*(x,\lambda), \mathbf{\Phi}(x,\lambda)\rangle$ does not depend on x. Using (3.1) we calculate

$$\langle \mathbf{\Phi}^*(x,\lambda), \mathbf{\Phi}(x,\lambda)\rangle_{|x=0} = M(\lambda) - M^*(\lambda).$$

Moreover, $\lim_{x\to\infty}\langle \mathbf{\Phi}^*(x,\lambda), \mathbf{\Phi}(x,\lambda)\rangle = 0$, and consequently,

$$M^*(\lambda) = M(\lambda). \tag{3.4}$$

Using (3.2) again, we get

$$\begin{gathered}
\varphi^*(x,\lambda)\mathbf{\Phi}'(x,\lambda) - \varphi^{*\prime}(x,\lambda)\mathbf{\Phi}(x,\lambda) = I_m,\\
\varphi^*(x,\lambda)\varphi'(x,\lambda) - \varphi^{*\prime}(x,\lambda)\varphi(x,\lambda) = 0,\\
\mathbf{\Phi}^{*\prime}(x,\lambda)\varphi(x,\lambda) - \mathbf{\Phi}^*(x,\lambda)\varphi'(x,\lambda) = I_m,\\
\mathbf{\Phi}^*(x,\lambda)\mathbf{\Phi}'(x,\lambda) - \mathbf{\Phi}^{*\prime}(x,\lambda)\mathbf{\Phi}(x,\lambda) = 0,
\end{gathered} \tag{3.5}$$

hence,

$$\begin{bmatrix} \varphi(x,\lambda) & \mathbf{\Phi}(x,\lambda) \\ \varphi'(x,\lambda) & \mathbf{\Phi}'(x,\lambda)\end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{\Phi}^{*\prime}(x,\lambda) & -\mathbf{\Phi}^*(x,\lambda) \\ -\varphi^{*\prime}(x,\lambda) & \varphi^*(x,\lambda)\end{bmatrix}. \tag{3.6}$$

Using the fundamental system of solutions $\{e(x,\rho), E(x,\rho)\}$ and the initial conditions on $\varphi(x,\lambda)$ we calculate

$$\varphi(x,\lambda) = e(x,\rho)A_1(\rho) + E(x,\rho)A_2(\rho), \qquad A_j(\rho) = \frac{1}{2}\big(I_m + O(\rho^{-1})\big),$$

$$|\rho|\to\infty, \qquad \rho\in\mathbf{\Omega},$$

and consequently, for $|\rho|\to\infty$, $\rho\in\boldsymbol{\Omega}$, $x\ge 0$, $\nu=0,1$, we obtain

$$\varphi^{(\nu)}(x,\lambda)=\frac{(-i\rho)^{\nu}}{2}\exp{(-i\rho x)}\big(I_m+O(\rho^{-1})\big)+\frac{(i\rho)^{\nu}}{2}\exp{(i\rho x)}\big(I_m+O(\rho^{-1})\big). \tag{3.7}$$

It follows from (2.2), (2.4) and (2.5) that for $|\rho|\to\infty$, $\rho\in\boldsymbol{\Omega}$, $x\ge 0$, $\nu=0,1$,

$$\boldsymbol{\Phi}^{(\nu)}(x,\lambda)=(i\rho)^{\nu-1}\exp{(i\rho x)}\big(I_m+O(\rho^{-1})\big). \tag{3.8}$$

Similarly, we calculate for $|\rho|\to\infty$, $\rho\in\boldsymbol{\Omega}$, $x\ge 0$, $\nu=0,1$,

$$\begin{gathered}\varphi^{*(\nu)}(x,\lambda)=\frac{(-i\rho)^{\nu}}{2}\exp{(-i\rho x)}\Big(I_m+O(\rho^{-1})\Big)+\frac{(i\rho)^{\nu}}{2}\exp{(i\rho x)}\big(I_m+O(\rho^{-1})\big),\\ \boldsymbol{\Phi}^{*(\nu)}(x,\lambda)=(i\rho)^{\nu-1}\exp{(i\rho x)}\big(I_m+O(\rho^{-1})\big).\end{gathered} \tag{3.9}$$

In particular, (3.7)–(3.9) yield for $\rho\in\boldsymbol{\Omega}$, $|\rho|\ge\rho^*$, $x\ge 0$, $\nu=0,1$,

$$\begin{gathered}\|\varphi^{(\nu)}(x,\lambda)\|,\ \|\varphi^{*(\nu)}(x,\lambda)\|\le C|\rho|^{\nu}\exp{(|\operatorname{Im}\rho|x)},\\ \|\boldsymbol{\Phi}^{(\nu)}(x,\lambda)\|,\ \|\boldsymbol{\Phi}^{*(\nu)}(x,\lambda)\|\le C|\rho|^{\nu-1}\exp{(-|\operatorname{Im}\rho|x)}.\end{gathered} \tag{3.10}$$

Now we consider the block-matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$ defined by

$$P(x,\lambda)\begin{bmatrix}\tilde\varphi(x,\lambda) & \tilde{\boldsymbol{\Phi}}(x,\lambda)\\ \tilde\varphi'(x,\lambda) & \tilde{\boldsymbol{\Phi}}'(x,\lambda)\end{bmatrix}=\begin{bmatrix}\varphi(x,\lambda) & \boldsymbol{\Phi}(x,\lambda)\\ \varphi'(x,\lambda) & \boldsymbol{\Phi}'(x,\lambda)\end{bmatrix}. \tag{3.11}$$

Taking (3.6) into account we calculate

$$\begin{gathered}P_{j1}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\tilde{\boldsymbol{\Phi}}^{*'}(x,\lambda)-\boldsymbol{\Phi}^{(j-1)}(x,\lambda)\tilde\varphi^{*'}(x,\lambda),\\ P_{j2}(x,\lambda)=\boldsymbol{\Phi}^{(j-1)}(x,\lambda)\tilde\varphi^{*}(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\tilde{\boldsymbol{\Phi}}^{*}(x,\lambda).\end{gathered} \tag{3.12}$$

It follows from (3.7)-(3.10) and (3.12) that

$$\|P_{jk}(x,\lambda)\|\le C|\rho|^{j-k},\qquad \rho\in\boldsymbol{\Omega},\quad |\rho|\ge\rho^*,\quad x\ge 0,\quad k=1,2, \tag{3.13}$$

$$P_{1k}(x,\lambda)=\delta_{1k}I_m+O(\rho^{-1}),\qquad |\rho|\to\infty,\quad \rho\in\Theta_\delta,\quad x>0,\quad k=1,2, \tag{3.14}$$

where $\Theta_\delta:=\{\rho\colon \arg\rho\in[\delta,\pi-\delta]\}$, $\delta>0$. Using (2.7), (3.3), (3.4) and (3.11) we obtain

$$\begin{aligned}P_{j1}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\tilde S^{*'}(x,\lambda)-S^{(j-1)}(x,\lambda)\tilde\varphi^{*'}(x,\lambda)\\ +\varphi^{(j-1)}(x,\lambda)(\tilde M(\lambda)-M(\lambda))\tilde\varphi^{*'}(x,\lambda),\end{aligned}$$

$$\begin{aligned}P_{j2}(x,\lambda)=S^{(j-1)}(x,\lambda)\tilde\varphi^{*}(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\tilde S^{*}(x,\lambda)\\ +\varphi^{(j-1)}(x,\lambda)(M(\lambda)-\tilde M(\lambda))\tilde\varphi^{*}(x,\lambda).\end{aligned}$$

Since $M(\lambda) \equiv \tilde{M}(\lambda)$, it follows that for each fixed x, the matrix-functions $P_{jk}(x,\lambda)$ are entire in λ of order 1/2. Together with (3.13)–(3.14) this yields

$$P_{11}(x,\lambda) \equiv I_m, \quad P_{12}(x,\lambda) \equiv 0.$$

By virtue of (3.11), we have $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$, for all x and λ, and consequently, $Q(x) = \tilde{Q}(x)$ a.e. on $(0,T)$, and $h = \tilde{h}$. Theorem 3.1 is proved. □

4. Solution of the inverse problem

Let the Weyl-matrix $M(\lambda)$ of the boundary value problem $L = L(Q,h)$ be given. We choose an arbitrary model boundary value problem $\tilde{L} = L(\tilde{Q},\tilde{h})$ (for example, one can take $\tilde{Q}(x) \equiv 0$, $\tilde{h} = 0$). Denote

$$D(x,\lambda,\mu) = \frac{\langle \varphi^*(x,\mu), \varphi(x,\lambda)\rangle}{\lambda-\mu} = \int_0^x \varphi^*(t,\mu)\varphi(t,\lambda)\,dt,$$

$$\tilde{D}(x,\lambda,\mu) = \frac{\langle \tilde{\varphi}^*(x,\mu), \tilde{\varphi}(x,\lambda)\rangle}{\lambda-\mu} = \int_0^x \tilde{\varphi}^*(t,\mu)\tilde{\varphi}(t,\lambda)\,dt,$$

$$r(x,\lambda,\mu) = \hat{M}(\mu)D(x,\lambda,\mu), \qquad \tilde{r}(x,\lambda,\mu) = \hat{M}(\mu)\tilde{D}(x,\lambda,\mu).$$

By the standard way (see [9, Ch. 2]) we get the estimates

$$\begin{gathered}\|D(x,\lambda,\mu)\|,\ \|\tilde{D}(x,\lambda,\mu)\| \le \frac{C_x \exp\left(|\operatorname{Im}\rho| x\right)}{|\rho \mp \theta| + 1}, \\ \lambda = \rho^2, \quad \mu = \theta^2 \ge 0, \quad \pm\theta \operatorname{Re}\rho \ge 0.\end{gathered} \tag{4.1}$$

It follows from (2.4) and (2.6) that

$$\hat{M}(\lambda) = O(\rho^{-2}), \qquad |\rho| \to \infty, \quad \rho \in \Omega. \tag{4.2}$$

In the λ-plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where γ' is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$, and γ'' is the two-sided cut along the arc $\{\lambda\colon \lambda > 0,\ \lambda \notin \operatorname{int} \gamma'\}$ (see Figure 1).

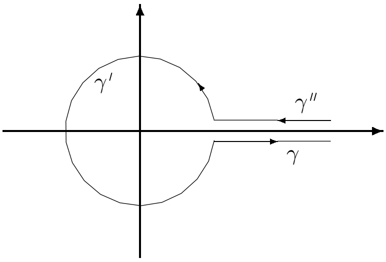

Figure 1.

Theorem 4.1. *The following relation holds*

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \varphi(x,\mu)\tilde{r}(x,\lambda,\mu)\,d\mu. \tag{4.3}$$

Proof. It follows from (3.10) and (4.1), (4.2) that the integrals in (4.3) converges absolutely and uniformly on γ, for each fixed $x \geq 0$. Denote $J_{\gamma} = \{\lambda:\ \lambda \notin \gamma \cup int\,\gamma'\}$. Consider the contour $\gamma_R = \gamma \cap \{\lambda:\ |\lambda| \leq R\}$ with counterclockwise circuit, and also consider the contour $\gamma_R^0 = \gamma_R \cup \{\lambda:\ |\lambda| = R\}$ with clockwise circuit. By Cauchy‘s integral formula [8, p. 84],

$$P_{1k}(x,\lambda) - \delta_{1k}I_m = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P_{1k}(x,\mu) - \delta_{1k}I_m}{\lambda - \mu}\,d\mu, \quad \lambda \in \text{int}\ \gamma_R^0,\ k = 1,2.$$

Using (3.13), (3.14) we get

$$\lim_{R\to\infty}\int_{|\mu|=R} \frac{P_{1k}(x,\mu) - \delta_{1k}I_m}{\lambda - \mu}\,d\mu = 0,\ k = 1,2,$$

and consequently,

$$P_{1k}(x,\lambda) = \delta_{1k}I_m + \frac{1}{2\pi i}\int_{\gamma} \frac{P_{1k}(x,\mu)}{\lambda - \mu}\,d\mu, \quad \lambda \in J_{\gamma},\ k = 1,2, \tag{4.4}$$

since

$$\frac{1}{2\pi i}\int_{\gamma} \frac{d\mu}{\lambda - \mu} = 0, \qquad \lambda \in J_{\gamma}$$

by Cauchy’s theorem. In (4.4) the integral is understood in the principal value sense:

$$\int_{\gamma} = \lim_{R\to\infty}\int_{\gamma_R}.$$

Since

$$\varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda),$$

it follows from (4.4) that

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \frac{P_{11}(x,\mu)\tilde{\varphi}(x,\lambda) + P_{12}(x,\mu)\tilde{\varphi}'(x,\lambda)}{\lambda - \mu}\,d\mu, \quad \lambda \in J_{\gamma}.$$

Taking (3.12) into account we get

$$\begin{aligned}\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \Big(&(\varphi(x,\mu)\widetilde{\mathbf{\Phi}^{*\prime}}(x,\mu) - \mathbf{\Phi}(x,\mu)\widetilde{\varphi^{*\prime}}(x,\mu))\tilde{\varphi}(x,\lambda) \\ &+ (\mathbf{\Phi}(x,\mu)\tilde{\varphi}^{*}(x,\mu) - \varphi(x,\mu)\tilde{\mathbf{\Phi}}^{*}(x,\mu))\tilde{\varphi}'(x,\lambda)\Big)\frac{d\mu}{\lambda - \mu}.\end{aligned}$$

In view of (2.7), (3.3) and (3.4) this yields (4.3), since the terms with $S(x,\mu)$ and $S^{*}(x,\mu)$ vanish by Cauchy‘s theorem. □

For each fixed $x \ge 0$, the relation (4.3) can be considered as a linear integral equation with respect to $\varphi(x,\lambda)$, $\lambda \in \gamma$. This equation is called the *main equation* of the inverse problem. Thus, the nonlinear inverse problem is reduced to the solution of this linear integral equation. Now we are going to prove the unique solvability of the main equation. For this purpose we need the following assertion.

Lemma 4.2. *The following relation holds*

$$\tilde r(x,\lambda,\mu) - r(x,\lambda,\mu) - \frac{1}{2\pi i}\int_\gamma r(x,\xi,\mu)\tilde r(x,\lambda,\xi)\,d\xi = 0. \tag{4.5}$$

Proof. Since

$$\frac{1}{\lambda-\mu}\Big(\frac{1}{\lambda-\xi} - \frac{1}{\mu-\xi}\Big) = \frac{1}{(\lambda-\xi)(\xi-\mu)}\ ,$$

we have by Cauchy‘s integral formula

$$\frac{P(x,\lambda)-P(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi, \quad \lambda,\mu \in \text{int}\ \gamma_R^0.$$

Using (3.13) we get

$$\lim_{R\to\infty}\int_{|\xi|=R} \frac{P(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi = 0,$$

and consequently,

$$\frac{P(x,\lambda)-P(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma} \frac{P(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi, \quad \lambda,\mu \in J_\gamma. \tag{4.6}$$

It follows from (4.6) that

$$[\varphi^{*'}(x,\mu), -\varphi^*(x,\mu)]\frac{P(x,\lambda)-P(x,\mu)}{\lambda-\mu}\left[\begin{array}{c}\tilde\varphi(x,\lambda)\\ \tilde\varphi'(x,\lambda)\end{array}\right]$$
$$= \frac{1}{2\pi i}\int_\gamma [\varphi^{*'}(x,\mu), -\varphi^*(x,\mu)]P(x,\xi)\left[\begin{array}{c}\tilde\varphi(x,\lambda)\\ \tilde\varphi'(x,\lambda)\end{array}\right]\frac{d\xi}{(\lambda-\xi)(\xi-\mu)}\ , \quad \lambda,\mu \in J_\gamma. \tag{4.7}$$

In view of (3.11) we infer

$$[\varphi^{*'}(x,\mu), -\varphi^*(x,\mu)]P(x,\lambda)\left[\begin{array}{c}\tilde\varphi(x,\lambda)\\ \tilde\varphi'(x,\lambda)\end{array}\right]$$
$$= [\varphi^{*'}(x,\mu), -\varphi^*(x,\mu)]\left[\begin{array}{c}\varphi(x,\lambda)\\ \varphi'(x,\lambda)\end{array}\right] = \langle \varphi^*(x,\mu), \varphi(x,\lambda)\rangle. \tag{4.8}$$

Taking (3.12) into account we calculate

$$[\varphi^{*'}(x,\mu),-\varphi^*(x,\mu)]P(x,\mu)=[\varphi^{*'}(x,\mu),-\varphi^*(x,\mu)]\times$$
$$\begin{bmatrix} \varphi(x,\mu)\tilde{\mathbf{\Phi}}^{*'}(x,\mu)-\mathbf{\Phi}(x,\mu)\tilde{\varphi}^{*'}(x,\mu), & \mathbf{\Phi}(x,\mu)\tilde{\varphi}^*(x,\mu)-\varphi(x,\mu)\tilde{\mathbf{\Phi}}^*(x,\mu) \\ \varphi'(x,\mu)\tilde{\mathbf{\Phi}}^{*'}(x,\mu)-\mathbf{\Phi}'(x,\mu)\tilde{\varphi}^{*'}(x,\mu), & \mathbf{\Phi}'(x,\mu)\tilde{\varphi}^*(x,\mu)-\varphi'(x,\mu)\tilde{\mathbf{\Phi}}^*(x,\mu) \end{bmatrix}.$$

According to (3.5) this yields

$$[\varphi^{*'}(x,\mu),-\varphi^*(x,\mu)]P(x,\mu)=[\tilde{\varphi}^{*'}(x,\mu),-\tilde{\varphi}^*(x,\mu)],$$

and consequently,

$$[\varphi^{*'}(x,\mu),-\varphi^*(x,\mu)]P(x,\mu)\begin{bmatrix}\tilde{\varphi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda)\end{bmatrix}$$
$$=[\tilde{\varphi}^{*'}(x,\mu),-\tilde{\varphi}^*(x,\mu)]\begin{bmatrix}\tilde{\varphi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda)\end{bmatrix}=\langle\tilde{\varphi}^*(x,\mu),\tilde{\varphi}(x,\lambda)\rangle. \quad (4.9)$$

Using (3.12), (2.7), (3.3) and (3.4) we get

$$\frac{1}{2\pi i}\int_\gamma P(x,\xi)\frac{d\xi}{(\lambda-\xi)(\xi-\mu)}$$
$$=\frac{1}{2\pi i}\int_\gamma\begin{bmatrix}-\varphi(x,\xi)\hat{M}(\xi)\tilde{\varphi}^{*'}(x,\xi) & \varphi(x,\xi)\hat{M}(\xi)\tilde{\varphi}^*(x,\mu)\\ -\varphi'(x,\xi)\hat{M}(\xi)\tilde{\varphi}^{*'}(x,\mu) & \varphi'(x,\xi)\hat{M}(\xi)\tilde{\varphi}^*(x,\mu)\end{bmatrix}\frac{d\xi}{(\lambda-\xi)(\xi-\mu)},$$

since the terms with $S(x,\xi)$ vanish by Cauchy‘s theorem. Therefore

$$\frac{1}{2\pi i}\int_\gamma[\varphi^{*'}(x,\mu),-\varphi^*(x,\mu)]P(x,\xi)\begin{bmatrix}\tilde{\varphi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda)\end{bmatrix}\frac{d\xi}{(\lambda-\xi)(\xi-\mu)}$$
$$=-\frac{1}{2\pi i}\int_\gamma\frac{\langle\varphi^*(x,\mu),\varphi(x,\xi)\rangle}{\xi-\mu}\hat{M}(\xi)\frac{\langle\tilde{\varphi}^*(x,\xi),\tilde{\varphi}(x,\lambda)\rangle}{\lambda-\xi}\,d\xi,\quad \lambda,\mu\in J_\gamma. \quad (4.10)$$

Substituting (4.8)–(4.10) into (4.7) we obtain

$$\frac{\langle\tilde{\varphi}^*(x,\mu),\tilde{\varphi}(x,\lambda)\rangle}{\lambda-\mu}-\frac{\langle\varphi^*(x,\mu),\varphi(x,\lambda)\rangle}{\lambda-\mu}$$
$$-\frac{1}{2\pi i}\int_\gamma\frac{\langle\varphi^*(x,\mu),\varphi(x,\xi)\rangle}{\xi-\mu}\hat{M}(\xi)\frac{\langle\tilde{\varphi}^*(x,\xi),\tilde{\varphi}(x,\lambda)\rangle}{\lambda-\xi}\,d\xi=0,$$

or, which is the same

$$\tilde{D}(x,\lambda,\mu)-D(x,\lambda,\mu)-\frac{1}{2\pi i}\int_\gamma D(x,\xi,\mu)\hat{M}(\xi)\tilde{D}(x,\lambda,\xi)\,d\xi=0.$$

Multiplying this relation by $\hat{M}(\mu)$, we arrive at (4.5). Lemma 4.2 is proved. □

Let us consider the Banach space B of continuous bounded on γ matrix-functions $z(\lambda) = [z_{jk}(\lambda)]_{j,k=\overline{1,m}}$, $\lambda \in \gamma$, with the norm $\|z\|_B = \sup\limits_{\lambda\in\gamma} \max\limits_{j,k=\overline{1,m}} |z_{jk}(\lambda)|$.

Theorem 4.3. *For each fixed* $x \geq 0$, *the main equation* (4.3) *has a unique solution* $\varphi(x,\lambda) \in B$.

Proof. For a fixed $x \geq 0$, we consider the following linear bounded operators in B:

$$\tilde{A}z(\lambda) = z(\lambda) + \frac{1}{2\pi i}\int_\gamma z(\mu)\tilde{r}(x,\lambda,\mu)\,d\mu,$$

$$Az(\lambda) = z(\lambda) - \frac{1}{2\pi i}\int_\gamma z(\mu)r(x,\lambda,\mu)z(\mu)\,d\mu.$$

Then

$$\tilde{A}Az(\lambda) = z(\lambda) + \frac{1}{2\pi i}\int_\gamma z(\mu)\Big(\tilde{r}(x,\lambda,\mu) - r(x,\lambda,\mu) \\ - \frac{1}{2\pi i}\int_\gamma r(x,\xi,\mu)\tilde{r}(x,\lambda,\xi)\,d\xi\Big)\,d\mu.$$

By virtue of (4.5) this yields

$$\tilde{A}Az(\lambda) = z(\lambda), \qquad z(\lambda) \in B.$$

Interchanging places for L and $\tilde{L}$, we obtain analogously $A\tilde{A}z(\lambda) = z(\lambda)$. Thus, $\tilde{A}A = A\tilde{A} = E$, where E is the identity operator. Hence the operator $\tilde{A}$ has a bounded inverse operator, and the main equation (4.3) is uniquely solvable for each fixed $x \geq 0$. □

Using the solution of the main equation one can construct the potential matrix $Q(x)$ and the coefficients of the boundary conditions h. Thus, we obtain the following algorithm for the solution of the inverse problem.

Algorithm. Let the matrix-function $M(\lambda)$ be given. Then
1. Choose $\tilde{L}$ and calculate $\tilde{\varphi}(x,\lambda)$ and $\tilde{r}(x,\lambda,\mu)$.
2. Find $\varphi(x,\lambda)$ by solving equation (4.3).
3. Construct $Q(x)$ and h via $Q(x) = \varphi''(x,\lambda)(\varphi(x,\lambda))^{-1} - \lambda I_m$, $h = \varphi'(0,\lambda)$.

Remark 4.4. Using the main equation one can also obtain necessary and sufficient conditions for the solvability of the inverse problem (see [9, sec. 2.2]).

Remark 4.5. For the case of a *locally integrable* complex-valued potential matrix $Q(x)$ the generalized Weyl-matrix can be introduced as a main spectral characteristic, and the inverse problem can be treated analogously to the scalar case (see [9, sec. 2.5]).

Remark 4.6. Similar results are obtained for the matrix Sturm–Liouville equation (1.1) on a *finite interval*. For this purpose the method of spectral mappings (see [20, 21]) has been used.

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