Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume x, Number x, pp. 1–22 (20xx)
`http://campus.mst.edu/adsa`

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# Determination of singular differential pencils from the Weyl function

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#### Abstract

The inverse spectral problem of recovering pencils of second-order differential operators on the half-line from the Weyl function is studied. We establish properties of the spectral characteristics, give a formulation of the inverse problem, prove an uniqueness theorem and provide a constructive procedure for the solution of the inverse problem by the method of spectral mappings

**AMS Subject Classifications:** 3.4A55 34B24 34L05 47E05
**Keywords:** differential equations, inverse spectral problems, Weyl function, method of spectral mappings.

## 1 Introduction

Consider the differential equation and the boundary condition $L = L(\ell, U)$:

$$\ell y := y'' + \Big(\rho^2 + i\rho q_1(x) + q_0(x)\Big)y = 0, \quad x > 0, \tag{1.1}$$

$$U(y) := P_1(\rho)y'(0) - P_0(\rho)y(0) = 0, \tag{1.2}$$

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`Received , 20; Accepted , 20`
`Communicated by`

where $\rho$ is the spectral parameter, $q_j(x)$ are complex-valued functions, and

$$P_k(\rho) = \sum_{j=0}^{p_k} P_{kj}\rho^{p_k-j}, \quad k = 0, 1, \quad p_k \geq 0,$$

are polynomials of degree $p_k$ with complex coefficients such that $P_1(\rho)$ and $P_0(\rho)$ has no common zeros.

In this paper we study the inverse problem of recovering the coefficients of the differential pencil $L$ from the given spectral characteristics. As the basic spectral characteristic we introduce and investigate the so-called Weyl function, which is an analogue of the classical Weyl function for the Sturm-Liouville operators [11]. Properties of the spectrum and the Weyl function are analyzed, a uniqueness theorem for the solution of the inverse problem is proved, and a constructive procedure for the solution of the inverse problem is provided.

Differential equations with linear or non-linear dependence on the spectral parameter arise in various problems of mathematics as well as in applications. In particular, several examples of spectral problems arising in mechanical engineering and having differential equations and boundary conditions depending on the spectral parameter are provided in the book [5] of Collatz; see also [13, 16, 18], where further references and links to applications can be found. Detailed studies on direct spectral problems for general classes of ordinary differential operators depending non-linearly on the spectral parameter can be found in various publications, see e.g. [13,16,18].

Classical inverse problems for the Sturm-Liouville equation

$$-y'' + q_0(x)y = \lambda y \tag{1.3}$$

(when $q_1(x) \equiv 0,\ \lambda = \rho^2$ in (1.1)) without the spectral parameter in the boundary conditions have been studied fairly completely (see the monographs [6, 10, 12, 14, 19] and the references therein). Inverse problems for differential pencils are more difficult to investigate, and nowadays there exists only a small number of papers in this direction. In particular, inverse problems for equation (1.3) with $\lambda$ – dependent boundary conditions were investigated in [1-4, 7-8]. All of these papers study such problems on a *finite interval*. The case of the half-line was considered in [20]. In [21-22] the inverse problem is investigated for a more general pencil (1.1) with boundary conditions linearly dependent on $\rho$. In particular, the solution of the inverse problem for the Regge boundary value problem [15] is provided. Some uniqueness results were also obtained in [21] for equation (1) on a finite interval with boundary conditions polynomially dependent on $\rho$. Inverse spectral problems for differential pencils (1.1)-(1.2) on the *half-line* have not been studied yet.

In this paper we consider singular non-self-adjoint pencils on the half-line and study the inverse spectral problem of recovering $L$ from the Weyl function. For this inverse problem we prove a uniqueness theorem and provide a procedure for constructing the solution. For this purpose we develop the ideas of the method of spectral mappings [19] and use ideas from [7] which help us to overcome difficulties connected with the appearance of $P_1(\rho)$ in the denominators in the main equation and in other important relations. Other difficulties in investigating the inverse problem for equation (1.1) (comparing with equation (1.3)) are produced by the more complicated form of the main equation. In particular, the corresponding linear operators (see (4.24)) are not inverses of each other as it was the case for equation (1.3). Nevertheless, the solvability of the main equation is proved which gives us an opportunity to construct the solution of the inverse problem considered. The obtained results are natural generalizations of the well-known results on the classical inverse problems (i.e. for $q_1 \equiv 0$).

## 2 Properties of the spectral characteristics

We consider a boundary value problem $L$ of the form (1.1)-(1.2). Fix $N \geq 0$. Denote by $W_N$ the set of functions $f(x)$ such that $f^{(\nu)}(x),\ \nu = \overline{0, N-1}$ are absolutely continuous on $[0,T]$ for each $T > 0$, and $f^{(\nu)}(x) \in L(0,\infty),\ \nu = \overline{0,N}$. We will say that $L \in V_N$, if $q_j(x) \in W_{N+j},\ j = 0,1$. We will seek the solution of the inverse problem in the classes $V_N$.

For definiteness, let $p_0 = p_1 + 1$. Other cases can be treated similarly (if $p_0 < p_1 + 1$, then all facts and formulae remain true with $P_{00} = 0$; the case $p_0 > p_1 + 1$ requires minor modifications). Without loss of generality we put $P_{10} = 1$. Suppose that $P_{00} \neq \pm i$. The last condition excludes from the consideration Regge-type problems [15] which require a separate investigation (see [21]). Denote by $Z_k = \{z_{ks}\}_{s=\overline{1,p_k}}$ the zeros of $P_k(\rho),\ k = 0,1$.

Let $S_1(x,\rho),\ S_2(x,\rho)$ and $\varphi(x,\rho)$ be solutions of equation (1.1) under the initial conditions

$$S_1(0,\rho) = S_2'(0,\rho) = 0, \quad S_1'(0,\rho) = S_2(0,\rho) = 1,$$

$$\varphi(0,\rho) = P_1(\rho),\ \varphi'(0,\rho) = P_0(\rho).$$

For each fixed $x$, and $\nu = 0,1$, the functions $S_j^{(\nu)}(x,\rho),\ j = 1,2$, and $\varphi^{(\nu)}(x,\rho)$ are entire in $\rho$, $U(\varphi) = 0$, and

$$\langle S_2(x,\rho), S_1(x,\rho)\rangle \equiv 1, \tag{2.1}$$

where $\langle y(x), z(x)\rangle := yz' - y'z$ is the Wronskian of $y$ and $z$. Clearly,

$$\varphi(x,\rho) = P_1(\rho) S_2(x,\rho) + P_0(\rho) S_1(x,\rho). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\langle \varphi(x,\rho), S_1(x,\rho)\rangle \equiv P_1(\rho). \tag{2.3}$$

We set

$$Q(x)=\frac{1}{2}\int_0^x q_1(s)\,ds,\quad \Pi_\pm=\{\rho:\ \pm Im\,\rho>0\},\quad \Pi=\Pi_+\cup\Pi_-,\quad \Pi_0=\{\rho:\ Im\,\rho=0\}.$$

By the well-known method (as in [6], Chapter 2 – see also, for example, [9], where slightly stronger assumptions are used) we get that for $x\ge 0,\ \rho\in\Pi_\pm$, there exists a solution $e(x,\rho)$ of equation (1.1) (which is called the Jost solution) with the following properties:

$1^o$. For each fixed $x\ge 0$, the functions $e^{(\nu)}(x,\rho),\ \nu=0,1$ are holomorphic for $\rho\in\Pi_+$ and $\rho\in\Pi_-$ (i.e. they are piecewise holomorphic).

$2^o$. The functions $e^{(\nu)}(x,\rho),\ \nu=0,1$ are continuous for $x\ge 0,\ \rho\in\overline{\Pi_+}\setminus\{0\}$ and $\rho\in\overline{\Pi_-}\setminus\{0\}$ (we distinguish between the sides of the cut $\Pi_0$). In other words, for real $\rho\ne 0$, there exist the finite limits

$$e_\pm^{(\nu)}(x,\rho)=\lim_{z\to\rho,\,z\in\Pi_\pm} e^{(\nu)}(x,z).$$

$3^o$. For $x\to\infty,\ \rho\in\overline{\Pi_\pm}\setminus\{0\},\ \nu=0,1,$

$$e^{(\nu)}(x,\rho)=(\pm i\rho)^\nu\exp(\pm(i\rho x-Q(x)))(1+o(1)).$$

For $\rho\in\Pi_\pm,\ \lim\limits_{x\to\infty} e(x,\rho)=0$. Moreover, $e(x,\rho)$ is the unique solution of (1.1) (up to a multiplicative constant) having this property.

$4^o$. For $|\rho|\to\infty,\ \rho\in\overline{\Pi_\pm},\ \nu=0,1$, uniformly in $x\ge 0$,

$$e^{(\nu)}(x,\rho)=(\pm i\rho)^\nu\exp(\pm(i\rho x-Q(x)))[1], \tag{2.4}$$

where $[1]:=1+O(\rho^{-1})$.

$5^o$. For real $\rho\ne 0$, the functions $e_+(x,\rho)$ and $e_-(x,\rho)$ form a fundamental system of solutions for (1), and

$$\langle e_+(x,\rho),e_-(x,\rho)\rangle=-2i\rho. \tag{2.5}$$

By virtue of Liouville‘s formula for the Wronskian, $\langle e_+(x,\rho),e_-(x,\rho)\rangle$ does not depend on $x$. Substituting $x=0$ into (2.5), we calculate for real $\rho\ne 0$:

$$e_-(0,\rho)e_+'(0,\rho)-e_+(0,\rho)e_-'(0,\rho)=2i\rho. \tag{2.6}$$

Denote

$$\Delta(\rho):=U(e(x,\rho))=P_1(\rho)e'(0,\rho)-P_0(\rho)e(0,\rho). \tag{2.7}$$

The function $\Delta(\rho)$ is called the *characteristic function* for the boundary value problem $L$. The function $\Delta(\rho)$ is holomorphic in $\Pi_+$ and $\Pi_-$, and for real $\rho\ne 0$ there exist the finite limits

$$\Delta_\pm(\rho)=\lim_{z\to\rho,\,z\in\Pi_\pm}\Delta(z).$$

In view of (2.7),

$$\Delta_\pm(\rho) = P_1(\rho)e'_\pm(0,\rho) - P_0(\rho)e_\pm(0,\rho). \tag{2.8}$$

Using (2.6) and (2.8) we obtain

$$e_-(0,\rho)\Delta_+(\rho) - e_+(0,\rho)\Delta_-(\rho) = 2i\rho P_1(\rho), \quad \rho \in \Pi_0 \setminus \{0\}. \tag{2.9}$$

According to (2.4) and (2.7) one has

$$\Delta(\rho) = \rho^{p_1+1}(\pm i - P_{00})[1], \quad |\rho| \to \infty, \quad \rho \in \overline{\Pi_\pm}. \tag{2.10}$$

Denote

$$\Lambda'_\pm = \{\rho \in \Pi_\pm :\ \Delta(\rho) = 0\}, \quad \Lambda' = \Lambda'_+ \cup \Lambda'_-,$$
$$\Lambda''_\pm = \{\rho \in \Pi_0 :\ \Delta_\pm(\rho) = 0\}, \quad \Lambda'' = \Lambda''_+ \cup \Lambda''_-,$$
$$\Lambda_\pm = \Lambda'_\pm \cup \Lambda''_\pm, \quad \Lambda = \Lambda_+ \cup \Lambda_-.$$

Then $\Lambda = \Lambda' \cup \Lambda''$, and $\Lambda'$ is at most countable set. It follows from (2.10) that $\Lambda$ is a bounded set.

Let us introduce the function

$$\Phi(x,\rho) = \frac{e(x,\rho)}{\Delta(\rho)}. \tag{2.11}$$

Clearly, $\Phi(x,\rho)$ satisfies (1.1) and the conditions $U(\Phi) = 1, \quad \Phi(x,\rho) = O(\exp(\pm(i\rho x - Q(x))),\ x \to \infty, \rho \in \Pi_\pm$ (while $\Delta(\rho) \neq 0$), where $U$ is defined by (1.2). Denote

$$M(\rho) := \Phi(0,\rho). \tag{2.12}$$

We will call $M(\rho)$ the *Weyl function* for $L$, since it is a generalization of the concept of the Weyl function for the classical Sturm-Liouville operator (see [11]). It follows from (2.11) and (2.12) that

$$M(\rho) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{2.13}$$

Using the initial conditions at $x = 0$ for $\Phi, S_1$ and $\varphi$, we get

$$\Phi(x,\rho) = \frac{1}{P_1(\rho)}\Big(S_1(x,\rho) + M(\rho)\varphi(x,\rho)\Big). \tag{2.14}$$

Together with (2.3) this yields

$$\langle \varphi(x,\rho), \Phi(x,\rho)\rangle \equiv 1. \tag{2.15}$$

Taking (2.11) and (2.15) into account we deduce

$$\langle \varphi(x,\rho), e(x,\rho)\rangle \equiv \Delta(\rho). \tag{2.16}$$

Let us obtain asymptotical formulae for $\varphi(x,\rho)$ and $\Phi(x,\rho)$ as $|\rho|\to\infty$. It is well-known (and can be proved as in [13], [17]) that there exists a fundamental system of solutions $\{Y_k(x,\rho)\}_{k=1,2}$ of equation (1.1) for $\rho\in\overline{\Pi_\pm},\ |\rho|>\rho^*,\ x\ge 0$ such that
1) the functions $Y_k^{(\nu)}(x,\rho),\ \nu=0,1,$ are continuous for $\rho\in\overline{\Pi_\pm},\ |\rho|>\rho^*,\ x\ge 0;$
2) for each fixed $x\ge 0$, the functions $Y_k^{(\nu)}(x,\rho)$ are holomorphic for $\rho\in\Pi_\pm,\ |\rho|>\rho^*;$
3) for $|\rho|\to\infty,\ \rho\in\overline{\Pi_\pm}$ uniformly in $x\ge 0,$

$$Y_k^{(\nu)}(x,\rho)=(\rho R_k)^\nu\exp(-iR_k(i\rho x-Q(x)))\Big(1+\frac{a_{k\nu}(x)}{\rho R_k}+\frac{\varepsilon_{k\nu}(x,\rho)}{\rho}\Big),$$

where $R_1=i,\ R_2=-i,$

$$a_{k0}(x)=\frac{iR_k}{4}\Big(q_1(0)-q_1(x)\Big)-\frac{1}{2}\int_0^x\Big(\frac{q_1^2(t)}{4}+q_0(t)\Big)\,dt,\ a_{k1}(x)=a_{k0}(x)+\frac{iR_kq_1(x)}{2},$$

$\varepsilon_{k\nu}(x,\rho)=o(1)$ for $N=0$, and $\varepsilon_{k\nu}(x,\rho)=O(\rho^{-1})$ for $N\ge 1$.

Using this fundamental system of solutions we calculate

$$\Phi(x,\rho)=\frac{Y_k(x,\rho)}{U(Y_k)},\quad k=\begin{cases}1 & \text{for}\quad \rho\in\Pi_+,\\ 2 & \text{for}\quad \rho\in\Pi_-,\end{cases}\tag{2.17}$$

$$\varphi(x,\rho)=\frac{U(Y_2)Y_1(x,\rho)-U(Y_1)Y_2(x,\rho)}{\langle Y_1(x,\rho),Y_2(x,\rho)\rangle}.\tag{2.18}$$

Taking (2.17) and (2.18) into account we arrive at the following propositions.

**Lemma 2.1.** *For* $|\rho|\to\infty,\ \rho\in\overline{\Pi_\pm}$, *uniformly in* $x\ge 0,$

$$\Phi^{(\nu)}(x,\rho)=\frac{(\pm i\rho)^\nu}{(\pm i-P_{00})\rho^{p_1+1}}\exp(\pm(i\rho x-Q(x))[1],\tag{2.19}$$

$$\varphi^{(\nu)}(x,\rho)=\frac{\rho^{p_1}}{2i}((i\rho)^\nu(i+P_{00})\exp(i\rho x-Q(x))[1]+(-i\rho)^\nu(i-P_{00})\exp(-i\rho x+Q(x))[1]).\tag{2.20}$$

**Theorem 2.2.** *The Weyl function* $M(\rho)$ *is holomorphic in* $\Pi_\pm\setminus\Lambda'_\pm$ *and continuous in* $\overline{\Pi_\pm}\setminus\Lambda_\pm$. *The set of singularities of* $M(\rho)$ *(as an analytic function) coincides with the set* $\Pi_0\cup\Lambda$. *For* $|\rho|\to\infty,\ \rho\in\overline{\Pi_\pm},$

$$M(\rho)=\frac{1}{(R_k-P_{00})\rho^{p_1+1}}\Big(1+\frac{M_k}{\rho R_k}+\frac{\varepsilon(\rho)}{\rho}\Big),\qquad k=\begin{cases}1 & \textit{for}\quad \rho\in\Pi_+,\\ 2 & \textit{for}\quad \rho\in\Pi_-,\end{cases}\tag{2.21}$$

*where* $\varepsilon(\rho)=o(1)$ *for* $N=0$, *and* $\varepsilon(\rho)=O(\rho^{-1})$ *for* $N\ge 1,$

$$M_k=\frac{1}{R_k-P_{00}}\Big(P_{11}+R_kP_{01}+\frac{iq_1(0)}{2}\Big).\tag{2.22}$$

**Definition 2.3.** The set of singularities of the Weyl function $M(\rho)$ is called the spectrum of $L$ (and is denoted by $\sigma(L)$). The values of the parameter $\rho$, for which equation (1.1) has nontrivial solutions satisfying (1.2) and the condition $y(\infty) = 0$ (i.e. $\lim\limits_{x\to\infty} y(x) = 0$), are called eigenvalues of $L$, and the corresponding solutions are called eigenfunctions of $L$.

Thus, $\sigma(L) = \Pi_0 \cup \Lambda$. The set $\Lambda$ is the discrete spectrum, and $\Pi_0$ is the continuous spectrum. Note that $\mathbf{C} \setminus \sigma(L)$ is the resolvent set of $L$.

**Theorem 2.4.** *$L$ has no eigenvalues in $\Pi_0 \setminus \{0\}$. Moreover, $|\Delta_+(\rho)| + |\Delta_-(\rho)| > 0$ in $\Pi_0 \setminus \{0\}$.*

*Proof.* Suppose that $\rho_0 \in \Pi_0\setminus\{0\}$ is an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. Since the functions $\{e_+(x,\rho_0), e_-(x,\rho_0)\}$ form a fundamental system of solutions for equation (1.1), we have $y_0(x) = C_1 e_+(x,\rho_0) + C_2 e_-(x,\rho_0)$. As $x \to \infty$, $y_0(x) \sim 0$, $e_\pm(x,\rho_0) \sim \exp(\pm(i\rho_0 x - Q(x)))$. But this is possible only if $C_1 = C_2 = 0$. Furthermore, if $\rho_0 \in \Pi_0 \setminus \{0\}$ and $\Delta_+(\rho_0) = \Delta_-(\rho_0) = 0$, then it follows from (2.9) that $P_1(\rho_0) = 0$, $P_0(\rho_0) \neq 0$. By virtue of (2.8), $\Delta_\pm(\rho_0) = -P_0(\rho_0)e_\pm(0,\rho_0)$, and consequently, $e_+(0,\rho_0) = e_-(0,\rho_0) = 0$, which is impossible in view of (2.6). Theorem 2.4 is proved. □

**Theorem 2.5.** The set $\Lambda'$ coincides with the set of non-zero eigenvalues of $L$, and

$$\varphi(x,\rho_k) = \beta_k e(x,\rho_k),\ \beta_k \neq 0 \quad \text{for all } \ \rho_k \in \Lambda'. \tag{2.23}$$

*Proof.* Let $\rho_k \in \Lambda'$. Then $U(e(x,\rho_k)) = \Delta(\rho_k) = 0$ and $\lim\limits_{x\to\infty} e(x,\rho_k) = 0$. Thus, $e(x,\rho_k)$ is an eigenfunction, and $\rho_k$ is an eigenvalue. Moreover, it follows from (2.16) that $\langle \varphi(x,\rho_k), e(x,\rho_k)\rangle = 0$, and consequently (2.23) is valid.

Conversely, let $\rho_k$, $\operatorname{Im}\rho_k \neq 0$ be an eigenvalue, and let $y_k(x)$ be a corresponding eigenfunction. Then $y_k(x) = C_k e(x,\rho_k)$, $C_k \neq 0$. Since $U(y_k) = 0$, it follows that $U(e(x,\rho_k)) = \Delta(\rho_k) = 0$, and $\varphi(x,\rho_k)$ and $e(x,\rho_k)$ are eigenfunctions. Theorem 2.5 is proved. □

Thus, the spectrum $\sigma(L)$ consists of the real line $\Pi_0$, and the discrete set $\Lambda = \Lambda' \cup \Lambda''$. Each element of $\Lambda'$ is an eigenvalue of $L$. According to Theorem 2.4, the points of $\Lambda''$ are not eigenvalues, they are called *spectral singularities*.

# 3 A uniqueness result

In this section we consider an inverse problem for $L$ and prove the corresponding uniqueness result. Let $Z_k$, $k = 0, 1$, be known and fixed. The inverse problem is formulated as follows.

**Inverse problem 3.1.** Given the Weyl function $M(\rho)$, construct the potentials $q_1(x), q_0(x)$ and the coefficient $P_{00}$.

We note that if $p_1 > 0$, then in general $M(\rho)$ does not uniquely determine all coefficients of the boundary condition.

For studying the inverse problem we agree that together with $L$ we consider a boundary value problem $\tilde L = L(\tilde\ell, \tilde U)$ of the same form but with $\tilde q_1, \tilde q_0, \tilde P_1, \tilde P_0$. If a certain symbol $\alpha$ denotes an object related to $L$, then the corresponding symbol $\tilde\alpha$ with tilde will denote the analogous object related to $\tilde L$, and $\hat\alpha := \alpha - \tilde\alpha$. Moreover, $Z_k = \tilde Z_k,\ k = 0, 1$.

**Theorem 3.2.** *If* $M(\rho) = \tilde M(\rho)$, *then* $q_j(x) = \tilde q_j(x),\ j = 0, 1,\ \ x > 0$, *and* $P_{00} = \tilde P_{00}$.

*Proof.* Taking (2.21) into account we get $P_{00} = \tilde P_{00}$. Since $P_{10} = \tilde P_{10} = 1$, and $Z_k = \tilde Z_k,\ k = 0, 1$, it follows that $P_k(\rho) = \tilde P_k(\rho),\ k = 0, 1$.

Let us define the matrix $Q(x, \lambda) = [Q_{jk}(x, \lambda)]_{j,k=1,2}$ by the formula

$$Q(x,\lambda)\begin{bmatrix} \tilde\varphi(x,\lambda) & \tilde\Phi(x,\lambda) \\ \tilde\varphi'(x,\lambda) & \tilde\Phi'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{3.1}$$

Using (2.15) and (3.1) we calculate

$$\left.\begin{aligned} Q_{j1}(x,\lambda) &= \varphi^{(j-1)}(x,\lambda)\tilde\Phi'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\tilde\varphi'(x,\lambda), \\ Q_{j2}(x,\lambda) &= \Phi^{(j-1)}(x,\lambda)\tilde\varphi(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde\Phi(x,\lambda), \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} \varphi(x,\lambda) &= Q_{11}(x,\lambda)\tilde\varphi(x,\lambda) + Q_{12}(x,\lambda)\tilde\varphi'(x,\lambda), \\ \Phi(x,\lambda) &= Q_{11}(x,\lambda)\tilde\Phi(x,\lambda) + Q_{12}(x,\lambda)\tilde\Phi'(x,\lambda). \end{aligned}\right\} \tag{3.3}$$

Substituting (2.14) into (3.2) we infer

$$Q_{11}(x,\rho) = \frac{1}{P_1(\rho)}(\tilde S'(x,\rho) + \tilde M(\rho)\tilde\varphi'(x,\rho))\varphi(x,\rho) - \frac{1}{P_1(\rho)}(S(x,\rho) + M(\rho)\varphi(x,\rho))\tilde\varphi'(x,\rho),$$

$$Q_{12}(x,\rho) = \frac{1}{P_1(\rho)}(S(x,\rho) + M(\rho)\varphi(x,\rho))\tilde\varphi(x,\rho) - \frac{1}{P_1(\rho)}(\tilde S(x,\rho) + \tilde M(\rho)\tilde\varphi(x,\rho))\varphi(x,\rho).$$

By virtue of (2.2), this yields

$$\left.\begin{aligned} Q_{11}(x,\rho) &= (C(x,\rho)\tilde S'(x,\rho) - \tilde C'(x,\rho)S(x,\rho)) - \frac{\hat M(\rho)}{P_1(\rho)}\varphi(x,\rho)\tilde\varphi'(x,\rho), \\ Q_{12}(x,\rho) &= (\tilde C(x,\rho)S(x,\rho) - C(x,\rho)\tilde S(x,\rho)) + \frac{\hat M(\rho)}{P_1(\rho)}\varphi(x,\rho)\tilde\varphi(x,\rho). \end{aligned}\right\} \tag{3.4}$$

Using (3.2), (2.19) and (2.20) we get for $|\rho| \to \infty$,

$$\left.\begin{aligned} Q_{11}(x,\rho) &= \Omega(x) + O\Big(\frac{1}{\rho}\Big), & Q_{12}(x,\rho) &= \frac{\Lambda(x)}{\rho} + O\Big(\frac{1}{\rho^2}\Big), \\ Q_{21}(x,\rho) &= -\rho\Lambda(x) + O(1), & Q_{22}(x,\rho) &= \Omega(x) + O\Big(\frac{1}{\rho}\Big), \end{aligned}\right\} \tag{3.5}$$

where

$$\Omega(x)=\frac{1}{2}\Big(\exp(\hat{Q}(x))+\exp(-\hat{Q}(x))\Big),\quad \Lambda(x)=-\frac{1}{2i}\Big(\exp(\hat{Q}(x))-\exp(-\hat{Q}(x))\Big). \tag{3.6}$$

Since $M(\rho)=\tilde{M}(\rho)$, it follows from (3.4) that the functions $Q_{11}(x,\rho)$ and $Q_{12}(x,\rho)$ are entire in $\rho$ for each fixed $x$. Together with (3.5) this yields $Q_{11}(x,\rho)\equiv\Omega(x)$, $Q_{12}(x,\rho)\equiv 0$. In view of (3.2), this yields

$$\tilde{\varphi}(x,\rho)\Phi(x,\rho)=\varphi(x,\rho)\tilde{\Phi}(x,\rho),$$

or

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)}=\frac{\Phi(x,\rho)}{\tilde{\Phi}(x,\rho)}. \tag{3.7}$$

Taking the asymptotical formulae (2.19) and (2.20) into account we obtain for $|\rho|\to\infty,\ \arg\rho\in[\delta,\pi-\varepsilon],\ \delta>0,$

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)}=\exp(\hat{Q}(x))[1],\quad \frac{\Phi(x,\rho)}{\tilde{\Phi}(x,\rho)}=\exp(-\hat{Q}(x))[1].$$

Comparing with (3.7) we conclude that $\exp(2\hat{Q}(x))\equiv 1$. Since $\hat{Q}(0)=0$, it follows that $\hat{Q}(x)\equiv 0$, i.e. $q_1(x)=\tilde{q}_1(x),\ x>0$. By virtue of (3.6), $\Omega(x)\equiv 1$, i.e. $Q_{11}(x,\rho)\equiv 1$. Since $Q_{12}(x,\rho)\equiv 0$, it follows from (3.3) that $\varphi(x,\rho)\equiv\tilde{\varphi}(x,\rho)$, $\Phi(x,\rho)\equiv\tilde{\Phi}(x,\rho)$, hence $q_0(x)=\tilde{q}_0(x)$ a.e. for $x>0$. Theorem 3.2 is proved. $\square$

**Remark 3.3.** The specification of the Weyl function $M(\rho)$ in only one half-plane $\Pi_+$ or $\Pi_-$ is insufficient for the unique determination of $L$. For example, if

$$q_0(x)=\frac{1}{2}q_1'(x)-\frac{1}{4}q_1^2(x),\quad P_1(\rho)=1,\quad P_0(\rho)=0,$$

then for $\rho\in\Pi_+$,

$$\Phi(x,\rho)\equiv\frac{1}{i\rho-q_1(0)/2}\exp\Big(i\rho x-Q(x)\Big),\quad M(\rho)\equiv\frac{1}{i\rho-q_1(0)/2}.$$

# 4 Solution of inverse problem 3.1

In this section we give a constructive procedure for the solution of inverse problem 3.1. The central role here is played by the so-called main equation of the inverse problem which connects the spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation which is a linear equation in a suitable Banach space. Moreover, we prove the unique solvability of the main equation. Using the solution of the main equation we provide an algorithm for the solution of the inverse problem considered.

Let $M(\rho)$ be the Weyl function for $L$. We find $P_{00}$ and $q_1(0)$ using (2.21)-(2.22), and construct $P_k(\rho)$ by the formula

$$P_k(\rho) = P_{k0}\prod_{s=1}^{p_k}(\rho - z_{ks}), \quad k = 0, 1, \quad P_{10} = 1.$$

Denote

$$D(x,\rho,\theta) = \frac{\langle\varphi(x,\rho),\varphi(x,\theta)\rangle}{\rho-\theta}. \tag{4.1}$$

Since by (1.1)

$$\frac{d}{dx}\langle\varphi(x,\rho),\varphi(x,\theta)\rangle = (\rho-\theta)(\rho+\theta+iq_1(x))\varphi(x,\rho)\varphi(x,\theta),$$

it follows that

$$D(x,\rho,\theta) = \int_0^x (\rho+\theta+iq_1(s))\varphi(s,\rho)\varphi(s,\theta)\,ds + \frac{P_1(\rho)P_0(\theta)-P_0(\rho)P_1(\theta)}{\rho-\theta}. \tag{4.2}$$

**Lemma 4.1.** *The following estimate holds*

$$|D(x,\rho,\theta)| \le C_x\frac{|\rho|+|\theta|+1}{|\rho-\theta|+1}(|\rho|^{p_1}+1)(|\theta|^{p_1}+1)\exp(|Im\,\rho|x)\exp(|Im\,\theta|x). \tag{4.3}$$

*Proof.* It follows from (2.20) that

$$|\varphi^{(\nu)}(x,\rho)| \le C(|\rho|^{p_1+\nu}+1)\exp(|Im\,\rho|x). \tag{4.4}$$

Fix $\delta_0 > 0$. Let $|\rho-\theta| \ge \delta_0$. Using (4.1), (4.4) and the equality

$$\langle\varphi(x,\rho),\varphi(x,\theta)\rangle = \varphi(x,\rho)\varphi'(x,\theta) - \varphi'(x,\rho)\varphi(x,\theta),$$

we obtain

$$|D(x,\rho,\theta)| \le C\frac{|\rho|+|\theta|+1}{|\rho-\theta|}(|\rho|^{p_1}+1)(|\theta|^{p_1}+1)\exp(|Im\,\rho|x)\exp(|Im\,\theta|x),$$

which brings us to (4.3).

Let now $|\rho-\theta| \le \delta_0$. Then, instead of (4.1), it is more convenient to use (4.2). It follows from (4.2) and (4.4) that

$$|D(x,\rho,\theta)| \le C(|\rho|^{p_1}+1)(|\theta|^{p_1}+1)(1+\exp(|Im\,\rho|x)\exp(|Im\,\theta|x)\int_0^x \Big(|\rho|+|\theta|+|q_1(s)|\Big)\,ds$$

$$\leq C_x(|\rho|^{p_1}+1)(|\theta|^{p_1}+1)(|\rho|+|\theta|+1)\exp(|Im\,\rho|x)\exp(|Im\,\theta|x),$$

and we arrive at (4.3) again. □

We now set

$$M^{\pm}(\rho)=\lim_{z\to 0, Re\,z>0} M(\rho\pm iz), \quad \rho\in\Pi_0\setminus\Lambda''_{\pm}.$$

It follows from (2.13) that

$$M^{\pm}(\rho)=\frac{e_{\pm}(0,\rho)}{\Delta_{\pm}(\rho)}.$$

Denote

$$m(\rho)=\rho^{p_1+2}M(\rho), \quad m^{\pm}(\rho)=\rho^{p_1+2}M^{\pm}(\rho).$$

We choose a model boundary value problem $\tilde{L}=L(\tilde{\ell},\tilde{U})$ such that $P_k(\rho)=\tilde{P}_k(\rho),\ k=0,1,\ q_1(0)=\tilde{q}_1(0)$, and $\int_{|\rho|\geq\rho^*}|\hat{m}^{\pm}(\rho)|^2\,d\rho<\infty$ for sufficiently large $\rho^*$. We note that for $N\geq 1$, the last relation is always fulfilled. Denote

$$A(x,\rho,\theta)=D(x,\rho,\theta)\frac{\hat{M}(\theta)}{P_1(\theta)}, \quad \tilde{A}(x,\rho,\theta)=\tilde{D}(x,\rho,\theta)\frac{\hat{M}(\theta)}{P_1(\theta)}.$$

In the $\rho$ - plane we consider the contour $\gamma=\gamma_{-1}\cup\gamma_0\cup\gamma_1$ (with counterclockwise circuit), where $\gamma_0$ is a bounded closed contour encircling the set $\Lambda\cup\tilde{\Lambda}\cup\{0\}$ (i.e. $\Lambda\cup\tilde{\Lambda}\cup\{0\}\subset\text{ int}\,\gamma_0$), and $\gamma_{\pm1}$ is the two-sided cut along the arc $\{\rho:\ \pm\rho>0,\ \rho\notin\text{ int}\,\gamma_0\}$. (see figure 1). Denote $J_{\gamma}=\{\rho:\ \rho\notin\gamma\cup\text{ int}\,\gamma_0\}$.

**Theorem 4.2.** *The following relations hold*

$$\Omega(x)\tilde{\varphi}(x,\rho)=\varphi(x,\rho)+\frac{1}{2\pi i}\int_{\gamma}\tilde{A}(x,\rho,\theta)\varphi(x,\theta)\,d\theta, \tag{4.5}$$

$$A(x,\rho,\theta)-\tilde{A}(x,\rho,\theta)+\frac{1}{2\pi i}\int_{\gamma}\tilde{A}(x,\rho,\xi)A(x,\xi,\theta)\,d\xi=\Lambda(x)\tilde{\varphi}(x,\rho)\varphi(x,\theta)\frac{\hat{M}(\theta)}{P_1(\theta)}. \tag{4.6}$$

*Proof.* It follows from Lemma 4.1 that

$$|A(x,\rho,\theta)|,|\tilde{A}(x,\rho,\theta)|\leq C_x\frac{(|\rho|+|\theta|)|\rho|^{p_1}}{(|\rho-\theta|+1)|\theta|^{p_1+2}}|\hat{m}(\theta)|, \quad \rho,\theta\in\gamma, \tag{4.7}$$

and the integrals in (4.5) and (4.6) converge absolutely.

Consider the contuor $\gamma_R:=\gamma\cap\{\rho:\ |\rho|\leq R\}$ oriented counterclockwise and also the contour $\gamma_R^0:=\gamma_R\cup\{\rho:\ |\rho|=R\}$ oriented clockwise. By Cauchy's integral formula we have

$$Q_{1k}(x,\rho)-\Omega(x)\delta_{1k}=\frac{1}{2\pi i}\int_{\gamma_R^0}\frac{Q_{1k}(x,\theta)-\Omega(x)\delta_{1k}}{\rho-\theta}\,d\theta, \quad k=1,2,$$

where $x \ge 0,\ \rho \in$ int $\gamma_R^0$, and $\delta_{jk}$ is the Kronecker delta. By virtue of (3.5),

$$Q_{1k}(x,\theta) - \Omega(x)\delta_{1k} = O\Big(\frac{1}{\theta}\Big), \quad |\theta| \to \infty.$$

Then

$$\lim_{R\to\infty} \int_{|\theta|=R} \frac{Q_{1k}(x,\theta) - \Omega(x)\delta_{1k}}{\rho-\theta}\, d\theta = 0,$$

and consequently,

$$Q_{1k}(x,\rho) = \Omega(x)\delta_{1k} + \frac{1}{2\pi i}\int_\gamma \frac{Q_{1k}(x,\theta)}{\rho-\theta}\, d\theta, \quad k=1,2,\ \rho \in J_\gamma,\ x \ge 0. \tag{4.8}$$

Here (and below where necessary) the integral is understood in the principal value sense: $\displaystyle\int_\gamma = \lim_{R\to\infty}\int_{\gamma_R}$. Next, in accordance with (3.3),

$$\varphi(x,\rho) = Q_{11}(x,\rho)\tilde\varphi(x,\rho) + Q_{12}(x,\rho)\tilde\varphi'(x,\rho). \tag{4.9}$$

Substituting (4.8) into (4.9) we obtain

$$\varphi(x,\rho) = \Omega(x)\tilde\varphi(x,\rho) + \frac{1}{2\pi i}\int_\gamma \frac{\tilde\varphi(x,\rho)Q_{11}(x,\theta) + \tilde\varphi'(x,\rho)Q_{12}(x,\theta)}{\rho-\theta}\, d\theta, \quad \rho \in J_\gamma,\ x \ge 0.$$

Using (3.2) we calculate

$$\varphi(x,\rho) = \Omega(x)\tilde\varphi(x,\rho) + \frac{1}{2\pi i}\int_\gamma \Big(\tilde\varphi(x,\rho)(\varphi(x,\theta)\tilde\Phi'(x,\theta) - \Phi(x,\theta)\tilde\varphi'(x,\theta))$$

$$+\tilde\varphi'(x,\rho)(\Phi(x,\theta)\tilde\varphi(x,\theta) - \varphi(x,\theta)\tilde\Phi(x,\theta)\Big)\frac{d\theta}{\rho-\theta}.$$

Taking (2.14) into account we obtain

$$\varphi(x,\rho) = \Omega(x)\tilde\varphi(x,\rho) - \frac{1}{2\pi i}\int_\gamma \frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle}{\rho-\theta}\,\frac{\hat M(\theta)}{P_1(\theta)}\varphi(x,\theta)\, d\theta$$

$$+\frac{1}{2\pi i}\int_\gamma \frac{1}{P_1(\theta)}\Big(\langle\tilde\varphi(x,\rho),\tilde S_1(x,\theta)\rangle\varphi(x,\theta) - \langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle S_1(x,\theta)\Big)\frac{d\theta}{\rho-\theta}.$$

According to (2.2),

$$\frac{1}{P_1(\theta)}\Big(\langle\tilde\varphi(x,\rho),\tilde S_1(x,\theta)\rangle\varphi(x,\theta) - \langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle S_1(x,\theta)\Big)$$

$$= \langle\tilde\varphi(x,\rho),\tilde S_1(x,\theta)\rangle S_2(x,\theta) - \langle\tilde\varphi(x,\rho),\tilde S_2(x,\theta)\rangle S_1(x,\theta).$$

Then the last integral vanishes by Cauchy's theorem, and we arrive at (4.5).

2) Since

$$\frac{1}{\rho-\theta}\Big(\frac{1}{\rho-\xi}-\frac{1}{\theta-\xi}\Big)=\frac{1}{(\rho-\xi)(\xi-\theta)}\,,$$

using Cauchy's integral formula and repeating the previous arguments, we obtain for $x\ge 0,\ \rho,\theta\in J_\gamma$:

$$\left.\begin{aligned}\frac{Q_{21}(x,\rho)-Q_{21}(x,\theta)}{\rho-\theta}&=-\Lambda(x)+\frac{1}{2\pi i}\int_\gamma\frac{Q_{21}(x,\xi)}{(\rho-\xi)(\xi-\theta)}\,d\xi,\\ \frac{Q_{jk}(x,\rho)-Q_{jk}(x,\theta)}{\rho-\theta}&=\frac{1}{2\pi i}\int_\gamma\frac{Q_{jk}(x,\xi)}{(\rho-\xi)(\xi-\theta)}\,d\xi,\quad (j,k)\ne(2,1).\end{aligned}\right\}\tag{4.10}$$

Let $y(x)$ be an arbitrary smooth function. Then, in accordance with (4.10),

$$\frac{Q(x,\rho)-Q(x,\theta)}{\rho-\theta}\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}=\begin{bmatrix}0\\ -\Lambda(x)y(x)\end{bmatrix}+\frac{1}{2\pi i}\int_\gamma Q(x,\xi)\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}\frac{d\xi}{(\rho-\xi)(\xi-\theta)}.\tag{4.11}$$

In view of (3.2),

$$Q(x,\rho)\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}=\langle y(x),\tilde\Phi(x,\rho)\rangle\begin{bmatrix}\varphi(x,\rho)\\ \varphi'(x,\rho)\end{bmatrix}-\langle y(x),\tilde\varphi(x,\rho)\rangle\begin{bmatrix}\Phi(x,\rho)\\ \Phi'(x,\rho)\end{bmatrix},$$

and equality (4.11) takes the form

$$\frac{Q(x,\rho)-Q(x,\theta)}{\rho-\theta}\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}=\begin{bmatrix}0\\ -\Lambda(x)y(x)\end{bmatrix}$$

$$+\frac{1}{2\pi i}\int_\gamma\Big(\langle y(x),\tilde\Phi(x,\xi)\rangle\begin{bmatrix}\varphi(x,\xi)\\ \varphi'(x,\xi)\end{bmatrix}-\langle y(x),\tilde\varphi(x,\xi)\rangle\begin{bmatrix}\Phi(x,\xi)\\ \Phi'(x,\xi)\end{bmatrix}\Big)\frac{d\xi}{(\rho-\xi)(\xi-\theta)}.$$

For $y(x)=\tilde\varphi(x,\rho)$ one has

$$\det\Big(\frac{Q(x,\rho)-Q(x,\theta)}{\rho-\theta}\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\theta)\\ \varphi'(x,\theta)\end{bmatrix}\Big)=-\Lambda(x)\tilde\varphi(x,\rho)\varphi(x,\theta)$$

$$+\frac{1}{2\pi i}\int_\gamma\Big(\frac{\langle\tilde\varphi(x,\rho),\tilde\Phi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\varphi(x,\xi),\varphi(x,\theta)\rangle}{\xi-\theta}-\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\Phi(x,\xi),\varphi(x,\theta)\rangle}{\xi-\theta}\Big)\,d\xi.\tag{4.12}$$

In view of (3.1),

$$Q(x,\rho)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix}=\begin{bmatrix}\varphi(x,\rho)\\ \varphi'(x,\rho)\end{bmatrix},$$

and consequently,

$$\det\Big(Q(x,\rho)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\theta)\\ \varphi'(x,\theta)\end{bmatrix}\Big)=\langle\varphi(x,\rho),\varphi(x,\theta)\rangle.\tag{4.13}$$

Furthermore, using (3.2) and (2.15) we calculate

$$\left.\begin{aligned} Q_{11}(x,\rho)\varphi'(x,\rho)-Q_{21}(x,\rho)\varphi(x,\rho)=\tilde\varphi'(x,\rho),\\ Q_{22}(x,\rho)\varphi(x,\rho)-Q_{12}(x,\rho)\varphi'(x,\rho)=\tilde\varphi(x,\rho),\end{aligned}\right\}$$

hence

$$\det\Big(Q(x,\theta)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\theta)\\ \varphi'(x,\theta)\end{bmatrix}\Big)=\tilde\varphi(x,\rho)\Big(Q_{11}(x,\theta)\varphi'(x,\theta)-Q_{21}(x,\theta)\varphi(x,\theta)\Big)$$

$$-\tilde\varphi'(x,\rho)\Big(Q_{22}(x,\theta)\varphi(x,\theta)-Q_{12}(x,\theta)\varphi'(x,\theta)\Big)=\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle. \qquad (47)$$

Substituting (4.13) and (4.14) into (4.12) we obtain

$$\frac{\langle\varphi(x,\rho),\varphi(x,\theta)\rangle}{\rho-\theta}-\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle}{\rho-\theta}=\Lambda(x)\tilde\varphi(x,\rho)\varphi(x,\theta)$$

$$+\frac{1}{2\pi i}\int_\gamma\Big(\frac{\langle\tilde\varphi(x,\rho),\tilde\Phi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\varphi(x,\xi),\varphi(x,\theta)\rangle}{\xi-\theta}-\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\Phi(x,\xi),\varphi(x,\theta)\rangle}{\xi-\theta}\Big)\,d\xi.$$

Taking (2.14) and (2.2) into account we get

$$\frac{\langle\varphi(x,\rho),\varphi(x,\theta)\rangle}{\rho-\theta}-\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\theta)\rangle}{\rho-\theta}=\Lambda(x)\tilde\varphi(x,\rho)\varphi(x,\theta)$$

$$-\frac{1}{2\pi i}\int_\gamma\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\varphi(x,\xi),\varphi(x,\theta)\rangle}{\xi-\theta}\,\frac{\hat M(\xi)}{P_1(\xi)}\,d\xi,$$

since the terms containing $S_j(x,\xi)$ and $\tilde S_j(x,\xi)$ vanish by Cauchy's theorem. Multiplying both parts of the last equality by $\hat M(\theta)/P_1(\theta)$, we arrive at (4.6), and (4.6) holds for all $x\ge 0,\ \rho\in\mathbf{C}$. Theorem 4.2 is proved. □

Analogously it can be proved that for $\rho\in J_\gamma$,

$$\Omega(x)\tilde\Phi(x,\rho)=\Phi(x,\rho)+\frac{1}{2\pi i}\int_\gamma\frac{\langle\tilde\Phi(x,\rho),\tilde\varphi(x,\theta)\rangle}{\rho-\theta}\frac{\hat M(\theta)}{P_1(\theta)}\varphi(x,\theta)\,d\theta. \qquad (4.15)$$

Interchanching $L$ and $\tilde L$, symmetrically to (4.5) and (4.6) we obtain

$$\Omega(x)\varphi(x,\rho)=\tilde\varphi(x,\rho)-\frac{1}{2\pi i}\int_\gamma A(x,\rho,\theta)\tilde\varphi(x,\theta)\,d\theta, \qquad (4.16)$$

$$A(x,\rho,\theta)-\tilde A(x,\rho,\theta)+\frac{1}{2\pi i}\int_\gamma A(x,\rho,\xi)\tilde A(x,\xi,\theta)\,d\xi=-\Lambda(x)\varphi(x,\rho)\tilde\varphi(x,\theta)\frac{\hat M(\theta)}{P_1(\theta)}. \qquad (4.17)$$

Denote

$$\psi(x,\rho)=\frac{\varphi(x,\rho)}{|\rho|^{p_1}},\quad \tilde\psi(x,\rho)=\frac{\tilde\varphi(x,\rho)}{|\rho|^{p_1}},$$

$$H(x,\rho,\theta)=A(x,\rho,\theta)\frac{|\theta|^{p_1}}{|\rho|^{p_1}},\quad \tilde H(x,\rho,\theta)=\tilde A(x,\rho,\theta)\frac{|\theta|^{p_1}}{|\rho|^{p_1}}.$$

By virtue of (4.4) and (4.7), one has for $\rho,\theta\in\gamma$,

$$|\psi(x,\rho)|,|\tilde\psi(x,\rho)|\le C,\quad |H(x,\rho,\theta)|,|\tilde H(x,\rho,\theta)|\le C_x\frac{(|\rho|+|\theta|)|\hat m(\theta)|}{(|\rho-\theta|+1)|\theta|^2}. \tag{4.18}$$

The relations (4.5), (4.6), (4.16) and (4.17) take the form

$$\Omega(x)\tilde\psi(x,\rho)=\psi(x,\rho)+\frac{1}{2\pi i}\int_\gamma \tilde H(x,\rho,\theta)\psi(x,\theta)\,d\theta, \tag{4.19}$$

$$\Omega(x)\psi(x,\rho)=\tilde\psi(x,\rho)-\frac{1}{2\pi i}\int_\gamma H(x,\rho,\theta)\tilde\psi(x,\theta)\,d\theta, \tag{4.20}$$

$$H(x,\rho,\theta)-\tilde H(x,\rho,\theta)+\frac{1}{2\pi i}\int_\gamma \tilde H(x,\rho,\xi)H(x,\xi,\theta)\,d\xi$$
$$=\Lambda(x)\tilde\psi(x,\rho)\psi(x,\theta)\frac{\hat m(\theta)|\theta|^{2p_1}}{P_1(\theta)\theta^{p_1+2}}, \tag{4.21}$$

$$H(x,\rho,\theta)-\tilde H(x,\rho,\theta)+\frac{1}{2\pi i}\int_\gamma H(x,\rho,\xi)\tilde H(x,\xi,\theta)\,d\xi$$
$$=-\Lambda(x)\psi(x,\rho)\tilde\psi(x,\theta)\frac{\hat m(\theta)|\theta|^{2p_1}}{P_1(\theta)\theta^{p_1+2}}. \tag{4.22}$$

It follows from (4.18) that for $\rho\in\gamma$,

$$\int_\gamma(|\tilde H(x,\rho,\theta)|+|H(x,\rho,\theta)|)d\theta\le C_x\int_\gamma\frac{(|\rho|+|\theta|)|\hat m(\theta)|}{(|\rho-\theta|+1)|\theta|^2}d\theta\le C_x\int_\gamma\frac{|\hat m(\theta)|}{|\theta|}d\theta\le C_x,$$

and consequently,

$$\int_\gamma(|\tilde H(x,\rho,\theta)|+|H(x,\rho,\theta)|)d\theta\le C_x,\quad \rho\in\gamma. \tag{4.23}$$

Let $C(\gamma)$ be the Banach space of all continuous bounded functions $\rho\to f(\rho)$ on $\gamma$ with the norm $\|f\|_{C(\gamma)}=\sup\limits_{\rho\in\gamma}|f(\rho)|$. According to (4.18), $\psi(x,\rho),\tilde\psi(x,\rho)\in C(\gamma)$ for each $x\ge 0$. For each fixed $x\ge 0$, we consider the following linear operators in $C(\gamma)$:

$$\left.\begin{aligned}\tilde By(\rho)&=y(\rho)+\frac{1}{2\pi i}\int_\gamma\tilde H(x,\rho,\theta)y(\theta)\,d\theta,\quad \rho\in\gamma,\\ By(\rho)&=y(\rho)-\frac{1}{2\pi i}\int_\gamma H(x,\rho,\theta)y(\theta)\,d\theta,\quad \rho\in\gamma.\end{aligned}\right\} \tag{4.24}$$

It follows from (4.23) that for each fixed $x \geq 0$, the linear operators $H$ and $\tilde{H}$ are bounded operators in $C(\gamma)$.

**Theorem 4.3.** *The following relations hold*

$$\tilde{B}\psi(x,\rho) = \Omega(x)\tilde{\psi}(x,\rho), \quad B\tilde{\psi}(x,\rho) = \Omega(x)\psi(x,\rho), \tag{4.25}$$

$$\tilde{B}By(\rho) = y(\rho) - \Lambda(x)\tilde{\psi}(x,\rho)\,\frac{1}{2\pi i}\int_\gamma \psi(x,\theta)\frac{\hat{m}(\theta)|\theta|^{2p_1}}{P_1(\theta)\theta^{p_1+2}}\,y(\theta)\,d\theta, \tag{4.26}$$

$$B\tilde{B}y(\rho) = y(\rho) - \Lambda(x)\psi(x,\rho)\,\frac{1}{2\pi i}\int_\gamma \tilde{\psi}(x,\theta)\frac{\hat{m}(\theta)|\theta|^{2p_1}}{P_1(\theta)\theta^{p_1+2}}\,y(\theta)\,d\theta. \tag{4.27}$$

*Proof.* The relations (4.25) follows from (4.19) and (4.20). According to (4.24),

$$\tilde{B}By(\rho) = \frac{1}{2\pi i}\int_\gamma \Big(\tilde{H}(x,\rho,\theta) - H(x,\rho,\theta) - \frac{1}{2\pi i}\int_\gamma \tilde{H}(x,\rho,\xi)H(x,\xi,\theta)\,d\xi\Big)y(\theta)\,d\theta.$$

Taking (4.21) into account we arrive at (4.26). Relation (4.27) is obtained similarly with the help of (4.22). ☐

**Remark 4.4.** We note that in the particular case of the Sturm-Liouville equation (when $q_1(x) \equiv 0$) we have $\Omega(x) \equiv 1,\ \Lambda(x) \equiv 0$, and consequently, the operators $B$ and $\tilde{B}$ are inverse to each other.

Denote $\omega = \{x :\ \Omega(x) \neq 0\},\ \omega_0 = \{x :\ \Omega(x) = 0\}$.

**Theorem 4.5.** *1) Let $x \in \omega$. Then the homogeneous equation*

$$\tilde{B}z(\rho) = 0, \quad z(\rho) \in C(\gamma) \tag{4.28}$$

*has only the trivial solution.*

*2) Let $x \in \omega_0$. Then the solutions of the homogeneous equation (4.28) form the one-dimensional subspace of functions of the form $z(\rho) = c\psi(x,\rho), \quad c - const.$*

*Proof.* 1) Let $x \in \omega$, and let $z_0(\rho)$ be a solution of (4.28), i.e. $\tilde{B}z_0(\rho) = 0.$. Then $B\tilde{B}z_0(\rho) = 0$. On the other hand, in accordance with (4.27),

$$B\tilde{B}z_0(\rho) = z_0(\rho) - c\psi(x,\rho),$$

and therefore, $z_0(\rho) = c\psi(x,\rho)$. Applying the operator $\tilde{B}$ to both parts of this equality we obtain

$$c\tilde{B}\psi(x,\rho) = 0.$$

By virtue of (4.25), this yields $c\Omega(x)\tilde{\psi}(x,\rho) = 0$. Therefore, $c = 0$, i.e. $z_0(\rho) = 0$.

2) Let $x \in \omega_0$. In view of (4.25) one has $\tilde{B}\psi(x,\rho) = 0$, hence the functions $z(\rho) = c\psi(x,\rho),\ c - const,$ are solutions of equation (4.28). On the other hand, let $z_0(\rho)$ be a solution of (61). Then $B\tilde{B}z_0(\rho) = 0$. Applying (4.27) we calculate

$$0 = B\tilde{B}z_0(\rho) = z_0(\rho) - c\psi(x,\rho),$$

i.e. $z_0(\rho) = c\psi(x,\rho)$. Theorem 4.5 is proved. ☐

Let $L$ and $\tilde{L}$ be such that $\omega_0 = \emptyset$. For example, this always holds if $\hat{q}_1(x)$ is real-valued. For each fixed $x \ge 0$, we consider in $C(\gamma)$ the linear equation

$$\tilde{\psi}(x,\rho) = z(x,\rho) + \frac{1}{2\pi i}\int_\gamma \tilde{H}(x,\rho,\theta) z(x,\theta)\,d\theta, \quad \rho \in \gamma \tag{4.29}$$

with respect to $z(x,\rho)$. Equation (4.29) is called the *main equation* of the inverse problem. The following result is a consequence of the above results.

**Theorem 4.6.** *For each fixed* $x \ge 0$, *equation (4.29) has a unique solution, namely* $z(x,\rho) = \psi(x,\rho)/\Omega(x)$.

Denote

$$w_1(x,\rho) = \frac{\varphi(x,\rho)}{\Omega(x)}, \quad w_2(x,\rho) = \frac{\Phi(x,\rho)}{\Omega(x)}.$$

Since the functions $\varphi(x,\rho)$ and $\Phi(x,\rho)$ are solutions of equation (1.1), it follows that the functions $w_1(x,\rho)$ and $w_2(x,\rho)$ satisfy the equation

$$w'' + a(x)w' + (\rho^2 + i\rho q_1(x) + h(x))w = 0,$$

where

$$a(x) = 2\frac{\Omega'(x)}{\Omega(x)}, \quad h(x) = q_0(x) + \frac{1}{2}a'(x) + \frac{1}{4}a^2(x). \tag{4.30}$$

Then

$$a(x)w_j'(x,\rho) + (i\rho q_1(x) + h(x))w_j(x,\rho) = -w_j''(x,\rho) - \rho^2 w_j(x,\rho), \quad j = 1,2. \tag{4.31}$$

Further it follows from (2.15) that

$$\langle w_1(x,\rho), w_2(x,\rho)\rangle \equiv \Omega^{-2}(x). \tag{4.32}$$

We rewrite (4.15) as follows

$$w_2(x,\rho) = \tilde{\Phi}(x,\rho) - \frac{1}{2\pi i}\int_\gamma \frac{\langle \tilde{\Phi}(x,\rho), \tilde{\varphi}(x,\theta)\rangle}{\rho - \theta}\frac{\hat{M}(\theta)}{P_1(\theta)} w_1(x,\theta)\,d\theta. \tag{4.33}$$

On the basis of the results obtained above, we arrive at the following procedure for the solution of inverse problem 3.1.

**Algorithm 4.7.** Let the Weyl function $M(\rho)$ and $Z_k,\ k = 0,1$, be given.
1) Find $P_{00}$ and $q_1(0)$ using (2.21)-(2.22), and construct $P_k(\rho)$ by the formula

$$P_k(\rho) = P_{k0}\prod_{s=1}^{p_k}(\rho - z_{ks}), \quad k = 0,1, \quad P_{10} = 1.$$

2) Choose $\tilde{L}$ and construct $\tilde{\psi}(x,\rho)$ and $\tilde{H}(x,\rho,\theta)$.
3) Find $z(x,\rho)$ by solving equation (4.29), and put $w_1(x,\rho)=z(x,\rho)|\rho|^{p_1}$.
4) Construct $w_2(x,\rho)$ by (4.33).
5) Calculate $\Omega^2(x)$ via (4.32).
6) Find $a(x), q_1(x)$ and $h(x)$ by solving the linear algebraic system (4.31) with the determinant $\Omega^{-2}(x)\neq 0$.
7) Construct $q_0(x)$ using (4.30).

# 5 Sturm-Liouville equation

Consider the boundary value problem $L_0$:

$$\ell y := -y'' + q(x)y = \lambda y, \quad x>0, \tag{5.1}$$

$$U(y) := P_1(\lambda)y'(0) - P_0(\lambda)y(0) = 0, \tag{5.2}$$

where $\lambda$ is the spectral parameter, $q(x)$ is a complex-valued function such that $q(x)\in L(0,\infty)$, and

$$P_k(\lambda)=\sum_{j=0}^{p_k} P_{kj}\lambda^{p_k-j}, \quad k=0,1, \quad p_k\geq 0,$$

are polynomials of degree $p_k$ with complex coefficients such that $P_1(\lambda)$ and $P_0(\lambda)$ have no common zeros. Boundary value problem (5.1)-(5.2) is a particular case of (1.1)-(1.2). In this section, we provide the solution of the inverse problem for $L_0$ using the results obtained above.

Let for definiteness, $p_1=p_0$. Other cases can be treated similarly. Without loss of generality we put $P_{10}=1$. Denote by $Z_k=\{z_{ks}\}_{s=\overline{1,p_k}}$ the zeros of $P_k(\lambda),\ k=0,1$. Let $\varphi(x,\lambda)$ be the solution of equation (1) under the initial conditions $\varphi(0,\lambda)=P_1(\lambda),\ \varphi'(0,\lambda)=P_0(\lambda)$.

Let $\lambda=\rho^2$, and let for definiteness $\operatorname{Im}\rho\geq 0$. Denote by $\Gamma_+$ the $\lambda$-plane with the two-sided cut $\Gamma_0$ along the arc $\Lambda_+ := \{\lambda:\ \lambda\geq 0\}$, and $\Gamma := \overline{\Gamma_+}\setminus\{0\}$; notice that here $\Gamma_+,\Gamma_0$ and $\Gamma$ must be considered as images of subsets of the Riemann surface of the square-root-function. Then, under the map $\rho\to\rho^2=\lambda$, $\Gamma_+$, $\Gamma_0$ and $\Gamma$ correspond to the domains $\Omega_+=\{\rho:\ \operatorname{Im}\rho>0\},\ \Omega_0=\{\rho:\ \operatorname{Im}\rho=0\}$ and $\Omega=\{\rho:\ \operatorname{Im}\rho\geq 0,\ \rho\neq 0\}$, respectively. Let $e(x,\rho),\ x\geq 0,\ \rho\in\Omega$, be the Jost solution of equation (5.1) (see section 2 and [6, ch.2]). Denote

$$\Delta(\rho) := P_1(\lambda)e'(0,\rho) - P_0(\lambda)e(0,\rho), \quad M(\lambda)=\frac{e(0,\rho)}{\Delta(\rho)}.$$

The function $M(\lambda)$ is called the Weyl function for $L_0$. The function $\Delta(\rho)$ is holomorphic in $\Omega_+$, continuous in $\Omega$, and

$$\Delta(\rho)=\lambda^{p_1}(i\rho)[1], \quad |\rho|\to\infty,\ \rho\in\Omega.$$

Denote

$$\Lambda = \{\lambda = \rho^2 : \rho \in \Omega,\ \Delta(\rho) = 0\}, \quad \Lambda' = \{\lambda = \rho^2 :\ \rho \in \Omega_+,\ \Delta(\rho) = 0\},$$

$$\Lambda'' = \{\lambda = \rho^2 :\ \rho \in \Omega_0,\ \rho \neq 0,\ \Delta(\rho) = 0\}.$$

Obviously, $\Lambda = \Lambda' \cup \Lambda''$ is a bounded set, $\Lambda'' \subset \Pi_0$, and $\Lambda'$ is a bounded and at most countable set.

**Theorem 5.1.** *The Weyl function $M(\lambda)$ is holomorphic in $\Gamma_+ \setminus \Lambda'$ and continuous in $\Gamma \setminus \Lambda$. The set of singularities of $M(\lambda)$ (as an analytic function) coincides with the set $\Lambda_0 := \Lambda_+ \cup \Lambda$. Moreover,*

$$M(\lambda) = \frac{1}{(i\rho)\lambda^{p_1}}\Big(1 + \frac{a_1}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big), \quad \rho \in \Omega,\ |\rho| \to \infty,\ a_1 = P_{00}. \tag{5.3}$$

Let $Z_k,\ k = 0, 1,$ be known and fixed. The inverse problem is formulated as follows.

**Inverse problem 5.2.** Given the Weyl function $M(\lambda)$, construct the potential $q(x)$ and the coefficient $P_{00}$.

The next theorem is a consequence of Theorem 3.2.

**Theorem 5.3.** *If $M(\lambda) = \tilde{M}(\lambda)$, then $q(x) = \tilde{q}(x)$ a.e. for $x > 0$, and $P_{00} = \tilde{P}_{00}$.*

Now we go on to constructing the solution of inverse problem 5.2. Let $M(\lambda)$ be the Weyl function for $L$. We find $P_{00}$ using (5.3), and construct $P_k(\lambda)$ by the formula

$$P_k(\lambda) = P_{k0} \prod_{s=1}^{p_k} (\lambda - z_{ks}), \quad k = 0, 1, \quad P_{10} = 1.$$

Denote

$$D(x, \lambda, \mu) := \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu}.$$

We choose a model pair $\tilde{L}_0$ such that $P_k(\lambda) = \tilde{P}_k(\lambda),\ k = 0, 1,$ and arbitrary in the rest. For example, one can take $\tilde{q}(x) \equiv 0$. In the $\lambda$ - plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where $\gamma'$ is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$, and $\gamma''$ is the two-sided cut along the arc $\{\lambda :\ \lambda > 0,\ \lambda \notin int\, \gamma'\}$. Denote

$$A(x, \lambda, \mu) = D(x, \lambda, \mu) \frac{\hat{M}(\mu)}{P_1(\mu)}, \quad \tilde{A}(x, \lambda, \mu) = \tilde{D}(x, \lambda, \mu) \frac{\hat{M}(\mu)}{P_1(\mu)}.$$

The next theorem is a consequence of Theorem 4.2.

**Theorem 5.4.** *The following relations hold*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \tilde{A}(x, \lambda, \mu) \varphi(x, \mu)\, d\mu, \tag{5.4}$$

$$A(x,\lambda,\mu)-\tilde A(x,\lambda,\mu)+\frac{1}{2\pi i}\int_\gamma \tilde A(x,\lambda,\xi)A(x,\xi,\mu)\,d\xi=0, \tag{5.5}$$

$$|\varphi(x,\lambda)|,\ |\tilde\varphi(x,\lambda)|\le C|\lambda|^{p_1}. \tag{5.6}$$

Denote

$$\psi(x,\lambda)=\varphi(x,\lambda)|\lambda|^{-p_1},\quad \tilde\psi(x,\lambda)=\tilde\varphi(x,\lambda)|\lambda|^{-p_1},$$

$$H(x,\lambda,\mu)=A(x,\lambda,\mu)|\mu|^{p_1}|\lambda|^{-p_1},\quad \tilde H(x,\lambda,\mu)=\tilde A(x,\lambda,\mu)|\mu|^{p_1}|\lambda|^{-p_1}.$$

In view of (5.6), one has for $\lambda,\mu\in\gamma$,

$$|\psi(x,\lambda)|,\ |\tilde\psi(x,\lambda)|\le C. \tag{5.7}$$

Let $C(\gamma)$ be the Banach space of continuous bounded functions $\lambda\to f(\lambda)$ on $\gamma$ with the norm $\|f\|_{C(\gamma)}=\sup\limits_{\lambda\in\gamma}|f(\lambda)|$. According to (5.7), $\psi(x,\lambda),\tilde\psi(x,\lambda)\in C(\gamma)$ for each $x\ge 0$. In view of our notations, relations (5.4)-(5.5) take the form

$$\tilde\psi(x,\lambda)=\psi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde H(x,\lambda,\mu)\varphi(x,\mu)\,d\mu, \tag{5.8}$$

$$H(x,\lambda,\mu)-\tilde H(x,\lambda,\mu)+\frac{1}{2\pi i}\int_\gamma \tilde H(x,\lambda,\xi)H(x,\xi,\mu)\,d\xi=0.$$

For each fixed $x\ge 0$, relation (5.8) can be considered as a linear integral equation with respect to $\psi(x,\lambda)$ in the Banach space $C(\gamma)$. Equation (5.8) is called the *main equation* for inverse problem 5.2.

**Theorem 5.5.** *For each fixed $x\ge 0$, the main equation (5.8) has a unique solution $\psi(x,\lambda)\in C(\gamma)$.*

Thus, we obtain the following algorithm for the solution of inverse problem 5.2.

**Algorithm 5.6.** *Let the function $M(\lambda)$ be given. Then*

*(1) Construct $P_{00}$ and $P_k(\lambda),\ k=0,1$.*
*(2) Choose $\tilde L$ such that $P_k(\lambda)=\tilde P_k(\lambda),\ k=0,1$.*
*(3) Find $\psi(x,\lambda)$ by solving equation (5.8).*
*(4) Construct $q(x)$ via the formula*

$$q(x)=\varphi''(x,\lambda)(\varphi(x,\lambda))^{-1}-\lambda,\quad \varphi(x,\lambda)=\psi(x,\lambda)(|\lambda|^{p_1}+1).$$

**Acknowledgment.** This research was supported in part by DAAD and Grants 10-01-00099 and 10-01-92001-NSC of the Russian Foundation for Basic Research and Taiwan National Science Council.

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