An inverse spectral problem for Sturm-Liouville operators with singular potentials on star-type graphs

G. Freiling, M. Ignatiev, and V. Yurko

Abstract. Sturm-Liouville differential operators with singular potentials on compact star-type graphs are studied. We establish properties of the spectral characteristics and investigate the inverse problem of recovering the operator from the so-called Weyl vector which is a generalization of the Weyl function (m - function) for the classical Sturm-Liouville operator on an interval. For this inverse problem we prove an uniqueness theorem and obtain a procedure for constructing the solution by the method of spectral mappings.

1. Introduction

This paper is devoted to the solution of the inverse spectral problem for Sturm-Liouville differential operators with singular potentials on compact star-type graphs. The inverse problem consists in recovering the potential on a graph from the given spectral characteristics. For the classical Sturm-Liouville operators on an *interval* inverse spectral problems have been studied fairly completely (see the monographs [1-8]). The case of singular potentials on an interval was investigated in [9-11] and other papers.

Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [12-13]). In recent years there has been considerable interest in the spectral theory of Sturm-Liouville equations on graphs (see a good review of such publications in [12]-[13]). Most of the works in this direction are devoted to the so-called direct problems of studying properties of the spectrum and the root functions. Inverse spectral problems, because of their nonlinearity, are more difficult for investigating, and nowadays there are only a number of papers in the inverse problem theory for differential operators with *integrable* coefficients on graphs (see [14-18] and the references therein). The inverse spectral problem for operators with *singular* potentials on graphs has not been studied yet.

In this paper we study the inverse spectral problem for Sturm-Liouville differential operators with singular potentials on compact star-type graphs. As the main spectral characteristic we introduce and investigate the so-called Weyl vector

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which is a generalization of the Weyl function (m-function) for the classical Sturm-Liouville operator on an *interval* (see [4], [19]). We show that the specification of the Weyl vector uniquely determines the coefficients of the differential equation on the graph, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl vector. For studying this inverse problem the ideas of the method of spectral mappings [8] are developed. The obtained results are natural generalizations of the well-known results on inverse problems for the differential operators on an interval.

The paper is structured as follows. In Section 2 we introduce the main notions and provide a formulation of the inverse problem. Section 3 contains some auxiliary propositions. Section 4 is devoted to the solution the so-called local inverse problems. In section 5 we provide a solution of the global inverse problem on the graph.

2. The Weyl vector. Formulation of the inverse problem

Let T be a star-type graph with the set of vertices $\{v_j\}_{j=0}^p$ and set of edges $\{e_j\}_{j=1}^p, e_j = [v_0, v_j]$. We suppose that the length of each edge is equal to 1. We consider each edge as a segment $[0,1]$ and parameterize it by the parameter $x \in [0,1]$. It is convenient for us to choose the orientation such that $x = 1$ corresponds to the vertex v_0.

A function Y on T is considered as a vector $Y(x) = [y_j(x)]_{j=1}^p, x \in [0,1]$. Let $q = [q_j(x)]_{j=1}^p$ be a real-valued function on T such that $q_j \in W_2^{-1}[0,1]$, i.e. $q_j(x) = \sigma_j'(x)$, $\sigma_j \in L_2[0,1]$ (the derivative here is considered in the sense of distributions). We call the function $\sigma = [\sigma_j(x)]_{j=1}^p$ the *potential.* The Sturm-Liouville differential operator on the edge e_j is defined by the following expression (see [9-11]):

$$\ell_j y_j = -(y_j^{[1]})' - \sigma_j(x) y_j^{[1]} - \sigma_j^2(x) y_j,$$

where $y_j^{[1]} := y_j' - \sigma_j y_j$ is a *quasi-derivative*, and

$$dom(\ell_j) = \{y_j \mid y_j \in W_2^1[0,1],\ y_j^{[1]} \in W_1^1[0,1],\ \ell_j y_j \in L_2[0,1]\}.$$

Consider the Sturm-Liouville equation on T:

$$\ell_j y_j = \lambda y_j, \quad x \in [0,1],\ j = \overline{1,p}, \tag{1}$$

where $y_j \in dom(\ell_j)$ and the functions y_j satisfy the following matching conditions at the internal vertex v_0:

$$y_j(1) = y_k(1), \quad j,k = \overline{1,p}, \tag{2}$$

$$\sum_{j=1}^p y_j^{[1]}(1) = 0. \tag{3}$$

The matching conditions (2)-(3) are called the standard conditions. In electrical circuits, they express Kirchhoff's law; in elastic string networks, they express the balance of tension, and so on. Denote $U_j(Y) := y_j^{[1]}(0),\ j = \overline{1,p}$, and consider the boundary value problem B_0 for equation (1) with the matching conditions (2)-(3) and with the boundary conditions

$$U_j(Y) = 0, \quad j = \overline{1,p}.$$

We also consider the boundary value problem $B_k,\ k=\overline{1,p}$ for equation (1) with the matching conditions (2)-(3) and with the boundary conditions

$$y_k(0)=0,\ U_j(Y)=0,\quad j=\overline{1,p}\setminus k.$$

Let $\Phi_k(x,\lambda)=[\Phi_{kj}(x,\lambda)]_{j=1}^p,\ k=\overline{1,p}$ be the solutions of the equation (1) satisfying (2), (3) and the boundary conditions:

$$U_j(\Phi_k)=\delta_{jk}, \tag{4}$$

where δ_{jk} is the Kronecker delta. Denote $M_k(\lambda):=\Phi_{kk}(0,\lambda)$, $M(\lambda)=[M_k(\lambda)]_{k=1}^{p-1}$. The function $M_k(\lambda)$ is called the *Weyl function* for (1) with respect to the vertex v_k. The vector $M(\lambda)$ is called the *Weyl vector*. The inverse problem is formulated as follows.

Inverse Problem 1. Given the Weyl vector M, construct the potential σ.

Remark 1. The notion of the Weyl vector is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator on an *interval* (see [4], [19]). As in the classical case [17] one can show that the functions $M_k(\lambda)$ are meromorphic in λ, namely:

$$M_k(\lambda)=\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}\,,$$

where $\Delta_k(\lambda)$ are the characteristic functions of the boundary value problems B_k. Zeros $\Lambda_k:=\{\lambda_{kn}\}_{n\ge 0}$ of the entire function $\Delta_k(\lambda)$ are real and coincide with the eigenvalues of B_k. The characteristic function $\Delta_k(\lambda)$ is uniquely determined by its zeros Λ_k. The specification of the Weyl vector $M(\lambda)$ is equivalent to the specification of the spectra $\Lambda_k,\ k=\overline{0,p}$. Thus, Inverse Problem 1 is equivalent to the Inverse Problem 2 of recovering σ from the system of spectra $\Lambda_k,\ k=\overline{0,p}$. Inverse Problems 1 and 2 are a generalization of the classical inverse Sturm-Liouville problems on an interval (see [4]).

Let us formulate the uniqueness theorem for the solution of Inverse Problem 1. For this purpose together with σ we consider a potential $\tilde\sigma$. Everywhere below if a symbol α denotes an object related to σ, then $\tilde\alpha$ will denote the analogous object related to $\tilde\sigma$, and $\hat\alpha:=\alpha-\tilde\alpha$.

Theorem 1. *If $M=\tilde M$, then $\sigma=\tilde\sigma$. Thus, the specification of the Weyl vector M uniquely determines the potential σ on T.*

This theorem will be proved in Section 5. Moreover, we will give there a constructive procedure for the solution of Inverse Problem 1. For this purpose we use the ideas of the method of spectral mappings [8].

3. Auxiliary propositions

Let $\varphi_j(x,\lambda),\psi_j(x,\lambda)$ and $s_j(x,\lambda)$ be the solutions of equation (1) on the edge e_j under the initial conditions

$$\varphi_j(0,\lambda)=\psi_j(1,\lambda)=s_j^{[1]}(0,\lambda)=1,\quad \varphi_j^{[1]}(0,\lambda)=\psi_j^{[1]}(1,\lambda)=s_j(0,\lambda)=0.$$

Let $\lambda=\rho^2$. The following asymptotical representations can be deduced directly from the results of [9-11]:

$$\varphi_j(x,\lambda)=\cos\rho x+\eta(x,\rho),\quad \varphi_j^{[1]}(x,\lambda)=-\rho\sin\rho x+\rho\eta(x,\rho), \tag{5}$$

$$(6)\quad \psi_j(x,\lambda)=\cos\rho(1-x)+\eta(1-x,\rho),\quad \psi_j^{[1]}(x,\lambda)=\rho\sin\rho(1-x)+\rho\eta(1-x,\rho),$$

Here and below the symbol $\eta(x,\rho)$ denotes (various) functions which are entire in ρ for all $x\in[0,1]$, and:

1) $\eta(x,\rho)=o(\exp(x|Im\rho|))$ for $\rho\to\infty$ and any fixed $x\in[0,1]$;

2) $\eta(x,\cdot)\in L_2(\gamma)$ for all $x\in[0,1]$ and real τ, where $\gamma=\gamma(\tau):=(-\infty+i\tau,+\infty+i\tau)$;

3) $\eta(\cdot,\cdot)\in L_2[0,1]\times\gamma$ and bounded uniformly on $[0,1]\times\gamma$ for any fixed real τ.

4) $\eta(x,\rho)$ depends continuously on the potential in the following sense: if $\sigma_n\to\sigma$ in $L_2(T)$ then the corresponding $\eta_n(x,\rho)$ converges to $\eta(x,\rho)$ uniformly on $[0,1]\times\gamma$, (for any fixed real τ), and

$$\max_{x\in[0,1]}\|\eta_n(x,\cdot)-\eta(x,\cdot)\|_{L_2(\gamma)}\to 0.$$

Throughout the paper we shall use the symbol $\kappa(\rho)$ for (various) meromorphic functions with the following properties:

1) for any $\delta>0$, one has $\kappa(\rho)\to 0,\ \rho\to\infty,\ \rho\in G_\delta$, where $G_\delta=\{\rho:|\rho-n\pi/2|>\delta\quad\forall n\in Z\}$;

2) $\kappa(\cdot)\in L_2(\gamma)$ for all $\tau>\tau_0$, where τ_0 may be different for different functions $\kappa(\rho)$;

3) $\kappa(\rho)$ depends continuously on $\sigma(x)$ in the following sense: if $\sigma_n\to\sigma$ in $L_2(T)$ then all the corresponding $\kappa_n(\cdot)\in L_2(\gamma)$ and $\kappa_n(\rho)$ converges to $\kappa(\rho)$ in $L_2(\gamma)$ and uniformly on γ for all $\tau>\tau_0$.

Also we shall use the notation $[1]:=1+\kappa(\rho)$.

Using the properties of the solutions φ_j,ψ_j, mentioned above, one can deduce the following asymptotics

$$(7)\qquad \varphi_j(1,\lambda)=\cos\rho[1],\quad \varphi_j^{[1]}(1,\lambda)=-\rho\sin\rho[1],$$

$$(8)\qquad \psi_j(0,\lambda)=\cos\rho[1],\quad \psi_j^{[1]}(0,\lambda)=\rho\sin\rho[1],$$

Lemma 1. *The following representations hold*

$$\Phi_{kk}(x,\lambda)=\frac{1-p}{p}\frac{[1]}{\rho\cos\rho\sin\rho}\cos\rho x+\frac{[1]}{\rho\sin\rho}\cos\rho(1-x)$$
$$+o(\rho^{-1}\exp(-x|Im\rho|))+o(\rho^{-1}\exp((x-2)|Im\rho|)),$$
$$\Phi_{kk}^{[1]}(x,\lambda)=\frac{p-1}{p}\frac{[1]}{\cos\rho\sin\rho}\sin\rho x+\frac{[1]}{\sin\rho}\sin\rho(1-x)$$
$$+o(\exp(-x|Im\rho|))+o(\exp((x-2)|Im\rho|))$$

for any fixed $x\in[0,1]$ *and* $\rho\to\infty,\ \rho\in G_\delta$. *Moreover,*

$$M_k(\lambda)=\frac{1-p}{p}\frac{[1]}{\rho\sin\rho\cos\rho}+\frac{[1]}{\rho}\cot\rho.$$

Proof. Using the fundamental system of solutions $\varphi_k(x,\lambda),\ \psi_k(x,\lambda)$, we get

$$(9)\qquad \Phi_{kk}(x,\lambda)=\beta_k(\lambda)\varphi_k(x,\lambda)+\gamma_k(\lambda)\psi_k(x,\lambda),$$

where $\beta_k(\lambda)$ and $\gamma_k(\lambda)$ do not depend on x. In view of (4), $\gamma_k(\lambda)=(U_k(\psi_k))^{-1}$. Taking (8) into account we calculate

$$(10)\qquad \gamma_k(\lambda)=(\rho\sin\rho)^{-1}[1].$$

Since Φ_k satisfies the matching conditions (2)-(3) one has

$$\sum_{j=1}^{p} \frac{\Phi_{kj}^{[1]}(1,\lambda)}{\Phi_{kj}(1,\lambda)} = 0.$$

According to (4), $\Phi_{kj}(x,\lambda)$ for $j \neq k$ are proportional to $\varphi_j(x,\lambda)$, hence

$$\sum_{j=\overline{1,p}\setminus k} \frac{\Phi_{kj}^{[1]}(1,\lambda)}{\Phi_{kj}(1,\lambda)} = \sum_{j=\overline{1,p}\setminus k} \frac{\varphi_{j}^{[1]}(1,\lambda)}{\varphi_{j}(1,\lambda)}.$$

By virtue of (7), this yields

$$\sum_{j=\overline{1,p}\setminus k} \frac{\Phi_{kj}^{[1]}(1,\lambda)}{\Phi_{kj}(1,\lambda)} = -(p-1)\rho\tan\rho[1].$$

Clearly,

$$\frac{\Phi_{kk}^{[1]}(1,\lambda)}{\Phi_{kk}(1,\lambda)} = \frac{\beta_k\varphi_{k}^{[1]}(1,\lambda)}{\beta_k\varphi_k(1,\lambda)+\gamma_k}.$$

Using (7) and (10) we calculate

$$\beta_k = \frac{1-p}{p}\,\frac{[1]}{\rho\sin\rho\cos\rho}. \tag{11}$$

Substituting (10) and (11) into (9) and using again the asymptotics (5)-(8) we arrive at the required representations. □

Let σ and $\tilde{\sigma}$ be two potentials. As in the classical case [4] we introduce the matrix

$$P^k(x,\lambda) := \begin{bmatrix} \varphi_k(x,\lambda) & \Phi_{kk}(x,\lambda) \\ \varphi_k^{[1]}(x,\lambda) & \Phi_{kk}^{[1]}(x,\lambda) \end{bmatrix} \cdot \begin{bmatrix} \tilde{\varphi}_k(x,\lambda) & \tilde{\Phi}_{kk}(x,\lambda) \\ \tilde{\varphi}_k^{[1]}(x,\lambda) & \tilde{\Phi}_{kk}^{[1]}(x,\lambda) \end{bmatrix}^{-1}.$$

LEMMA 2. *For any fixed* $x \in [0,1]$ *and* $\rho \to \infty,\ \rho \in G_\delta$

$$P_{js}^k(x,\lambda) - \delta_{js} = o(1), \quad (j,s) \neq (2,1),$$

$$P_{21}^k(x,\lambda) = o(\rho).$$

PROOF. Since $\langle \tilde{\varphi}_k(x,\lambda), \tilde{\Phi}_{kk}(x,\lambda)\rangle = 1$ (where $\langle y,z\rangle := yz^{[1]} - y^{[1]}z$) one has

$$P_{11}^k = \varphi_k\tilde{\Phi}_{kk}^{[1]} - \Phi_{kk}\tilde{\varphi}_k^{[1]}, \quad P_{12}^k = \Phi_{kk}\tilde{\varphi}_k - \varphi_k\tilde{\Phi}_{kk},$$

$$P_{21}^k = \varphi_k^{[1]}\tilde{\Phi}_{kk}^{[1]} - \Phi_{kk}^{[1]}\tilde{\varphi}_k^{[1]}, \quad P_{22}^k = \tilde{\varphi}_k\Phi_{kk}^{[1]} - \varphi_k^{[1]}\tilde{\Phi}_{kk}.$$

The assertion of lemma now follows directly from the asymptotics (5) and lemma 1. □

In the ρ–plane consider the contour $\gamma = \gamma(\tau) := (-\infty+i\tau, +\infty+i\tau)$ where $\tau > 0$ is such that $inf\{\Lambda_k \cup \tilde{\Lambda}_k\} > -\tau^2$. Let Γ be the contour in the λ– plane which is the image of γ under the mapping $\lambda = \rho^2$. Denote by D^+ the image of the half-plane $\{Im\rho > \tau\}$, and $D^- := \mathbf{C}\setminus D^+$. Let $C_N := \{|\lambda| = (N+1/4)^2\}$ and $C_N^- := C_N \cap D^-$ be the contours with clockwise orientation. Denote $\Gamma_N = \Gamma \cap \operatorname{int} C_N$, $\Gamma_N^- = \Gamma_N \cup C_N^-$. Define the functions

$$D_k(x,\lambda,\mu) := \frac{\langle \varphi_k(x,\lambda), \varphi_k(x,\mu)\rangle}{\lambda-\mu} = \int_0^x \varphi_k(t,\lambda)\varphi_k(t,\mu)\,dt,$$

$$\tilde{D}_k(x,\lambda,\mu) := \frac{\langle \tilde{\varphi}_k(x,\lambda), \tilde{\varphi}_k(x,\mu)\rangle}{\lambda-\mu} = \int_0^x \tilde{\varphi}_k(t,\lambda)\tilde{\varphi}_k(t,\mu)\,dt.$$

Lemma 3. *For all $x \in [0,1]$ and $\lambda \in D^+$ the following relation holds*

$$\varphi_k(x,\lambda) = \tilde{\varphi}_k(x,\lambda) + \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\varphi_k(x,\mu)d\mu. \tag{12}$$

Proof. The asymtotics established in lemma 2 allow us to repeat the arguments from [4] that give the following relation

$$\varphi_k(x,\lambda) = \tilde{\varphi}_k(x,\lambda) + \frac{1}{2\pi i}\int_{\Gamma_N^-} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\varphi_k(x,\mu)d\mu + \varepsilon_N(x,\lambda), \tag{13}$$

where $\varepsilon_N(x,\lambda) \to 0$ for $N \to \infty$. The functions $\tilde{D}_k(x,\lambda,\mu)$ and $\varphi_k(x,\mu)$ are bounded for fixed λ and $\mu \in D^-$. Since $\hat{M}_k(\lambda) = \hat{\kappa}(\rho)\rho^{-1} = o(\rho^{-1}),\ \rho\to\infty,\ \rho \in G_\delta$, it follows that

$$\int_{C_N^-} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\varphi_k(x,\mu)d\mu \to 0, \quad N\to\infty.$$

Taking in (13) the limit as $N\to\infty$, we arrive at (12). □

Define

$$\tilde{r}_k(x,\rho,\theta) := \tilde{D}_k(x,\lambda,\mu)\theta\hat{M}_k(\mu),\ r_k(x,\rho,\theta) := D_k(x,\lambda,\mu)\theta\hat{M}_k(\mu),\ \lambda=\rho^2,\ \mu=\theta^2.$$

Everywhere below we assume that τ in the contour $\gamma=\gamma(\tau)$ is chosen such that $\theta\hat{M}_k(\mu) \in L_2(\gamma)$.

Lemma 4. *1) For any $\lambda \in \overline{D^+}$ and $x \in [0,1]$,*

$$\int_\gamma |D_k(x,\lambda,\theta^2)|^2|\,d\theta| < C(x,\rho),$$

where $C(x,\rho)$ is bounded uniformly on any compact set.

2) The following estimates hold

$$\int_\gamma\int_\gamma |r_k(x,\rho,\theta)|^2\,|d\theta|\,|d\rho| < \infty, \qquad \int_\gamma\int_\gamma |\tilde{r}_k(x,\rho,\theta)|^2\,|d\theta|\,|d\rho| < \infty.$$

Proof. 1) Using (5) we split $D_k(x,\lambda,\mu)$ into four terms:

$$D_k(x,\lambda,\mu) = D_{k,11}(x,\lambda,\mu) + D_{k,12}(x,\lambda,\mu) + D_{k,21}(x,\lambda,\mu) + D_{k,22}(x,\lambda,\mu),$$

where

$$D_{k,11}(x,\lambda,\mu) = \int_0^x \cos\rho t\cos\theta t\,dt, \quad D_{k,12}(x,\lambda,\mu) = \int_0^x \cos\rho t\eta(t,\theta)\,dt,$$

$$D_{k,21}(x,\lambda,\mu) = \int_0^x \cos\theta t\eta(t,\rho)\,dt, \quad D_{k,22}(x,\lambda,\mu) = \int_0^x \eta(t,\rho)\eta(t,\theta)\,dt.$$

Direct calculations show

$$|D_{k,11}(x,\lambda,\mu)| < C(x)\exp((|Im\rho|+\tau)x)\left(\frac{1}{|\rho-\theta|+1} + \frac{1}{|\rho+\theta|+1}\right). \tag{14}$$

Using the Cauchy inequality we obtain

$$|D_{k,12}(x,\lambda,\mu)| \le C_0(x,\rho)\|\eta(\cdot,\theta)\|, \tag{15}$$

$$|D_{k,22}(x,\lambda,\mu)| \le \|\eta(\cdot,\rho)\|\|\eta(\cdot,\theta)\|. \tag{16}$$

Moreover, Parseval's identity gives

$$\int_\gamma |D_{k,21}(x,\lambda,\mu)|^2 |\,d\theta| = 2\pi \|\eta(\cdot,\rho)\|^2_{L_2[0,x]}. \tag{17}$$

Note that $C_0(x,\rho) = \left\{\int_0^x |\cos\rho t|^2 dt\right\}^{1/2}$ is bounded uniformly on any compact set. The first part of the lemma follows directly from (14)-(17).

2) It follows from (16), (17) and the relation $\hat{M}_k(\mu) = o(\theta^{-1})$ (see Lemma 1) that

$$\int_\gamma\int_\gamma |D_{k,2s}(x,\lambda,\mu)\theta\hat{M}_k(\mu)|^2\,|d\theta||d\rho| < \infty, \quad s = 1,2.$$

Then (14) yields

$$\int_\gamma |D_{k,11}(x,\lambda,\mu)|^2\,|d\rho| < C_1(x).$$

Taking $\theta\hat{M}_k(\mu) = \kappa(\theta) \in L_2(\gamma)$ into account we conclude that

$$\int_\gamma\int_\gamma |D_{k,11}(x,\lambda,\mu)\theta\hat{M}_k(\mu)|^2\,|d\rho||d\theta| < \infty.$$

Let us now return to $D_{k,12}(x,\lambda,\mu)$. From it's definition and the Parseval's identity it follows that

$$\int_\gamma |D_{k,12}(x,\lambda,\mu)|^2\,|d\rho| = 2\pi\|\eta(\cdot,\theta)\|^2_{L_2[0,x]},$$

and as above we arrive at

$$\int_\gamma\int_\gamma |D_{k,12}(x,\lambda,\mu)\theta\hat{M}_k(\mu)|^2\,|d\rho||d\theta| < \infty.$$

Lemma 4 is proved. ☐

4. Local inverse problems

Fix $k \in \overline{1,p-1}$, and consider the following auxiliary inverse problem.

Problem IP(k). Given $M_k(\lambda)$, construct $\sigma_k(x)$, $x \in [0,1]$.

Recall that the Weyl function $M_k(\lambda)$ is determined in terms of boundary problems B_0 and B_k which are considered on the whole tree T. This means that the problem IP(k) cannot be reduced to the inverse spectral problem on a finite segment in any sense. Nevertheless $M_k(\lambda)$ is found to bring enough information for reconstructing the potential on the edge e_k.

Theorem 2. *If $M_k = \tilde{M}_k$ then $\sigma_k = \tilde{\sigma}_k$.*

Proof. Consider again the matrix $P^k(x,\lambda)$ introduced in previous section. Since, according to (4), we have $\Phi_{kk}(\lambda) = s_k(x,\lambda) + M_k(\lambda)\varphi_k(x,\lambda)$, the elements $P^k_{1s}(x,\lambda)$ can be rewritten in the form:

$$P^k_{11}(x,\lambda) = \varphi_k(x,\lambda)\tilde{s}^{[1]}_k(x,\lambda) - s_k(x,\lambda)\tilde{\varphi}^{[1]}_k(x,\lambda) - \hat{M}_k(\lambda)\varphi_k(x,\lambda)\tilde{\varphi}^{[1]}_k(x,\lambda),$$

$$P^k_{12}(x,\lambda) = \tilde{\varphi}_k(x,\lambda)s_k(x,\lambda) - \tilde{s}_k(x,\lambda)\varphi_k(x,\lambda) + \hat{M}_k(\lambda)\varphi_k(x,\lambda)\tilde{\varphi}_k(x,\lambda).$$

If $M_k = \tilde{M}_k$ then the functions $P^k_{1s}(x,\lambda)$ are entire in λ. Using lemma 2 and Liouville's theorem we get $P^k_{11}(x,\lambda) \equiv 1$, $P^k_{12}(x,\lambda) \equiv 0$. By virtue of definition of the matrix $P^k(x,\lambda)$ this yields $\varphi_k(x,\lambda) \equiv \tilde{\varphi}_k(x,\lambda)$ and, consequently, $\sigma_k = \tilde{\sigma}_k$. ☐

Our next goal is a constructive procedure for solving the problem IP(k). Let the Weyl function $M_k(\lambda)$ be given. Take a potential $\tilde{\sigma} = 0$, and consider relation (12). We are going to transfer (12) to an equation in the Hilbert space $L_2(\gamma)$. For this purpose we first split the integral term into two parts:

$$\int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\varphi_k(x,\mu)\,d\mu = \int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\hat{\varphi}_k(x,\mu)\,d\mu$$

$$+ \int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\tilde{\varphi}_k(x,\mu)\,d\mu.$$

Note that

$$\lim_{N\to\infty} \int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\hat{\varphi}_k(x,\mu)\,d\mu = \int_{\Gamma} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\hat{\varphi}_k(x,\mu)\,d\mu,$$

and by virtue of the first part of Lemma 4, the integral in the right hand side converges absolutely and uniformly with respect to λ on any compact subset of $\overline{D^+}$. Then take the limit in (12) as $\lambda \to \Gamma$, and define

$$\Psi_k(x,\rho) := \varphi_k(x,\lambda) - \tilde{\varphi}_k(x,\lambda),$$

$$\tilde{F}_k(x,\rho) := \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\Gamma_N} \tilde{D}_k(x,\lambda,\mu)\hat{M}_k(\mu)\tilde{\varphi}_k(x,\mu)d\mu, \quad \lambda = \rho^2. \tag{18}$$

It follows from (5) that $\Psi_k(x,\cdot) \in L_2(\gamma)$. Making the change of variables $\mu = \theta^2$ and using the notation $\tilde{r}_k(x,\rho,\theta)$ defined above we arrive at the relation

$$\Psi_k(x,\rho) = \frac{1}{\pi i} \int_{\gamma} \tilde{r}_k(x,\rho,\theta)\Psi_k(x,\theta)\,d\theta + \tilde{F}_k(x,\rho). \tag{19}$$

For each fixed $x \in [0,1]$, relation (19) can be considered as a linear integral equation in $L_2(\gamma)$ with respect to $\Psi_k(x,\rho)$. Equation (19) is called the *main equation* for IP(k). It is convenient for us to rewrite the main equation in the operator form. For each fixed $x \in [0,1]$, we define the linear operators $\tilde{H}_k(x)$ and $H_k(x)$ in $L_2(\gamma)$ as follows:

$$\tilde{H}_k(x)f(\rho) := \frac{1}{\pi i} \int_{\gamma} \tilde{r}_k(x,\rho,\theta)f(\theta)\,d\theta, \quad H_k(x)f(\rho) := \frac{1}{\pi i} \int_{\gamma} r_k(x,\rho,\theta)f(\theta)\,d\theta.$$

Then the main equation becomes

$$\Psi_k(x) = \tilde{H}_k(x)\Psi_k(x) + \tilde{F}_k(x). \tag{20}$$

By virtue of the second part of Lemma 4, $\tilde{H}_k(x)$ is a Hilbert-Schmidt operator in $L_2(\gamma)$.

THEOREM 3. *For each fixed* $x \in [0,1]$, *equation (20) is uniquely solvable in* $L_2(\gamma)$.

PROOF. Using estimates from lemma 2 and repeating the arguments from [4] we obtain for any $\lambda, \mu \in \overline{D^+}$:

$$D_k(x,\lambda,\mu) - \tilde{D}_k(x,\lambda,\mu) = \frac{1}{2\pi i} \int_{\Gamma_N^-} \hat{M}_k(\xi)\tilde{D}_k(x,\lambda,\xi)D_k(x,\xi,\mu)\,d\xi + \varepsilon_N^1(x,\lambda,\mu),$$

where $\lim_{N\to\infty} \varepsilon_N^1(x,\lambda,\mu) = 0$. Similarly to the proof of Lemma 2 one gets

$$D_k(x,\lambda,\mu) - \tilde{D}_k(x,\lambda,\mu) = \frac{1}{2\pi i} \int_{\Gamma} \hat{M}_k(\xi)\tilde{D}_k(x,\lambda,\xi)D_k(x,\xi,\mu)d\xi.$$

By virtue of Lemmas 1 and 3 the integral in the right-hand side converges absolutely and uniformly in λ, μ on any compact subsets of Γ. Make the change of variables $\xi = \zeta^2$ in the integral term and multiply both sides by $\theta \hat{M}_k(\mu)$. This yields

$$r_k(x,\rho,\theta) - \tilde{r}_k(x,\rho,\theta) = \frac{1}{\pi i}\int_\gamma \tilde{r}_k(x,\rho,\zeta) r_k(x,\zeta,\theta)\, d\zeta,$$

which is equivalent to the operator relation

$$(E - \tilde{H}_k(x))(E + H_k(x)) = E,$$

where E is the identity operator. Interchange the places for σ and $\tilde{\sigma}$ and repeating the arguments above we obtain

$$(E + H_k(x))(E - \tilde{H}_k(x)) = E.$$

Thus, the operator $(E - \tilde{H}_k(x))$ is invertible, and the theorem is proved. □

Using the solution $\Psi_k(x,\rho)$ of the main equation, one can calculate the function $\varphi_k(x,\lambda)$ and then construct $\sigma_k(x)$, according to the next theorem.

THEOREM 4. *The solution* $\sigma_k(x)$ *of the Problem IP(k) can be found by the formula*

$$\sigma_k(x) = -m_k(x) - \frac{1}{\pi i}\int_\Gamma \hat{\varphi}_k(x,\lambda)\tilde{\varphi}_k(x,\lambda)\hat{M}_k(\lambda)\, d\lambda, \tag{21}$$

where

$$m_k(x) = \frac{1}{\pi i} l.i.m._{N\to\infty} \int_{\gamma_N} \rho \hat{M}_k(\rho^2)\cos 2\rho x\, d\rho. \tag{22}$$

PROOF. In view of (5) and Lemma 1, the integral in the right-hand side in (21) converges absolutely and uniformly for all $x \in [0,1]$. The function $m_k(x)$, defined in (22) via the cosine Fourier transform of the function $\rho\hat{M}_k(\rho^2) \in L_2(\gamma)$, is $L_2[0,1]$– function. Hence, the right-hand side in (21) is defined correctly as a function $\sigma_k^*(x) \in L_2[0,1]$. Our goal now is to prove that $\sigma_k^*(x) = \sigma_k(x)$.

Consider sequences $\{\sigma_{nj}\}_{n=\overline{1,\infty}}$, $j \in \overline{1,p}$ such that $\sigma_{nj}(x) \in W_2^1[0,1]$, $\sigma_{nj} \to \sigma_j$ for $n \to \infty$ in $L_2(T)$ and $\sigma_{nj}(0) = 0$. Then (see [17]) $\sigma_{nj}(x)$ is given by (21), and

$$\sigma_k^*(x) - \sigma_{nk}(x) = m_k^{(n)}(x) - m_k(x)$$
$$+\frac{1}{\pi i}\int_\Gamma \tilde{\varphi}_k(x,\lambda)\left(\hat{\varphi}_k^{(n)}(x,\lambda)\hat{M}_k^{(n)}(\lambda) - \hat{\varphi}_k(x,\lambda)\hat{M}_k(\lambda)\right) d\lambda.$$

(Here if a symbol α denotes an object related to $\{\sigma_j\}$, then $\alpha^{(n)}$ denotes the analogous object related to $\{\sigma_{nj}\}$). One can rewrite this relation as follows

$$\sigma_k^*(x) - \sigma_{n,k}(x) = m_k^{(n)}(x) - m_k(x)$$
$$+\frac{1}{\pi i}\int_\Gamma \tilde{\varphi}_k(x,\lambda)\hat{M}_k(\lambda)\left(\hat{\varphi}_k^{(n)}(x,\lambda) - \hat{\varphi}_k(x,\lambda)\right) d\lambda +$$
$$\frac{1}{\pi i}\int_\Gamma \tilde{\varphi}_k(x,\lambda)\hat{\varphi}_k^{(n)}(x,\lambda)\left(\hat{M}_k^{(n)}(\lambda) - \hat{M}_k(\lambda)\right) d\lambda.$$

By virtue of (5) and Lemma 1, this yields

$$\|\sigma_k^* - \sigma_{n,k}\| < \|m_k^{(n)} - m_k\|$$
$$+C\left(\max_{x\in[0,1]} \|\varphi_k^{(n)}(x,\rho^2) - \varphi_k(x,\rho^2)\|_{L_2(\gamma)} + \|M_k^{(n)} - M_k\|_{L_2(\Gamma)}\right).$$

Using again (5), Lemma 1 and the continuity of the Fourier transform we deduce that $\|\sigma_k^* - \sigma_{n,k}\| \to 0$ for $n \to \infty$. Since $\sigma_k(x) = \lim_{n\to\infty} \sigma_{n,k}(x)$ in $L_2[0,1]$, we arrive at the required identity $\sigma_k^*(x) = \sigma_k(x)$. ☐

Thus, the solution of the auxiliary problem IP(k) can be constructed by the following algorithm.

Algorithm 1. Given $M_k(\lambda)$.
1) Take $\tilde\sigma = 0$ and calculate $\tilde\varphi_k(x,\lambda)$, $\tilde M_k(\lambda)$, $\tilde D_k(x,\lambda,\mu)$ and $\tilde r_k(x,\rho,\theta)$.
2) Construct $\tilde F_k(x,\rho)$ by (18).
3) Find $\Psi_k(x,\rho)$ by solving the main equation (19) for each $x \in [0,1]$.
4) Construct $\sigma_k(x)$ using (21), (22) where $\hat\varphi_k(x,\lambda) = \Psi_k(x,\rho)$.

5. Solution of Inverse Problem 1

Algorithm 1 described in the previous section allows us to reconstruct the potentials $\sigma_k(x), k = \overline{1,p-1}$ from the given Weyl vector $M(\lambda)$. The final step in solving the Inverse Problem 1 is to reconstruct the potential $\sigma_p(x)$. For this purpose we shall formulate and solve some inverse spectral problem on the remaining edge e_p.

Define $\Phi_0(x,\lambda)$ as the solution of the following boundary value problem:

$$\ell_p\Phi_0 = \lambda\Phi_0,\ \Phi_0^{[1]}(1,\lambda) = 1,\ U_1(\Phi_0) = 0, \tag{23}$$

and let $M_0(\lambda) := -\Phi_0(1,\lambda)$.

Problem IP(0). Given $M_0(\lambda)$ construct $\sigma_p(x)$.

It is clear that the problem IP(0) can be treated in the same way as the problem IP(k). The analogous considerations show that it has a unique solution and can be solved using the following algorithm.

Algorithm 2. Given $M_0(\lambda)$.
1) Calculate $\hat M_0(\lambda) := M_0(\lambda) - \rho^{-1}\cot\rho$, $\tilde D_0(x,\lambda,\mu) := \int_0^x \cos\rho t \cos\theta t\, dt$,

$$\tilde F_0(x,\rho) := \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\Gamma_N} \tilde D_0(x,\lambda,\mu)\hat M_0(\mu)\tilde\varphi_0(x,\mu)d\mu, \quad \lambda = \rho^2, \mu = \theta^2.$$

2) Find $\Psi_0(x,\rho)$ by solving for each $x \in [0,1]$ the main equation (19) with $k = 0$.

3) Construct $\sigma_p(x)$ by the formulae

$$-\sigma_p(1-x) = -m_0(x) - \frac{1}{\pi i}\int_\Gamma \Psi_0(x,\rho)\cos\rho x \hat M_0(\lambda)\, d\lambda,$$

$$m_0(x) = \frac{1}{\pi i} l.i.m._{N\to\infty}\int_{\gamma_N} \rho\hat M_0(\rho^2)\cos 2\rho x\, d\rho.$$

Now we show how to obtain the input data for the Problem IP(0) from $\sigma_k(x),\ k = \overline{1,p-1}$. It follows from (23) and (4) that $\Phi_0(x,\lambda) = \alpha_{0k}(\lambda)\Phi_{kp}(x,\lambda)$ for any $k \in \overline{1,p-1}$. This yields

$$M_0(\lambda) = -\frac{\Phi_0(1,\lambda)}{\Phi_0^{[1]}(1,\lambda)} = -\frac{\Phi_{kp}(1,\lambda)}{\Phi_{kp}^{[1]}(1,\lambda)}. \tag{24}$$

In order to calculate the right-hand side of (24) we use the matching conditions (2), (3). Recalling that for $j \neq k$ the function $\Phi_{kj}(x,\lambda)$ is proportional to $\varphi_j(x,\lambda)$, we obtain from (2):

$$\Phi_{kj}(x,\lambda) = \frac{\Phi_{kk}(1,\lambda)}{\varphi_j(1,\lambda)}\varphi_j(x,\lambda), \quad j \in \overline{1,p-1} \setminus k, \tag{25}$$

$$\Phi_{kp}(1,\lambda) = \Phi_{kk}(1,\lambda), \tag{26}$$

where $k \in \overline{1,p-1}$ is arbitrarily fixed. Analogously (3) gives

$$\Phi^{[1]}_{kp}(1,\lambda) = -\sum_{j=1}^{p-1} \Phi^{[1]}_{kj}(1,\lambda). \tag{27}$$

Relations (24)-(27) allow us to find $M_0(\lambda)$ in terms of the known $\sigma_k(x),\ k = \overline{1,p-1}$.

The considerations above show that the solution of Inverse Problem 1 is uniquely determined, and Theorem 1 is proved. The constructive procedure for solving the problem is described in the following algorithm.

Algorithm 3. Given the Weyl vector $M(\lambda)$.

1) For each fixed $k = \overline{1,p-1}$, construct $\sigma_k(x)$, $x \in [0,1]$, by solving the inverse problem IP(k) according to Algorithm 1.

2) Calculate $\varphi_k(x,\lambda), \Phi_{kk}(x,\lambda),\ k = \overline{1,p-1}$, as the corresponding solutions of the Sturm-Liouville equation on the edge e_k with the known potential $\sigma_k(x)$.

3) Fix $k = \overline{1,p-1}$. For $j = \overline{1,p-1} \setminus k$, find $\Phi_{kj}(x,\lambda)$ from (25).

4) Calculate $\Phi_{kp}(1,\lambda)$ and $\Phi^{[1]}_{kp}(1,\lambda)$ via (26), (27).

5) Find $M_0(\lambda)$ using (24).

6) Construct $\sigma_p(x),\ x \in [0,1]$ by solving the problem IP(0) according to Algorithm 2.

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Fachbereich Mathematik, Universität Duisburg-Essen, Campus Duisburg, D-47048 Duisburg, Germany
E-mail address: `gerhard.freiling@uni-due.de`

Department of Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410026, Russia
E-mail address: `ignatievmu@info.sgu.ru`

Department of Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410026, Russia
E-mail address: `yurkova@info.sgu.ru`