

# COUPLED RICCATI DIFFERENTIAL EQUATIONS ARISING IN CONNECTION WITH NASH DIFFERENTIAL GAMES

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**Abstract:** In this note we provide iterative procedures for the numerical computation of stabilizing solutions of two types of coupled matrix Riccati differential equations arising in connection with Nash differential games using open loop or feedback strategies. Here we assume that these equations are associated with positive systems. The proposed procedures are based on solutions of uncoupled symmetric or nonsymmetric Lyapunov equations and complement the procedure proposed in [4].

Keywords: noncooperative differential game, coupled Riccati differential equations, Nash strategies, Lyapunov iterations .

## 1. INTRODUCTION AND PROBLEM FORMULATION

A dynamic game could be considered as a decision problem where several decision-makers (players or controllers) are acting on the same system. Each player has to choose a control to optimize his cost functional. When the system is described by a set of differential equations, we will be dealing with dynamic differential games. The study of such decision problems was initiated by Isaacs [10]. Two main classes of dynamic games are considered in the literature: cooperative and non-cooperative games. In cooperative games all players are trying to achieve the same goal. Standard optimal control problems may be considered as a special case of cooperative games. In non-cooperative games each player is optimizing his individual performance criterion. Zero-sum games were the first type of non-cooperative games to be studied in relation with the pursuit-evasion problem. In such differential games there is a single performance index which is minimized by one player and maximized by a second player. Zero-sum games are intimately related to the  $H_\infty$  robust control approach for disturbance rejection. This topic and the associated symmetric/Hermitian Riccati equation are discussed in detail in Chapter 6 of [1] and also in [3] and other textbooks. For studies on generalized Riccati equations appearing in stochastic control problems we refer to the recent textbook [6].

In this lecture (which is based on [5]) we are concerned with coupled Riccati- equations related to non-cooperative games. More precisely we study the existence of stabilizing

solutions of two pairs of coupled matrix Riccati differential equations associated with linear-quadratic games of the following form:

$$\dot{x} = A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t); \quad x(0) = x_0,$$

where  $x \in \mathbf{R}^n$ ,  $u_i \in \mathbf{R}^{r_i}$  ( $i = 1, 2$ ),

and where the cost functionals associated with each player are

$$J_1 = \frac{1}{2}x_f^T X_{1f} x_f$$

$$+ \frac{1}{2} \int_{t_0}^{t_f} (x^T Q_1(t)x + u_1^T R_{11}(t)u_1 + u_2^T R_{12}(t)u_2) dt,$$

$$J_2 = \frac{1}{2}x_f^T X_{2f} x_f$$

$$+ \frac{1}{2} \int_{t_0}^{t_f} (x^T Q_2(t)x + u_1^T R_{21}(t)u_1 + u_2^T R_{22}(t)u_2) dt,$$

$x_f = x(t_f)$ .

The matrices are assumed to be real, symmetric with  $Q_i \geq 0$  and  $R_{ii} > 0$  ( $i = 1, 2$ ).

The Riccati equations examined in this paper are associated to two types of strategies of the two players: the feedback Nash strategies and the open-loop Nash strategies.

It is known from the literature (see [1, 3, 7] for precise definitions and further details on this topic) that the optimal feedback and open-loop Nash strategies have the form

$$u_1(t) = -R_{11}^{-1}(t)B_1^T(t)X_1(t)x(t),$$

$$u_2(t) = -R_{22}^{-1}(t)B_2^T(t)X_2(t)x(t),$$

where  $x(t)$  can be determined from the initial value problem

$\dot{x} = [A(t) - S_1(t)X_1(t) - S_2(t)X_2(t)]x(t)$ ;  $x(0) = x_0$ , provided it is possible to determine for all  $t \in [t_0, t_f]$  the solutions  $(X_1(t), X_2(t))$  of the following coupled matrix Riccati differential equations (1) and (2), respectively, with terminal values  $X_i(t_f) = X_{if}$ ,  $i = 1, 2$ .

Using the abbreviations

$$S_i(t) = B_i(t)R_{ii}^{-1}(t)B_i^T(t), \quad 1 \leq i \leq 2,$$

and

$$S_{ij}(t) = B_j(t)R_{jj}^{-1}(t)R_{ij}(t)R_{jj}^{-1}(t)B_j^T(t), \quad 1 \leq i, j \leq 2,$$

we have to determine in the case of feedback Nash strategies the solution  $(X_1, X_2)$  of the coupled system

$$\begin{aligned} \frac{d}{dt}X_1 + A^T(t)X_1 + X_1A(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2 \\ - X_2S_2(t)X_1 + X_2S_{12}(t)X_2 + Q_1(t) = 0, \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{d}{dt}X_2 + A^T(t)X_2 + X_2A(t) - X_2S_2(t)X_2 - X_2S_1(t)X_1 \\ - X_1S_1(t)X_2 + X_1S_{21}(t)X_1 + Q_2(t) = 0, \end{aligned}$$

and in the case of open-loop Nash strategies the solution  $(X_1, X_2)$  of the system

$$\begin{aligned} \frac{d}{dt}X_1 + A^T(t)X_1 + X_1A^T(t) - X_1S_1(t)X_1 \\ - X_1S_2(t)X_2 + Q_1(t) = 0, \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d}{dt}X_2 + A^T(t)X_2 + X_2A(t) - X_2S_1(t)X_1 \\ - X_2S_2(t)X_2 + Q_2(t) = 0, \end{aligned}$$

where we assume for convenience that  $A : \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$ ;  $Q_i, S_i : \mathbf{R} \rightarrow \mathcal{S}_n, i = 1, 2$ ;  $S_{ij} : \mathbf{R} \rightarrow \mathcal{S}_n, (ij) \in \{(1, 2), (2, 1)\}$  are bounded and continuous matrix valued functions; here, as usual,  $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$  is the linear subspace of all symmetric  $n \times n$  matrices.

If the differential game is considered on an infinite time horizon (i.e.  $t_f = +\infty$ ), then the optimal strategy is constructed using a special global solution of the equations (1) and (2), respectively. Such solutions have to achieve the exponentially stable behavior of the trajectories of the closed-loop system. In this paper we are interested to derive procedures for numerical computation of a such global solutions of (1) and (2), respectively.

The equations (1) and (2) were investigated either as mathematical objects with interest in themselves in [1], Chapter 6, or in connection with several aspects of two players Nash differential games (see [2, 3, 7, 13, 14, 15] and references therein).

We mention that the system (2) can be rewritten as a non-symmetric (rectangular) matrix Riccati differential equation for the block matrix  $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ , therefore we can use for its solution all results and methods known for this type of equations (see [1], Chapter 6, [9] and [12]) – it is known that the global existence of the solutions of

such differential equations is only guaranteed under rather restrictive conditions.

Existence results for the nonlinear coupled system (1) are also rare; although the solutions  $X_j, 1 \leq j \leq 2$ , of (1) are symmetric, if the terminal (or initial) values  $X_{jf}, 1 \leq j \leq 2$ , are symmetric, the existence of the corresponding solutions can frequently only be guaranteed locally (see [11]).

The situation becomes better if one confines to differential equations (1) or (2) under assumptions leading to positive systems; in particular equation (1) and (2) were studied under these restrictions in [2], [4] and in [13], respectively.

In the present paper we assume that the equations (1), (2) are also in the case of positive systems. Therefore, according with the assumptions from [13, 2, 4] we make the following hypothesis concerning the coefficients of (1) and (2):

**H<sub>1</sub>**) (i) For each  $t \in \mathbf{R}$ ,  $A(t) = (a_{ij}(t))$  is a Metzler matrix, i.e.  $a_{ij}(t) \geq 0$  for  $i \neq j$ .

(ii)  $S_i(t) \preceq 0, i = 1, 2, \forall t \in \mathbf{R}$ .

(iii)  $S_{ij}(t) \succeq 0, (i, j) \in \{(1, 2), (2, 1)\}, \forall t \in \mathbf{R}$ .

(iv)  $Q_l(t) \succeq 0, t \in \mathbf{R}, l = 1, 2$ .

Here and below  $\preceq$  and  $\succeq$  are denoting the corresponding componentwise ordering.

Our aim is to construct sequences of iterates which converge towards the stabilizing solution of (1) and (2) respectively.

At each step we will have to solve two uncoupled symmetric Lyapunov differential equations or uncoupled nonsymmetric Lyapunov equations (Sylvester equations), respectively.

## 2. STABILIZING SOLUTIONS

Since (1) and (2) are nonstandard (coupled) Riccati differential equations, we consider that the obtained results could be useful to clarify the concept of stabilizing solutions of such equations.

To this end we regard these equations as nonlinear differential equations on an Hilbert space  $\mathcal{X}$ . For equation (1) we take  $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n$  while for equation (2) we take  $\mathcal{X} = \mathbf{R}^{n \times n} \oplus \mathbf{R}^{n \times n}$ . The usual inner product is given by

$$\langle X, Y \rangle = Tr [Y_1^T X_1] + Tr [Y_2^T X_2] \quad (3)$$

for all  $X = (X_1, X_2), Y = (Y_1, Y_2)$  in  $\mathcal{X}$ .

Let  $Q(t) = (Q_1(t), Q_2(t)), \mathcal{R}(t, X) = (\mathcal{R}_1(t, X), \mathcal{R}_2(t, X))$  then on  $\mathcal{X}$  the equations (1) and (2) may be written in a compact form as follows:

$$\frac{d}{dt}X + \mathcal{R}(t, X) + Q(t) = 0, \quad (4)$$

with

$$\begin{aligned} \mathcal{R}_1(t, X) &= A^T(t)X_1 + X_1A(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2 \\ &\quad - X_2S_2(t)X_1 + X_2S_{12}(t)X_2, \\ \mathcal{R}_2(t, X) &= A^t(t)X_2 + X_2A(t) - X_2S_2(t)X_2 - X_2S_1(t)X_1 \end{aligned}$$

$$-X_1 S_1(t) X_2 + X_1 S_{21}(t) X_1,$$

in case of equation (1), and

$$\mathcal{R}_1(t, X) = A^T(t) X_1 + X_1 A^T(t) - X_1 S_1(t) X_1 - X_1 S_2(t) X_2,$$

$$\mathcal{R}_2(t, X) = A^T(t) X_2 + X_2 A(t) - X_2 S_1(t) X_1 - X_2 S_2(t) X_2,$$

in case of equation (2).

For each solution  $X(t) = (X_1(t), X_2(t))$  of equation (4) we may construct the following operator valued function  $\mathcal{L}_X : \mathbf{R} \rightarrow \mathcal{B}[X]$  by  $\mathcal{L}_X(t)U = (\mathcal{L}_{1X}(t)U, \mathcal{L}_{2X}(t)U)$  where

$$\begin{aligned} \mathcal{L}_{1X}(t)U &= (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_1 \\ &\quad + U_1(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T \\ &\quad - (S_1(t)X_2(t) - S_{21}(t)X_1(t))U_2 \\ &\quad - U_2(X_2(t)S_1(t) - X_1(t)S_{21}(t)), \end{aligned} \quad (5)$$

$$\begin{aligned} \mathcal{L}_{2X}(t)U &= -(S_2(t)X_1(t) - S_{12}(t)X_2(t))U_1 \\ &\quad - U_1(X_1(t)S_2(t) - X_2(t)S_{12}(t)) \\ &\quad + (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_2 \\ &\quad + U_2(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T, \end{aligned}$$

in case of equation (1), and

$$\begin{aligned} \mathcal{L}_{1X}(t)U &= (A(t) - S_1(t)X_1^T(t))U_1 \\ &\quad + U_1(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T - S_1(t)X_2^T(t)U_2, \end{aligned} \quad (6)$$

$$\begin{aligned} \mathcal{L}_{2X}(t)U &= (A(t) - S_2(t)X_2^T(t))U_2 \\ &\quad + U_2(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T - S_2(t)X_1^T(t)U_1, \end{aligned}$$

in case of equation (2).

It is easy to see that

$$\mathcal{R}'(t, X(t)) = \mathcal{L}_X^*(t), \quad (7)$$

where  $\mathcal{R}'(t, \cdot)$  is the Fréchet derivative of the function  $X \rightarrow \mathcal{R}(t, X)$  while  $\mathcal{L}_X^*(t)$  is the adjoint operator of  $\mathcal{L}_X(t)$  with respect to the inner product (3).

**Definition 2.1** We say that a solution  $\tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$  of (4) is:

a) a *stabilizing solution* if the zero state equilibrium of the linear differential equation on  $\mathcal{X}$ :

$$\frac{d}{dt}Z = \mathcal{L}_{\tilde{X}}(t)Z \quad (8)$$

is exponentially stable.

b) a *closed-loop stabilizing solution* if the zero state equilibrium of the linear differential equation on  $\mathbf{R}^n$ :

$$\frac{d}{dt}x = A_{cl}(t)x \quad (9)$$

is exponentially stable, where  $A_{cl}(t) = A(t) - S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t)$ .

**Remark 2.2** a) Based on (7) it follows that in the time invariant case, the concept for a stabilizing solution introduced above can be characterized by the fact that the

eigenvalues of the operator  $\mathcal{R}'(X)$  are located in the open left half-plane  $Re \lambda < 0$ .

b) In [4] it was shown that if  $\tilde{X}(t)$  is a stabilizing solution of (1) then it is also a closed-loop stabilizing solution of the same equation.

Reasoning as in Lemma 8.1 (ii), (iii) in [4] one obtains that if  $\tilde{X}(t)$  is a stabilizing solution of (2) then the solution  $Z_k = 0$  of the linear differential equations

$$\frac{d}{dt}Z_k = \Lambda_{k, \tilde{X}}(t)Z_k, \quad k = 1, 2, \quad (10)$$

is exponentially stable, where

$$\Lambda_{k, \tilde{X}}(t)Z_k = (A(t) - S_k(t)\tilde{X}_k^T(t))Z_k \quad (11)$$

$$+ Z_k(A(t) - S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t))^T$$

is a nonsymmetric Lyapunov operator (i.e. a Sylvester operator).

Unfortunately we are not able to show that the exponential stability of the evolution generated by the Sylvester operator (11) implies the exponential stability of the corresponding closed-loop matrix  $A_{cl}(t)$  defined by (9).

c) Necessary and sufficient conditions under which a closed-loop stabilizing solution of (4) is also a stabilizing solution can be derived using the developments from section 6 in [4].

In [13, 2, 4] the sequences of iterates  $X^j = (X_1^j, X_2^j)$  converging towards the stabilizing solution were provided.

In each step  $X^j$  is obtained either as solution of the linear differential equations on  $\mathcal{X}$ :

$$\frac{d}{dt}X^j + \mathcal{L}_{X^{j-1}}^*(t)X^j + Q^j(t) = 0 \quad (12)$$

in the time-varying case or as solution of the algebraic linear equations on  $\mathcal{X}$ :

$$\mathcal{L}_{X^{j-1}}^* X^j + Q^j = 0 \quad (13)$$

in the time invariant case.

In this paper we replace equations (12) and (13) respectively, by uncoupled Lyapunov differential equations or uncoupled algebraic Lyapunov equations, respectively.

At the end of this section we introduce the following set of functions related to equation (4):

$$\Omega(\mathcal{R}, Q) = \quad (14)$$

$$\{P : \mathbf{R} \rightarrow \mathcal{X} | P(t) \succeq 0, \frac{d}{dt}P(t) + \mathcal{R}(t, P(t)) + Q(t) \preceq 0\}.$$

We recall that if  $H : \mathbf{R} \rightarrow \mathcal{X}$  we shall write  $H(t) \succ \succ 0$  if there exists a positive constant  $\delta$  such that  $H(t) \succeq \delta \mathbf{1}_n \succ 0$ , where  $\mathbf{1}_n$  is the  $n \times n$  matrix with all ones (for details see Ex. 2.5 (ii) in [4]).

We shall write  $H(t) \prec \prec 0$  if  $-H(t) \succ \succ 0$ .

**Remark 2.3** In (14) the operator  $\mathcal{R}(\cdot, \cdot)$  takes different forms according to the fact that the set  $\Omega(\mathcal{R}, Q)$  is associated either to equation (1) or to equation (2).

### 3. LYAPUNOV TYPE ITERATIONS FOR (1)

Let  $\{X^j(t)\}_{j \geq 0}$  be the sequence of functions  $X^j : \mathbf{R} \rightarrow \mathcal{X}$ ,  $X^j(t) = (X_1^j(t), X_2^j(t))$  with  $X_l^j(t)$  being the unique bounded on  $\mathbf{R}$  solution of the Lyapunov differential equation (here and also below we suppress  $(t)$ ):

$$\begin{aligned} \frac{d}{dt} X_l^j + Q_l^{j-1} + [A - S_1 X_1^{j-1} - S_2 X_2^{j-1}]^T X_l^j \\ + X_l^j [A - S_1 X_1^{j-1} - S_2 X_2^{j-1}] = 0, \end{aligned} \quad (15)$$

$l = 1, 2, X_l^0(t) = 0, t \in \mathbf{R}$ , where

$$Q_1^{j-1} = Q_1 + X_1^{j-1} S_1 X_1^{j-1} + X_2^{j-1} S_{12} X_2^{j-1}, \quad (16)$$

$$Q_2^{j-1} = Q_2 + X_2^{j-1} S_2 X_2^{j-1} + X_1^{j-1} S_{21} X_1^{j-1}. \quad (17)$$

Before stating the main result of this section we make the following assumption:

**H<sub>2</sub>** (i) The zero state equilibrium of the linear differential equation on  $\mathbf{R}^n$ :

$$\frac{d}{dt} x(t) = A(t)x(t)$$

is exponentially stable.

(ii) The set  $\Omega(\mathcal{R}, Q)$  is not empty.

Now we prove:

**Theorem 3.1** *Under the assumptions **H<sub>1</sub>** and **H<sub>2</sub>** the sequence  $\{X^j(t)\}_{j \geq 0}$  defined by (15)-(17) is well defined and convergent.*

If  $\tilde{X}(t) := \lim_{j \rightarrow \infty} X^j(t)$  then  $\tilde{X}(t)$  is the stabilizing solution of (1). Moreover  $\tilde{X}(t)$  is the minimal solution of (1) with respect to the class of global bounded nonnegative solutions of (1).

**Proof:** We shall show iteratively the following items.

$a_j$ )  $0 \preceq X^j(t) \preceq P(t)$  for all  $P(t) \in \Omega(\mathcal{R}, Q)$ .

$b_j$ ) The zero state equilibrium of the linear differential equation:

$$\frac{d}{dt} x(t) = A_j(t)x(t)$$

is exponentially stable, where

$$A_j(t) = A(t) - S_1(t)X_1^j(t) - S_2(t)X_2^j(t). \quad (18)$$

$c_j$ )  $X^j(t) \preceq X^{j+1}(t)$  for all  $t \in \mathbf{R}$ .

From assumption **H<sub>2</sub>**) together with  $X_l^0(t) = 0$  one obtains that items  $a_j$ ) and  $b_j$ ) are fulfilled for  $j = 0$ .

To check that  $c_0$ ) is also true let us remark that  $X_l^1(t)$  is the unique bounded solution of the Lyapunov differential equation:

$$\frac{d}{dt} X_l^1(t) + A^T(t)X_l^1(t) + X_l^1(t)A(t) + Q_l(t) = 0.$$

Since  $Q_l(t) \succeq 0$ , one obtains via Theorem 4.7 (iv) of [4] that  $X_l^1(t) \succeq 0 = X_l^0(t), t \in \mathbf{R}$ . This is just  $c_0$ ).

Let us assume next that  $a_i$ ),  $b_i$ ),  $c_i$ ) are fulfilled for  $0 \leq i \leq j-1$  and prove that then they also hold for  $i = j$ .

If  $b_{j-1}$ ) is fulfilled then from Theorem 4.7 (i) of [4] it follows that equation (15) has an unique bounded on  $\mathbf{R}$  solution and thus  $X^j(t)$  is well defined.

If  $P(t) = (P_1(t), P_2(t)) \in \Omega(\mathcal{R}, Q)$  one can see that it verifies the following differential equation:

$$\frac{d}{dt} P(t) + \tilde{\mathcal{R}}(t, P(t)) + Q(t) + \hat{Q}(t) = 0, \quad (19)$$

where  $\hat{Q}(t) = (\hat{Q}_1(t), \hat{Q}_2(t)) \succ \succ 0$ .

It is easy to check that  $P_l(t)$  verifies the following Lyapunov equations:

$$\frac{d}{dt} P_l + A_{j-1}^T P_l + P_l A_{j-1} + H_l^{j-1} = 0, \quad l = 1, 2, \quad (20)$$

where  $A_{j-1}(t)$  is as in (18) with  $X_l^j(t)$  replaced by  $X_l^{j-1}(t)$  and where

$H^{j-1}(t) = (H_1^{j-1}(t), H_2^{j-1}(t))$  with

$$\begin{aligned} H_1^{j-1} &= -[P_1 - X_1^{j-1}]S_1[P_1 - X_1^{j-1}] \\ &\quad - [P_2 - X_2^{j-1}]S_2P_1 - P_1S_2[P_2 - X_2^{j-1}] \\ &\quad + P_2S_{12}P_2 + X_1^{j-1}S_1X_1^{j-1} + Q_1 + \hat{Q}_1, \end{aligned} \quad (21)$$

$$\begin{aligned} H_2^{j-1} &= -[P_2 - X_2^{j-1}]S_2[P_2 - X_2^{j-1}] \\ &\quad - [P_1 - X_1^{j-1}]S_1P_2 - P_2S_1[P_1 - X_1^{j-1}] \\ &\quad + P_1S_{21}P_1 + X_2^{j-1}S_2X_2^{j-1} + Q_2 + \hat{Q}_2. \end{aligned} \quad (22)$$

From (15) and (20) one obtains

$$\begin{aligned} \frac{d}{dt} (P_l(t) - X_l^j(t)) + A_{j-1}^T(t)(P_l(t) - X_l^j(t)) \\ + (P_l(t) - X_l^j(t))A_{j-1}(t) + M_l^{j-1}(t) = 0, \end{aligned} \quad (23)$$

where  $M_l^{j-1}(t) = H_l^{j-1}(t) - Q_l^{j-1}(t)$ .

Since  $a_{j-1}$ ) is fulfilled one obtains from (16), (17) and (21)-(22) that  $M_l^{j-1}(t) \succ \succ 0, t \in \mathbf{R}$ .

Applying Theorem 4.7 (iv) in [4] to the equation (23) one concludes that for  $t \in \mathbf{R}$

$$P_l(t) - X_l^j(t) \succ c \mathbf{1}_n, \quad (24)$$

where  $c$  is a positive constant. Thus we deduce that  $a_j$ ) is fulfilled.

To check that  $b_j$ ) is fulfilled we rewrite equation (20) in the form:

$$\frac{d}{dt} P_l(t) + A_j^T(t)P_l(t) + P_l(t)A_j(t) + H_l^j(t) = 0, \quad (25)$$

where  $A_j(t)$  is as in (18) and the matrices  $H_l^j(t)$  are as in (21)-(22) with  $X_l^{j-1}(t)$  replaced by  $X_l^j(t)$ .

Equation (20) can be rewritten as

$$\frac{d}{dt} X_l^j(t) + A_j^T(t)X_l^j(t) + X_l^j(t)A_j(t) + G_l^j(t) = 0, \quad (26)$$

where  $A_j(t)$  is given by (18) and

$$\begin{aligned}
G_1^j &= Q_1 + X_1^{j-1}S_1X_1^{j-1} + X_2^{j-1}S_{12}X_2^{j-1} \\
&- (X_1^{j-1} - X_1^j)S_1X_1^j - X_1^jS_1(X_1^{j-1} - X_1^j) \\
&- X_2^jX_2(X_2^{j-1}X_2^j) - (X_2^{j-1} - X_2^j)S_2X_2^j, \quad (27)
\end{aligned}$$

$$\begin{aligned}
G_2^j &= Q_2 + X_1^{j-1}S_{21}X_1^{j-1} + X_2^{j-1}S_2X_2^{j-1} \\
&- X_2^jS_1(X_1^{j-1} - X_1^j) - (X_1^{j-1} - X_1^j)S_1X_2^j \\
&- X_2^jS_2(X_2^{j-1} - X_2^j) - (X_2^{j-1} - X_2^j)S_2X_2^j. \quad (28)
\end{aligned}$$

Subtracting (26) from (25) and taking into account (24) one obtains that the function  $t \rightarrow P_l(t) - X_l^j(t)$  is a bounded and uniform positive solution of the Lyapunov equation

$$\frac{d}{dt}Y_l(t) + A_j^T(t)Y_l(t) + Y_l(t)A_j(t) + \Theta_l^j(t) = 0 \quad (29)$$

with  $\Theta_l^j(t) = H_l^j(t) - G_l^j(t)$ .

It is easy to see that  $\Theta_l^j(t) \succ \succ 0$ .

Applying the implication  $(vi) \rightarrow (i)$  of Theorem 4.5 in [4] to equation (29) one concludes that the zero state equilibrium of the equation

$$\frac{d}{dt}x(t) = A_j(t)x(t)$$

is exponentially stable. Thus we obtained that item  $b_j$ ) is fulfilled.

To check the validity of item  $c_j$ ) one subtracts equation (26) from equation (15) written for  $X_l^{j+1}(t)$  and obtains:

$$\frac{d}{dt}(X_l^j(t) - X_l^{j+1}(t)) = A_j^T(t)(X_l^{j+1}(t) - X_l^j(t)) \quad (30)$$

$$-X_l^j(t) + (X_l^{j+1}(t) - X_l^j(t))A_j(t) + \Delta_l^j(t),$$

where  $\Delta_l^j(t) = Q_l^j(t) - G_l^j(t)$ ,  $l = 1, 2$ .

Combining (16)-(17) written for  $j + 1$  instead of  $j$  with (27)-(28) one can see that  $\Delta_l^j(t) \gg 0$ .

Applying Theorem 4.7 (iv) of [4] to equation (30) we conclude that

$$X_l^{j+1}(t) - X_l^j(t) \geq 0, \quad t \in \mathbf{R}.$$

This shows that  $c_j$ ) is fulfilled.

From  $a_j$ ) and  $c_j$ ),  $j \geq 0$  it follows that the sequences  $\{X_l^j(t)\}_{j \geq 0}$ ,  $l = 1, 2, t \in \mathbf{R}$  are convergent. Set  $\tilde{X}_l(t) = \lim_{j \rightarrow \infty} X_l^j(t)$ ,  $l = 1, 2, t \in \mathbf{R}$ . By standard arguments one obtains that  $t \rightarrow \tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$  is a solution of (1). As in [4] one proves that  $\tilde{X}(t)$  is just the stabilizing solution of (1).

In the same way as in the proof of item  $a_j$ ) one shows that  $X_l^j(t) \preceq Y_l(t)$  for arbitrary  $Y(t) = (Y_1(t), Y_2(t))$  which verifies

$$\frac{d}{dt}Y(t) + \mathcal{R}(t, Y(t)) + Q(t) \preceq 0, Y_l(t) \succeq 0.$$

This allows us to conclude that  $\tilde{X}(t)$  is the minimal solution of (1), thus the proof ends.

**Remark 3.2** a) Applying Theorem 4.7 (iii) in [4] one deduces that in the time invariant case the unique bounded solution of (15) is constant. Therefore in the time invariant case, for each iteration we have to solve two algebraic Lyapunov equations

$$A_{j-1}^T X_l^j + X_l^j A_{j-1} + Q_l^j = 0, \quad l = 1, 2,$$

with  $A_{j-1} = A - S_1 X_1^{j-1} - S_2 X_2^{j-1}$  and  $Q_l^{j-1}$  as in (16)-(17).

b) If  $A(\cdot), S_j(\cdot), S_{kl}(\cdot), Q_l(\cdot)$  are periodic functions with period  $\theta > 0$ , then the unique bounded solution of (15) is a periodic function with the same period  $\theta$ .

The initial value  $X_l^j(0)$  is obtained as solution of the Stein equation:

$$\begin{aligned}
X_l^j(0) &= \Phi_{j-1}^T(\theta, 0)X_l^j(0)\Phi_{j-1}(\theta, 0) \\
&+ \int_0^\theta \Phi_{j-1}^T(s, 0)Q_l^{j-1}(s)\Phi_{j-1}(s, 0)ds,
\end{aligned}$$

where  $\Phi_{j-1}(t, \tau)$  is the fundamental matrix solution of  $\frac{d}{dt}x(t) = A_{j-1}(t)x(t)$ .

c) An alternative algorithm for the iterative solution of (1), which is based on the solution of a sequence of two coupled Lyapunov differential equations, has been derived in [4], where further details can be found.

#### 4. LYAPUNOV TYPE ITERATIONS FOR (2)

We construct the following sequence of functions  $\{X^j(t)\}_{j \geq 0}$ ,  $X^j(t) = (X_1^j(t), X_2^j(t))$ , where  $X_l^j : \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$  is the unique bounded solution of the following nonsymmetric Lyapunov differential equations:

$$\begin{aligned}
&\frac{d}{dt}X_l^j(t) + (A^T(t) - X_l^{j-1}(t)S_l(t))X_l^j(t) \\
&+ X_l^j(t)(A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t)) \\
&+ Q_l(t) + X_l^{j-1}(t)S_l(t)X_l^{j-1}(t) = 0, \\
&l = 1, 2, j \geq 1, X_l^0(t) = 0, l = 1, 2. \quad (31)
\end{aligned}$$

The main result of this section is:

**Theorem 4.1** *Under the assumptions  $\mathbf{H}_1$ ) and  $\mathbf{H}_2$ ) the sequence  $\{X^j(t)\}_{j \geq 0}$  is well defined and convergent. If*

$$\tilde{X}(t) = \lim_{j \rightarrow \infty} X^j(t), \quad t \in \mathbf{R} \quad (32)$$

then  $\tilde{X}(t)$  is the stabilizing and minimal solution of (2).

The proof follows the same line as in the case of Theorem 3.1 and it is omitted for shortness.

However we remark that instead of the item  $b_j$ ) one proves the following new item

$b_j^*$ ) The zero state equilibrium of the linear differential equation on  $\mathbf{R}^{n \times n}$ :

$$\frac{d}{dt}Z_l(t) = [A(t) - S_l(t)(X_l^j(t))^T]Z_l(t)$$

$+Z_l(t)[A(t) - S_1(t)X_1^j(t) - S_2(t)X_2^j(t)]^T$   
is exponentially stable.

**Remark 4.2** a) If in (2),  $A(\cdot), S_j(\cdot), Q_l(\cdot)$  are constants, then one obtains inductively that the unique solution of (31) is constant. Therefore in the time invariant case at each iteration we solve the following nonsymmetric algebraic Lyapunov equations:

$$(A^T - X_l^{j-1}(t)S_l)X_l^j + X_l^j(A - S_1X_1^{j-1} - S_2X_2^j) + Q_l + X_l^{j-1}S_lX_l^{j-1} = 0.$$

b) If in (2)  $A(\cdot), S_j(\cdot), Q_l(\cdot)$  are period functions with period  $\theta > 0$ , then one obtains via Theorem 4.7 (ii) in [4] that the unique bounded solution of (31) is periodic function with the same period  $\theta$ . In this case the initial values  $X_l^j(0)$  are obtained as solutions of the following nonsymmetric Stein equations

$$X_l^j(0) = \Theta_{j-1,l}^T(\theta, 0)X_l^j(0)\Phi_{j-1}(\theta, 0) + \int_0^\theta \Theta_{j-1,l}^T(s, 0)[Q_l(s) + X_l^{j-1}(s)S_l(s)X_l^{j-1}(s)]\Phi_{j-1}(s, 0)ds$$

where  $\Theta_{j-1,l}(t, \tau)$  is the fundamental matrix solution of the differential equation

$$\frac{d}{dt}x(t) = [A(t) - S_l(t)(X_l^{j-1}(t))^T]x(t)$$

and  $\Phi_{j-1}(t, \tau)$  is the fundamental matrix solution of the differential equation

$$\frac{d}{dt}x(t) = [A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t)]x(t).$$

At the end of this section we provided the time varying counterpart of Corollary 1 from [13]:

**Corollary 4.3** *If there exists  $P(t) = (P_1(t), P_2(t)) \in \Omega(\mathcal{R}, Q)$  such that the zero solution of the linear differential equation*

$$\frac{d}{dt}z(t) = [A(t) - S_1(t)P_1(t) - S_2(t)P_2(t)]z(t)$$

*is exponentially stable then  $\tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$  defined by (31) is a closed-loop stabilizing solution of equation (2).*

**Proof.** From Theorem 4.1 one obtains that  $\tilde{X}_l(t) \preceq P_l(t)$ . The conclusion follows from Proposition 4.1 (ii) in [4].

## REFERENCES

- [1] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations in control and systems theory*, Birkhäuser Verlag, Basel, (2003).
- [2] T. P. Azevedo-Perdicoulis, G. Jank, *Linear quadratic Nash games and positive linear systems*, European Journal of Control 11, (2005), 632-644.
- [3] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. Academic Press, London, (1995).
- [4] V. Dragan, T. Damm, G. Freiling, T. Morozan, *Differential equations with positive evolutions and some applications*, Result. Math. 48, (2005), 206-236.
- [5] V. Dragan, T. Damm, G. Freiling, Lyapunov iterations for coupled Riccati differential equations in connection with Nash differential games. Mathematical Reports of the Roumanian Academy 9, (2007), 35-46.
- [6] V. Dragan, T. Morozan, A. Stoica, *Mathematical methods in robust control of linear stochastic systems*, Springer Verlag, Berlin, (2006).
- [7] J. Engwerda, *LQ Dynamic Optimization and Differential Games*, John Wiley, New York, (2005).
- [8] J. Engwerda, *Uniqueness condition for the infinite-planning horizon linear quadratic differential game*, In: Proceedings of CDC-ECC'05, Seville, Spain, 2005, (in CD ROM pp. 3507-3512).
- [9] S. Fital, C.-H. Guo, *Convergence of the solution of nonsymmetric matrix Riccati differential equations to its stable equilibrium solution*, J. Math. Anal. Appl. 318, (2006), 648-657.
- [10] Isaacs, R. (1965). *Differential games*. John Wiley, New York.
- [11] G. Freiling, G. Jank, H. Abou-Kandil, *On global existence of solutions to coupled matrix Riccati equations in closed-loop Nash games*, IEEE Transactions Aut. Control 41, (1996), 264-269.
- [12] G. Freiling, *A survey of nonsymmetric Riccati equations* Linear Algebra Appl. 251/252, (2002), 243-270.
- [13] G. Jank, D. Kremer, *Open loop Nash games and positive systems- solvability conditions for nonsymmetric Riccati equations*, Proceedings of MTNS 2004, Katolieke Universiteit, Leuven, Belgium, July 2004 (in CD ROM).
- [14] D. Kremer, *Nonsymmetric Riccati theory and noncooperative games*, Aachener Beiträge zur Mathematik 30, Wissenschaftsverlag Mainz, Aachen, 2003.
- [15] D. Kremer, R. Stefan, *Nonsymmetric Riccati theory and linear quadratic Nash games*, Proceedings of MTNS 2002, Notre Dame University, USA, 2002 (in CD ROM).