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NUMERICAL SOLUTION OF THE INVERSE SYNTHESIS PROBLEM FOR STRATIFIED MEDIA FROM INCOMPLETE SPECTRAL INFORMATION

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Abstract

A numerical algorithm for solving an inverse synthesis problem from incomplete spectral information is described. Some results of numerical experiments are also presented.

1. Introduction

The paper deals with the following system of differential equations

$$\frac{dy_1}{dx} = i\rho R(x) y_2, \quad \frac{dy_2}{dx} = i\rho \frac{1}{R(x)} y_1, \quad x \in [0, T], \tag{1}$$

with the initial conditions $y_1(0, \rho) = 1$, $y_2(0, \rho) = -1$. Here $\rho = k + i\tau$ is

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the spectral parameter, and $R(x)$ is a real function which is called the wave resistance.

System (1) is a canonical form for many problems in natural sciences. It describes the wave propagation in a stratified medium and often appears in optics, spectroscopy, in electrodynamic and acoustic problems. Radio engineering problems of the design of directional couplers for non-uniform electronics lines and synthesizing transitions between acoustic wave guides can also be reduced to studying system (1) (see [5], [7], [9] and the references therein).

Some of these classes of synthesis problems relate to inverse problems of spectral analysis with incomplete spectral information. Several aspects of synthesis problems for system (1) were studied in [2], [5] and [10]. In this paper we shall use partially the results derived in [2], [10]. We note that the general theory of inverse spectral problems for ordinary differential equations is reflected in the monographs [1], [3], [4], [6], [8] and [11].

In the present paper we formulate a numerical algorithm for recovering the wave resistance from incomplete spectral information. Moreover we carry out some numerical experiments using a computer program that we developed and which is based on this numerical algorithm. This program is available from the authors (for using the program the user should be familiar with the C++ language and with the .NET platform).

We shall say that $R(x) \in B_{0,}$ if $R(x)$, $R'(x)$ are absolutely continuous on $[0, T]$, $R'' \in L_2(0, T)$, $R(x) > 0$, $R(0) = 1$, $R'(0) = 0$. Let $R(x) \in B_0$. Consider the functions

$$f_1(\rho) = \frac{y_1(T, \rho) - R^0 y_2(T, \rho)}{2\sqrt{R^0}}, \quad f_2(\rho) = \frac{y_1(T, \rho) + R^0 y_2(T, \rho)}{2\sqrt{R^0}}, \quad R^0 := R(T),$$

which are called the transmission coefficients. Denote

$$\alpha_j(k) = \left| f_j(k) \right|, \quad k := \operatorname{Re} \rho.$$

It is known (see [2], [5], [10]) that

$$\alpha_1^2(k) - \alpha_2^2(k) \equiv 1. \tag{2}$$

In many cases of practical interest, the phase is difficult or impossible to measure, while the amplitude is easily accessible to measurement. Such cases lead us to inverse spectral problems with incomplete information. In this paper we study the following incomplete inverse problem:

Inverse Problem 1. Given $\alpha_1(k)$, construct $R(x)$.

In order to solve this inverse problem we first should reconstruct the transmission coefficients from their modulus $\alpha_j(k)$ on the real line. Here we face with a lack of information which leads to the non-uniqueness of the solution. For the construction of the transmission coefficients we use an additional information about their zeros. Further we calculate the so-called characteristic function:

$$\Delta(\rho) = f_1(\rho) + f_2(\rho), \tag{3}$$

and obtain the solution of the inverse problem of recovering $R(x)$ from the characteristic function $\Delta(\rho)$.

The paper is organized as follows. For convenience of the reader, in Section 2 we provide briefly the results from [2], [5], [10] which will be used for constructing a numerical algorithm for the solution of Inverse Problem 1. In Section 3 we describe a numerical algorithm for the solution of Inverse Problem 1 and provide the results of some numerical experiments.

2. Synthesis of $R(x)$ from the Modulus of the Transmission Coefficients

In this section we briefly formulate necessary results that have been derived in [2, 5, 10]. Let us introduce the functions

$$u(x, \rho) = \frac{1}{\sqrt{R(x)}} y_1(x, \rho), \quad v(x, \rho) = \sqrt{R(x)} y_2(x, \rho), \quad h(x) = \frac{R'(x)}{2R(x)}.$$

Then the transmission coefficients and the characteristic function can be written in the form

$$f_1(\rho) = \frac{u(T, \rho) - v(T, \rho)}{2}, \quad f_2(\rho) = \frac{u(T, \rho) + v(T, \rho)}{2}, \quad \Delta(\rho) = u(T, \rho).$$

The function $u(x, \rho)$ is the solution of the Cauchy problem

$$-u'' + q(x)u = \rho^2 u, \quad u(0, \rho) = 1, \quad u'(0, \rho) = -i\rho,$$

where $q(x) = h^2(x) - h'(x)$. Similarly,

$$-v'' + g(x)v = \rho^2 v, \quad v(0, \rho) = -1, \quad v'(0, \rho) = i\rho,$$

where $g(x) = h^2(x) + h'(x)$. Moreover, $u' + h(x)u = i\rho v$, $v' - h(x)v = i\rho u$. For definiteness, in the sequel, let $h := h(T) \neq 0$. Denote $\prod_+ := \{\rho : \operatorname{Im} \rho > 0\}$.

Theorem 1. *The functions* $f_1(\rho)$ *and* $\Delta(\rho)$ *are entire in* ρ, *and have the form*

$$f_1(\rho) = \exp(-i\rho T) + \int_{-T}^{T} g_1(t)\exp(-i\rho t)dt,$$

$$g_1(t) \in AC[-T, T], \quad g_1'(t) \in L_2(-T, T), \tag{4}$$

$$f_2(\rho) = \int_{-T}^{T} g_2(t)\exp(-i\rho t)dt,$$

$$g_2(t) \in AC[-T, T], \ g_2'(t) \in L_2(-T, T), \tag{5}$$

$$\Delta(\rho) = \exp(-i\rho T) + \int_{-T}^{T} \eta(t)\exp(-i\rho t)dt,$$

$$\eta(t) \in AC[-T, T], \ \eta'(t) \in L_2(-T, T), \tag{6}$$

where $g_j(t)$, $\eta(t)$ *are real, and*

$$\eta(T) = \frac{1}{2}\int_0^T q(t)dt, \quad g_1(T) = \frac{1}{2}\int_0^T h^2(t)dt,$$

$$g_2(T) = -\frac{h}{2}, \quad g_j(-T) = \eta(-T) = 0.$$

Moreover, the functions $f_1(\rho)$ *and* $\Delta(\rho)$ *have no zeros in* $\overline{\prod_+}$.

It follows from (4) that

$$\alpha_1^2(k) = 1 + \frac{h^2}{4k^2} + \frac{\omega(k)}{k^2}, \quad \omega(k) \in L_2(-\infty, \infty). \tag{7}$$

Let us consider an auxiliary inverse problem of recovering the wave resistance $R(x) \in B_0$ from the given characteristic function $\Delta(\rho)$. The following results can be found in [2].

Theorem 2. *The specification of the characteristic function* $\Delta(\rho)$ *uniquely determines the wave resistance* $R(x)$.

Theorem 3. *For a function* $\Delta(\rho)$ *of the form* (6) *with* $\eta(-T) = 0$, *to be the characteristic function for a certain* $R(x) \in B_0$, *it is necessary and sufficient that* $\Delta(\rho)$ *has no zeros in* $\overline{\Pi_+}$.

The wave resistance $R(x)$ can be constructed from $\Delta(\rho)$ by the following algorithm.

Algorithm 1. Let a function $\Delta(\rho)$ satisfying the hypothesis of Theorem 3 be given. Then

(1) Construct the function $F(t), t \in (0, 2T)$ either from the linear integral equation

$$\theta(t) + F(t) + \int_t^{2T} \theta(s - t)F(s)ds = 0, \quad t \in (0, 2T). \tag{8}$$

where $\theta(t) = \eta(T - t)$, or directly by the formula

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - s(\rho))\exp(i\rho t)\, d\rho, \tag{9}$$

where $s(\rho) := (\Delta(\rho))^{-1}\Delta(-\rho)(-2i\rho T)$.

(2) For each fixed $x \in [0, T]$, find $G(x, t)$ by solving the linear integral equation

$$G(x, t) + F(x + t) + \int_x^{2T-t} F(t + s)G(x, s)ds = 0,$$

$$0 \le x \le T, \; x < t < 2T - x. \tag{10}$$

(3) Calculate $R(x)$ by

$$R(x) = \frac{1}{e^2(T-x)}, \quad e(x) = 1 + \int_x^{2T-x} G(x, t)dt. \tag{11}$$

For constructing the transmission coefficients from their modulus we will use the following theorems from [2].

Theorem 4. *Let*

$$f_1(k) = \alpha_1(k)e^{-i\delta_1(k)}. \tag{12}$$

Then

$$\delta_1(k) = kT + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - k}d\xi. \tag{13}$$

In (13) (*and everywhere below, where necessary*) *the integral is understood in the principal value sense.*

Theorem 5. *Let*

$$f_2(k) = \alpha_2(k)e^{-i\delta_2(k)}, \tag{14}$$

and let $\rho_j = k_j + i\tau_j,\ \tau_j > 0,\ j = \overline{1, m}$ *be zeros of* $f_2(\rho)$ *in* $\prod_+$. *Then*

$$\delta_2(k) = \frac{\pi}{2}\operatorname{sign}\left(\frac{k}{h}\right) + kT + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln\left|\frac{2\xi}{h}\alpha_2(\xi)\right|}{\xi - k}d\xi + 2\sum_{j=1}^{m} \operatorname{arctg}\frac{\tau_j}{k - k_j}. \tag{15}$$

In particular, if $f_2(\rho)$ *has no zeros in* $\prod_+$, *then*

$$\delta_2(k) = \frac{\pi}{2}\operatorname{sign}\left(\frac{k}{h}\right) + kT + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln\left|\frac{2\xi}{h}\alpha_2(\xi)\right|}{\xi - k}d\xi. \tag{16}$$

Now we provide a procedure for the construction of $R(x)$ from the given $\alpha_1(k)$. Let the function $\alpha_1(k)$ $(\alpha_1(k) \geq 1,\ \alpha_1(k) = \alpha_1(-k))$ be given.

Our scheme of calculation is as follows:

Step 1. Construct $f_1(k)$ by (12), (13). Find $g_1(t)$ from (4).

Step 2. Construct $f_2(k)$ by (14), (15) or, if $f_2(\rho)$ has no zeros in $\prod_+$, by (14), (16). Find $g_2(t)$ from (5).

Step 3. In view of (3), calculate $\eta(t) = g_1(t) + g_2(t)$ and $\Delta(\rho)$ by (6).

Step 4. Construct $R(x)$ using Algorithm 1.

3. Numerical Algorithm and its Realization

In this section, using the results from Section 2, we provide a concrete numerical algorithm which realizes the scheme from Section 2.

For $T \in [0, T]$ we consider the functions

$$\varphi_j(t) = g_j(t) + g_j(-t), \quad \psi_j(t) = g_j(t) - g_j(-t), \quad j = 1, 2, \tag{17}$$

$$\varphi(t) = \eta(t) + \eta(-t), \quad \psi(t) = \eta(t) - \eta(-t). \tag{18}$$

Since $\eta(t) = g_1(t) + g_2(t)$, we get

$$\psi(t) = \varphi_1(t) + \varphi_2(t), \quad \psi(t) = \psi_1(t) + \psi_2(t). \tag{19}$$

Solving (17) and (18) with respect to $g_j(t)$ and $\eta(t)$ we obtain

$$g_j(t) = \begin{cases} \frac{1}{2}(\varphi_j(t) + \psi_j(t)), & t > 0, \\ \frac{1}{2}(\varphi_j(-t) - \psi_j(-t)), & t < 0, \end{cases} \qquad \eta(t) = \begin{cases} \frac{1}{2}(\varphi(t) + \psi(t)), & t > 0, \\ \frac{1}{2}(\varphi(-t) + \psi(-t)), & t < 0, \end{cases} \tag{20}$$

It follows from (17) that

$$\varphi_1(T) = \psi_1(T) = -w_1, \quad \varphi_2(T) = \psi_2(T) = -\frac{h}{2}, \quad \psi_1(0) = \psi_2(0) = 0, \tag{21}$$

where

$$w_1 = -\frac{1}{2}\int_0^T h^2(\xi)\,d\xi.$$

By virtue of (4)-(6),

$$f_1(k) = (\cos kT + C_1(k)) - i(\sin kT + S_1(k)), \quad f_2(k) = C_2(k) - iS_2(k),$$

where

$$C_j(k) = \int_0^T \varphi_j(t)\cos kt\,dt, \qquad S_j(k) = \int_0^T \psi_j(t)\sin kt\,dt,$$

For calculating the arguments of the transmission coefficients we will use (13) and (16), i.e.,

$$\delta_1(k) = kT + \tilde{\delta}_1(k),\ \delta_2(k) = \frac{\pi}{2}w + kT + \tilde{\delta}_2(k),\ k > 0,$$

where $w = \operatorname{sign} h$,

$$\tilde{\delta}_1(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - k}\, d\xi, \tag{22}$$

$$\tilde{\delta}_2(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln \tilde{\alpha}_2(\xi)}{\xi - k}\, d\xi, \quad \tilde{\alpha}_2(\xi) := \left|\frac{2\xi}{h}\alpha_2\xi\right|. \tag{23}$$

Furthermore, consider the functions

$$\left.\begin{aligned} \varphi_1^*(t) &= \varphi_1(t) + w_1, \quad \psi_1^*(t) = \psi_1(t) + w_1\frac{t}{T}, \\ \varphi_2^*(t) &= \varphi_2(t) + \frac{h}{2}, \quad \psi_2^*(t) = \psi_2(t) + \frac{ht}{2T}. \end{aligned}\right\} \tag{24}$$

Then in view of (21),

$$\varphi_j^*(T) = \psi_j^*(T) = \psi_j^*(0) = 0, \quad j = 1, 2. \tag{25}$$

Denote

$$C_j^*(k) = \int_0^T \varphi_j^*(t)\cos kt\, dt, \quad S_j^*(k) = \int_0^T \psi_j^*(t)\sin kt\, dt. \tag{26}$$

Integrating the integrals in (26) by parts and taking (25) into account, we obtain

$$C_j^*(k) = -\int_0^T \varphi_j^{*\prime}(t)\frac{\sin kt}{k}\, dt, \quad S_j^*(k) = \int_0^T \psi_j^{*\prime}(t)\frac{\cos kt}{k}\, dt. \tag{27}$$

Then, taking into account the asymptotics for $C_j(k)$ and $S_j(k)$, we arrive at

$$C_1(k) = C_1^*(k) - w_1\frac{\sin kT}{k}, \quad S_1(k) = S_1^*(k) + w_1\left(\frac{\cos kT}{k} - \frac{\sin kT}{Tk^2}\right), \tag{28}$$

$$C_2(k) = C_2^*(k) - \frac{h}{2}\frac{\sin kT}{k}, \quad S_2(k) = S_2^*(k) + \frac{h}{2}\left(\frac{\cos kT}{k} - \frac{\sin kT}{Tk^2}\right). \tag{29}$$

Now let $\alpha_1(k)$ $(\alpha_1(k) \geq 1, \alpha_1(k) = \alpha_1(-k))$ be given for $|k| \leq B$, and put

$$\alpha_1^2(k) = 1 + \frac{h^2}{4k^2}, \quad |k| > B, \tag{30}$$

where $B > |h|$ is chosen sufficiently large, such that (see (7)) $\alpha_1(k)$ is sufficiently accurate for $|k| > B$. Using (2) we calculate $\alpha_2(k)$. Then $\alpha_2(k) = \alpha_2(-k) > 0$, and

$$\alpha_2^2(k) = \frac{h^2}{4k^2}, \; |k| > B. \tag{31}$$

In order to calculate $\tilde{\delta}_1(k)$ we use (22). First let $|k| > B + \chi$, $\chi > 0$. In view of (31), equality (25) takes the form

$$\tilde{\delta}_1(k) = \frac{1}{\pi}\int_{-B}^{B} \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi + \frac{1}{2\pi}\int_{|\xi|>B} \frac{\ln\left(1 + \dfrac{h^2}{4\xi^2}\right)}{\xi - k} d\xi.$$

Representing integrals in form of sums and separating the terms of order $1/k$, we arrive at

$$\tilde{\delta}_1(k) = \frac{w_1}{k} + \tilde{\delta}_1^*(k), \tag{32}$$

where

$$\tilde{\delta}_1^*(k) = \frac{1}{\pi}\int_{-B}^{B} \frac{\ln \alpha_1(\xi)}{k(\xi - k)} d\xi + \frac{1}{2\pi}\ln\left(1 + \frac{2B}{k - B}\right)\ln\left(1 + \frac{h^2}{4k^2}\right)$$

$$+ \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{h^2}{4}\right)^j \sum_{\mu=0}^{j-1}\frac{1}{(2j - 2\mu - 1)B^{2j-2\mu-1}k^{2\mu+1}}, \quad |k| > B + \chi, \tag{33}$$

$$w_1 = -\frac{1}{\pi}\int_{-B}^{B} \ln \alpha_1(\xi) d\xi \, \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{h^2}{4B^2}\right)^j \cdot \frac{B}{2j - 1} < 0. \tag{34}$$

Now let $|k| \leq B + \chi$. In this case we rewrite (22) in the form

$$\tilde{\delta}_1(k) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\ln \alpha_1(\xi + k) - \ln \alpha_1(\xi - k)}{\xi} d\xi. \tag{35}$$

Transforming the integral in (35), we obtain

$$\tilde{\delta}_1(k) = \frac{1}{\pi}\int_0^r \frac{\ln\alpha_1(\xi + k) - \ln\alpha_1(\xi - k)}{\xi}\,d\xi + \frac{h^2}{8\pi}\left(\frac{1}{k^2}\ln\frac{r + k}{r - k} - \frac{1}{k}\left(\frac{1}{r + k} + \frac{1}{r - k}\right)\right)$$

$$+ \varepsilon(k),\ \ \tilde{\delta}_1(0) = 0,\ \ |k| \le B + \chi,\ r \ge 2B + \chi,\ \ |\varepsilon(k)| \le \frac{h^4}{96\pi r(r - B - \chi)^3}. \quad (36)$$

In order to calculate $\delta_2(k)$ we use (23). Let $|k| > B + \chi,\ \chi > 0$. According to (31) we have $\tilde{\alpha}_2(k) = 1$ for $|k| > B$. Hence

$$\tilde{\delta}_2(k) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\tilde{\alpha}_2(\xi)}{\xi - k}\,d\xi,\ \ |k| > B + \chi. \quad (37)$$

For $|k| \le B + \chi$ it is more convenient to use another formula:

$$\tilde{\delta}_2(k) = \frac{1}{\pi}\int_0^r \frac{\ln\tilde{\alpha}_2(\xi + k) - \ln\tilde{\alpha}_2(\xi - k)}{\xi}\,d\xi,\ \ |k| \le B + \chi,\ r = 2B + \chi. \quad (38)$$

Let us calculate $\psi_j^*(t)$, φ_j^*. By virtue of (26),

$$\psi_j^*(t) = \frac{2}{T}\sum_{n=1}^{\infty} S_j^*\left(\frac{n\pi}{T}\right)\sin\frac{n\pi}{T}t,$$

$$\varphi_j^*(t) = \frac{1}{T}C_j^*(0) + \frac{2}{T}\sum_{n=1}^{\infty} C_j^*\left(\frac{n\pi}{T}\right)\cos\frac{n\pi}{T}t. \quad (39)$$

According to (27), the series in (39) converge absolutely and uniformly. By virtue of (28), (29), the coefficients $S_j^*\left(\frac{n\pi}{T}\right)$ and $C_j^*\left(\frac{n\pi}{T}\right)$ can be calculated via the formulae

$$\left.\begin{aligned}
S_1^*(k) &= (-1)^n\left(\alpha_1(k)\sin\tilde{\delta}_1(k) - \frac{w_1}{k}\right), && k = \frac{n\pi}{T},\ n \ge 1,\\
S_2^*(k) &= (-1)^n\left(\alpha_2(k) w\cos\tilde{\delta}_2(k) - \frac{h}{2k}\right), && k = \frac{n\pi}{T},\ n \ge 1,\\
C_1^*(k) &= (-1)^n(\alpha_1(k)\cos\tilde{\delta}_1(k) - 1) + \delta_{n0}w_1 T, && k = \frac{n\pi}{T},\ n \ge 0,\\
C_2^*(k) &= (-1)^{n+1} w\sin\tilde{\delta}_2(k) + \delta_{n0}\frac{hT}{2}, && k = \frac{n\pi}{T},\ n \ge 0,
\end{aligned}\right\} \quad (40)$$

where δ_{nj} is the Kronecker delta.

Using the formulas obtained above we arrive at the following numerical algorithm for the solution of Inverse Problem 1.

Algorithm 2. Let the constants B, h and χ be given along with a desired accuracy ε. The function $\alpha_1(k)$ is given for $k_i = \frac{B}{N} i$, $i = -N, \dots, N$. Take $\alpha_1(k)$ according to (30) for $|k| > B$. Then

(1) Construct $\alpha_2(k)$ by (2).

(2) Find the number w_1 by (34).

(3) Calculate $\tilde{\delta}_1(k)$, $\tilde{\delta}_2(k)$ by (32), (33), (37) for $|k| > B + \chi$, and by (36), (38) for $|k| \le B + \chi$.

(4) Construct $\varphi_j^*(t)$, $\psi_j^*(t)$ by (39), where $S_j^*\left(\frac{n\pi}{T}\right)$, $C_j^*\left(\frac{n\pi}{T}\right)$ is defined by (40).

(5) Find $\varphi_j(t)$, $\psi_j(t)$ using (24).

(6) Calculate $\eta(t)$, $t \in [-T, T]$ by (19), (20).

(7) Find $F(t)$, $0 < t < 2T$ from the integral equation (8), where $\theta(t) = \eta(T - t)$, or by (9).

(8) Calculate $G(x, t)$ from the integral equation (10).

(9) Construct $R(x)$, $x \in [0, T]$ via (11).

We note that the solution of inverse problem 1 is not unique. Algorithm 2 finds one of the possible solutions of the inverse problem, namely it finds the wave resistance $R(x)$ for which the transmission coefficient $f_2(\rho)$ has no zeros in the upper half-plane Π_+. Other solutions of inverse problem 1 with the same input data $\alpha_1(k)$ can be found by using the more general formula (15) instead of (16).

Let us take $T = 1$, and $R(x) = (\cos x)^{-2}$, $x \in [0, 1]$. Then $q(x) \equiv 1$, $h(x) = \tan x$, $h = \tan 1$. Moreover

$$u(x, k) = \cos \mu x - ik \frac{\sin \mu x}{\mu},$$

where $\mu^2 = k^2 + 1$. Since $ikv(x, k) = u'(x, k) + h(x)u(x, k)$,

$$ikv(x, k) = -\mu \sin \mu x - ik \cos \mu x + \tan x \left(\cos \mu x - ik \frac{\sin \mu x}{\mu} \right).$$

Using the relations $f_1(k) = (u(1, k) + v(1, k))/2$, $\alpha_1(k) = |f_1(k)|$, we calculate

$$\alpha_1(k) = \left| \cos \mu \left(1 - \frac{\tan 1}{2ik} \right) + \frac{\sin \mu}{\mu} \left(-ik + \frac{\tan 1}{2} + \frac{1}{2ik} \right) \right|. \tag{41}$$

Let us now describe the results of some numerical experiments. For the solution of inverse problem 1 we will use Algorithm 2. In order to check our calculations we will solve numerically the direct problem by the following algorithm.

Algorithm 3. Let the function $R(x)$, $x \in [0, T]$, be given. Then

(1) Find $y_1(x, k)$ and $y_2(x, k)$ by solving system (1) for real $\rho = k$ with the initial conditions $y_1(0, k) = 1$, $y_2(0, k) = -1$.

(2) Construct $f_1(k)$ by the formula

$$f_1(k) = (y_1(T, k) - R(T) y_2(T, k)) (2\sqrt{R(T)})^{-1}.$$

(3) Calculate $\alpha_1(k) = |f_1(k)|$.

One of the important decisions while implementing the numerical algorithm is the choice of the discretization. Since the scheme of calculation on each step is different, it is convenient to use its own discretization for each step. This approach gives us flexibility to choose both the type (uniform/non-uniform) and the number of points of division optimal for each particular step of the algorithm.

In the numerical experiments described below a uniform discretization was used along with the trapezoid rule. The number of points of division was chosen based on the desired accuracy. Modification of these settings can provide better accuracy and/or calculation time.

Example 1. As an input data for Algorithm 2 we take $\alpha_1(k)$ from (41). This input data are displayed in Figure 1 as a solid line.

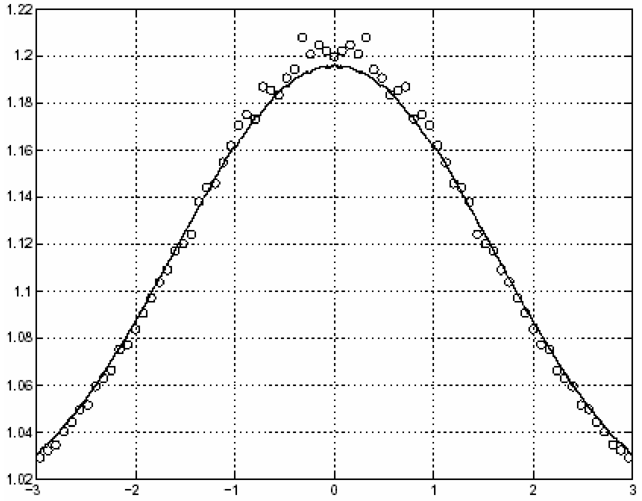

Figure 1. Input data for Example 1.

Applying Algorithm 2 with $T = 1$, $B = 3$, $h = 0.9$ and $\chi = 2.5$, we obtain the wave resistant $R(x)$ which is shown on Figure 2.

For checking our calculations, we apply Algorithm 3 with the input data $R(x)$ from Figure 2. Algorithm 3 gives us the modulus of the transition coefficient $\alpha_1(k)$ which is displayed in Figure 1 as a (o)-line.

Example 2. As input data for Algorithm 2 we take $\alpha_1(k)$ displayed on Figure 3 as a solid line. Applying Algorithm 2 with $T = 1$, $B = 3$, $h = 0.18$ and $\chi = 0.3$, we obtain the wave resistant $R(x)$ which is shown on Figure 4.

For checking our calculations, we apply Algorithm 3 with the input data $R(x)$ from Figure 4. Algorithm 3 gives us the modulus of the transition coefficient $\alpha_1(k)$ which is displayed in Figure 3 as a (o)-line.

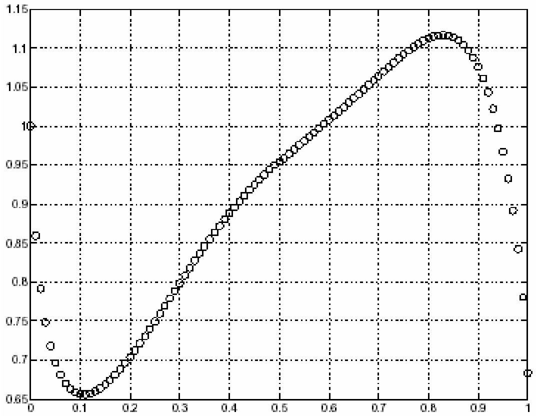

Figure 2. Solution of the inverse problem for Example 1.

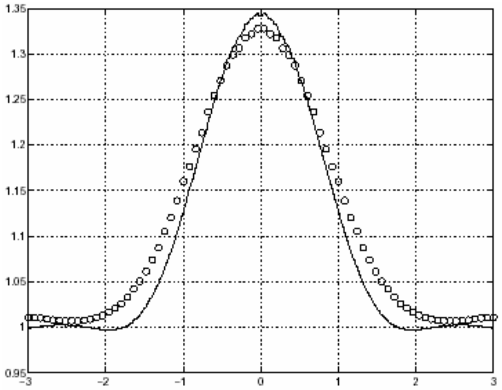

Figure 3. Input data for Example 2.

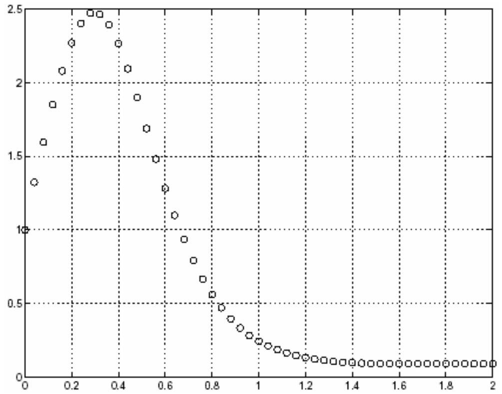

Figure 4. Solution of the inverse problem for Example 2.

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