SOLUTION AND ASYMPTOTIC BEHAVIOR OF COUPLED RICCATI EQUATIONS IN JUMP LINEAR SYSTEMS *

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H. ABOU-KANDIL

Ecole Normale Supérieure de Cachan - LURPA, 61, Av. Président Wilson - 94230 Cachan - France, email: hisham@lurpa9.ens-cachan.fr

G. FREILING and G. JANK

Lehrstuhl II für Mathematik - RWTH Aachen, Templergraben 55, D-5100 Aachen - Germany, email: sf010fr@dacth11.bitnet

Abstract:

I. Introduction

We consider a class of linear stochastic systems where the parameter values are subject to random jumps. This class of systems was introduced in [1] and further studied in [2,3,4]. The optimal jump linear quadratic problem can be defined as follows:

Plant state:

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t); \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathbf{R^n}$ is the plant state and $u(t) \in \mathbf{R^m}$ is the control vector. $A(r(t))$ and $B(r(t))$ are matrices of dimensions $n \times n$ and $n \times m$ respectively. $r(t)$ is a finite state Markov jump process representing the plant mode, i.e. $r(t)$ takes discrete values in a

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given finite set $\mathcal{S} = \{1, 2, \ldots, N\}$.

Plant mode:

$$\mathbf{Prob}(r(t+\Delta) = j \mid r(t) = i) = \begin{cases} \pi_{ij}\Delta + o(\Delta) & \text{for} \quad i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{for} \quad i = j, \end{cases} \tag{2}$$

here $o(\Delta)$ is the Landau symbol.
The jump transition density matrix is therefore

$$\Pi = (\pi_{ij}) \qquad\qquad i, j = 1, 2, \ldots N$$

with

$$\pi_{ii} < 0 \ ; \quad \pi_{ij} > 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad \sum_{j=1}^{N} \pi_{ij} = 0 \qquad \text{for} \quad i = 1, 2, \ldots N.$$

When the system operates in the $i-$th mode ($r(t) = i$), the pair $[A(r(t)), B(r(t))]_{r(t)=i}$ is denoted $[A_i, B_i]$. It is required to minimize the performance criterion

$$J(u, t_0, r(t_0), x_0) = \mathbf{E}\left\{\frac{1}{2}\int_{t_0}^{t_f} (x^T(t)Q(r(t))x(t) + u^T(t)R(r(t))u(t))dt \mid t_0, x_0, r(t_0)\right\}. \tag{3}$$

The mode dependent weighting matrices are $[Q(r(t)), R(r(t))]_{r(t)=i} =: [Q_i, R_i]$ with $Q_i \geq 0, \quad R_i > 0 \quad (i = 1, 2, \ldots, N)$. The optimal control is obtained as [2,3]:

$$u^*(x(t), r(t)) = -R_i^{-1} B_i^T K_i(t) x(t) \qquad\qquad \text{for} \ \ r(t) = i, \tag{4}$$

where $K_i(t) \ \ (i = 1, 2, \ldots, N)$ must satisfy the set of coupled differential matrix Riccati equations:

$$\dot{K}_i = -A_i^T K_i - K_i A_i - Q_i + K_i S_i K_i - \sum_{j=1}^{N} \pi_{ij} K_j \tag{5}$$

with $\quad S_i = B_i R_i^{-1} B_i^T \quad$ and $\quad K_i(t_f) = 0.$

As $t_f \to \infty$ and under stochastic controllability and observability assumptions it is shown in[4] that the matrices $K_i(t)$ tend to constant matrices K_i^∞ verifying the set of coupled algebraic Riccati equations:

$$A_i^T K_i^\infty + K_i^\infty A_i + Q_i - K_1^\infty S_i K_i^\infty + \sum_{j=1}^{N} \pi_{ij} K_j^\infty = 0. \tag{6}$$

For convenience equation (6) is rewritten as

$$\tilde{A}_i^T K_i^\infty + K_i^\infty \tilde{A}_i + Q_i - K_i^\infty S_i K_i^\infty + \sum_{j=1, j\neq i}^{N} \pi_{ij} K_j^\infty = 0 \tag{7}$$

with $\tilde{A}_i = (A_i + \frac{1}{2}\pi_{ii} I)$.
Mariton and Bertrand [5] have proposed a homotopy method for the solution of (7). The main goal of this paper is to derive two algorithms for the solution of (7) being more efficient and numerically stable.

II. Preliminaries

In this section we recall some of the basic facts for Riccati differential equations of the form

$$\dot{P} = -A^T P - PA - Q + PSP \tag{8}$$

and for the corresponding algebraic Riccati equation

$$-A^T P - PA - Q + PSP = 0, \tag{9}$$

with $S = BR^{-1}B^T$. In the whole section we assume that $Q \geq 0, R > 0$ and that (A, B) is stabilizable.

The following results are known from the literature.

(i) There exists a unique *strong* solution $\hat{P} \geq 0$ of (9) [6, Theorem 1]. We recall that a solution of (9) is called strong if the closed-loop state transition matrix $A - BR^{-1}B^T K$ has all its eigenvalues in the closed left half-plane.
(ii) The strong solution of (9) depends monotonically on S and Q (see, [7, Theorem 1]): If $\tilde{P}$ is the strong solution of the algebraic Riccati equation obtained from (9) by replacing therein A, Q and S by $\tilde{A}, \tilde{Q}$ and $\tilde{S}$ then

$\begin{pmatrix} Q & A^T \\ A & -S \end{pmatrix} \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^T \\ \tilde{A} & -\tilde{S} \end{pmatrix}$ implies $\hat{P} \geq \tilde{P} \geq 0$.

(iii) Further we shall use the well known fact that the solutions of (8) are also depending monotonically on S and Q (see[8, Satz 10.2] for details).
(iv) Let $P(\cdot, P_T)$ be the solution of (8) with $P(T, P_T) = P_T \geq 0$ then $P(t, P_T)$ exists, is bounded and positive semidefinite on the interval $(-\infty, T]$.
In addition $\lim_{t \to -\infty} P(t, P_T) = \hat{P}$ if

- either $P_T \geq \hat{P}$
- or $P_T > 0$ and A has no purely imaginary $(\sqrt{Q}, A)$ -unobservable eigenvalues
- or $\mathcal{N}(P_T)$ (the null-space of P_T) is contained in the sum of all $(\sqrt{Q}, A)$ unobservable eigenspaces corresponding to purely imaginary eigenvalues of A.

If $(\sqrt{Q}, A)$ is detectable then $A - S\hat{P}$ is stable; $\hat{P}$ is positive definite if $(\sqrt{Q}A)$ is observable. (See [9, Theorem 2.1] and [10, Theorems 1,2,3]).

It should be noted that (5) and (7) could be written as a simple higher dimensional system, i.e.

$$\dot{\mathcal{K}} = -\mathcal{A}^T \mathcal{K} - \mathcal{K}\mathcal{A} - \mathcal{Q} + \mathcal{K}\mathcal{S}\mathcal{K} - \sum_{k=1}^{N-1} \mathcal{T}_k \mathcal{K} \mathcal{T}_k^T \tag{10}$$

with

$$\mathcal{K} = \mathrm{diag}\{K_i\}_{i=1,\ldots N}, \quad \mathcal{A} = \mathrm{diag}\{\tilde{A}_i\}_{i=1,\ldots N}, \quad \mathcal{Q} = \mathrm{diag}\{Q_i\}_{i=1,\ldots N}, \quad \mathcal{S} = \mathrm{diag}\{S_i\}_{i=1,\ldots N},$$

and where $\mathcal{T}_k$ are appropriate permutation matrices to introduce coupling terms. For example when $N = 3$

$$\mathcal{T}_1 = \begin{bmatrix} 0 & I\sqrt{\pi_{12}} & 0 \\ 0 & 0 & I\sqrt{\pi_{23}} \\ I\sqrt{\pi_{31}} & 0 & 0 \end{bmatrix} \quad \mathcal{T}_2 = \begin{bmatrix} 0 & 0 & I\sqrt{\pi_{13}} \\ I\sqrt{\pi_{21}} & 0 & 0 \\ 0 & I\sqrt{\pi_{32}} & 0 \end{bmatrix},$$

I being the identity matrix.

According to [9, Theorem 2.1] the solution $\mathcal{K}$ of (10), satisfying the terminal condition $\mathcal{K}(t_f) = \text{diag}\{K_{if}\}_{i=1,\dots N}$ with $K_{t_f} \geq 0$ for $1 \leq i \leq N$, exists on the whole interval $(-\infty, t_f]$.

In the following section we shall give sufficient conditions for the convergence of $\mathcal{K}(t)$ for $t \to -\infty$ to a positive semidefinite limit. Wonham [3] has obtained similar results under stronger conditions.

III. Main results

First we show that the solutions of (10) are monotonically dependent on their value in a fixed point t_0.

Lemma 1: Let $K_{i1}, K_{i2}, 1 \leq i \leq N$, be solutions of the coupled system (5) on the interval $\mathcal{I}$ with $K_{i1}(t_0) < K_{i2}(t_0), 1 \leq i \leq N$, for some $t_0 \in \mathcal{I}$. Then $K_{i1}(t) < K_{i2}(t)$ for $1 \leq i \leq N$ and $t \in \mathcal{I} \cap (-\infty, t_0]$.

Proof: Let $D_i = K_{i2} - K_{i1}, 1 \leq i \leq N$, then $D_i(t_0) > 0$, and we have to prove that $D_i(t) > 0$ for $t \in \mathcal{I} \cap (-\infty, t_0]$. $K_{i2} = K_{i1} + D_i$ verifies (5), hence

$$\dot{K}_{i1}+\dot{D}_i = -\tilde{A}_i^T(K_{i1}+D_i)-(K_{i1}+D_i)\tilde{A}_i-Q_i+(K_{i1}+D_i)S_i(K_{i1}+D_i)-\sum_{j=1,j\neq 1}^{N}\pi_{ij}(K_{j1}+D_j)$$

and

$$\dot{K}_{i1} = -\tilde{A}_i^T K_{i1} - K_{i1}\tilde{A}_i^T - Q_i + K_{i1}S_iK_{i1} - \sum_{j=1,j\neq i}^{N}\pi_{ij}K_{j1}.$$

Subtracting these equations we infer that

$$\begin{aligned} \dot{D}_i &= -\tilde{A}_i^T D_i - D_i\tilde{A}_i + D_iS_iD_i + D_iS_iK_{i1} + K_{i1}S_iD_i - \sum_{j=1,j\neq i}^{N}\pi_{ij}D_j \\ &\leq D_iM_i + M_i^TD_i \ \text{sinceaccordingto[11, Theorem2.1(ii)} \sum_{j=1,j\neq i}^{N}\pi_{ij}D_j(t) \geq 0, \end{aligned}$$

and where $M_i = (-\tilde{A}_i + \frac{1}{2}S_iD_i + S_iK_{i1})$. Using $D_i(t_0) > 0$ we infer from [8, Hilfssatz 10.3], that $D_i(t) > 0$ for $t \in \mathcal{I} \cap (-\infty, t_0]$; this proves Lemma 1. □

Remark 1. Using [9, Theorem 2.1 (ii)] it is easy to see that the statement of Lemma 1 remains valid if we replace therein both $<$ symbols by $\leq$.

In the next Lemma we prove that the solutions of (5) are bounded below by the corresponding solutions of the decoupled Riccati equations

$$\dot{P}_i = \tilde{A}_i^T P_i + P_i\tilde{A}_i - P_iS_iP_i,\ 1 \leq i \leq N. \tag{11}$$

In the remaining part of this section we denote by K_i and P_i, $1 \leq i \leq N$, the solutions of (5) respectively (11) satisfying the terminal conditions $K_i(t_f) = P_i(t_f) = K_{i0} \geq 0$, $1 \leq i \leq N$. Notice that (5) is a special case of the problem considered here.

Lemma 2: If $K_i(t)$, $1 \leq i \leq N$, exist for $t \in [t_o, t_f]$ then

$$0 \leq P_i(t) \leq K_i(t) \;\text{ for }\; 1 \leq i \leq N,\; t \in [t_0, t_f]. \tag{12}$$

Proof: According to [9, Theorem 2.1] $P_i(t)$ and $K_i(t)$ are positive semidefinite and consequently

$$Q_i(t) \leq Q_i(t) + \sum_{j=1;j\neq i}^{N} \pi_{ij} K_j(t) \;\text{ for }\; t \in [t_0, t_f].$$

Hence (12) follows if we apply [8, Satz 10.2] to the functions P_i and K_i. □

Next we give sufficient conditions for the monotonicity of $K_i(t), 1 \leq i \leq N$, and for the existence of positive semidefinite solutions of the coupled algebraic Riccati equations (7).

Theorem 1. If $K_{i0} \geq 0$, $1 \leq i \leq N$, can be chosen such that

$$(\dot{K}_{i0}(t_f) :=) - \tilde{A}_i^T K_{i0} - K_{i0} \tilde{A}_i - Q_i + K_{i0} S_i K_{i0} - \sum_{j=1,j\neq i}^{N} \pi_{ij} K_{j0} \geq 0, \tag{13}$$

then $P_i(t)$ and $K_i(t)$ exist for $t \in (-\infty, t_f]$ with

$$\dot{K}_i(t) \geq 0,\; \dot{P}_i(t) \geq 0 \tag{14}$$

and

$$0 \leq P_i(t) \leq K_i(t) \leq K_{i0} \tag{15}$$

for $1 \leq i \leq N$ and $t \in (-\infty, t_f]$. In addition

$$K_i^\infty := \lim_{t\to-\infty} K_i(t) \;\text{ and }\; P_i^\infty := \lim_{t\to-\infty} P_i(t),\; 1 \leq i \leq N$$

exist.
If in addition $(\sqrt{Q_i}, \tilde{A}_i)$, $1 \leq i \leq N$, is detectable then the matrices $\tilde{A}_i - S_i K_i^\infty$, $1 \leq i \leq N$, are stable.

Proof. Differentiating (5) we obtain (with the notations of (7))

$$\begin{aligned} \ddot{K}_i &= -\tilde{A}_i^T \dot{K}_i - \dot{K}_i \tilde{A}_i + \dot{K}_i S_i K_i + K_i S_i \dot{K}_i - \sum_{j=1;j\neq i}^{N} \pi_{ij} \dot{K}_j \\ &= (-\tilde{A}_i + S_i K_i)^T \dot{K}_i + \dot{K}_i(-\tilde{A}_i + S_i K_i) - \sum_{j=1;j\neq i}^{N} \pi_{ij} \dot{K}_j. \end{aligned}$$

According to (13) $\dot{K}_i(t_f) \geq 0$ for $1 \leq i \leq N$, hence we infer from [9, Theorem 2.1 (ii)] that $\dot{K}_i(t) \geq 0$ for $t \in (-\infty, t_f]$. Similarly we obtain $\dot{P}_i(t) \geq 0$ for $t \in (-\infty, t_f]$. This proves

(14) and with Lemma 2 we get (15) and the existence of the limits $K_i^\infty \geq 0$ and $P_i^\infty \geq 0$. Applying [9, Lemma 4.2] we infer that $\tilde{A}_i - S_i K_i$, $1 \leq i \leq N$, is stable if $(\sqrt{Q_i}, A_i)$ is detectable. □

The following Corollary is an immediate consequence of Theorem 1.

Corollary 1. The following statements are equivalent:

(i) (7) has solutions $K_i^\infty \geq 0,\ 1 \leq i \leq N$.

(ii) (13) has positive semidefinite solutions $K_{i0} \geq 0,\ 1 \leq i \leq N$.

Condition (13) is a necessary and sufficient condition for the existence of a positive semidefinite solution K_i^∞, $1 \leq i \leq N$ of (7) and it ensures the existence of a solution of the differential equation (1) which is nonincreasing as t decreases and has the limit K_i^∞, $1 \leq i \leq N$. On the other hand we infer from Theorem 1 and [9]:

Corollary 2. (i) The solutions K_i^0, $1 \leq i \leq N$ of the terminal value problem (5) are monotone nondecreasing as t decreases.
(ii) $K_i^{0,\infty} := \lim_{t \to -\infty} K_i^0(t)$ exists if there are solutions $K_{i0} \geq 0,\ 1 \leq i \leq N$, of (13) or if $(\tilde{A}_i, B_i)$ is stabilizable, $1 \leq i \leq N$, and (with $B = \mathrm{diag}(B_1, \ldots B_N)$)

$$\inf_{\mathcal{K}} || \int_0^\infty e^{t(\mathcal{A}-\mathcal{B}\mathcal{K})^T} \sum_{k=1}^{N-1} \mathcal{T}_k \mathcal{T}_k^T e^{t(\mathcal{A}-\mathcal{B}\mathcal{K})} || < 1. \tag{16}$$

Proof. Assertion (i) corresponds to [9, Lemma 5.2 (ii)].
If (13) has positive semidefinite solutions (or if (16) and stabilizability holds) then it follows from Lemma 1, Remark 1 and Theorem 1 (respectively from [9, Lemma 5.2 (i)]) that $K_i^0(t)$ is bounded for $t \in (-\infty, t_f]$. Hence assertion (ii) follows from (1).

Remarks 2. (i) An equivalent formulation of f condition (16) is given in [3, p. 194].
(ii) If $K_{i0} \geq 0,\ 1 \leq i \leq N$, can be chosen such that

$$\dot{K}_i(t_f) = -\tilde{A}_i^T K_{i0} - K_{i0}\tilde{A}_i - Q_i + K_{i0} S_i K_{i0} - \sum_{j=1, j\neq i}^{N} \pi_{ij} K_{j0} \leq 0,$$

then we infer as with the proof of Theorem 1 that $P_i(t)$ and $K_i(t)$ exist for $t \in (-\infty, t_f]$ with

$$\dot{K}_i(t) \leq 0 \quad \text{and} \quad \dot{P}_i(t) \leq 0 \quad \text{for} \quad t \in (-\infty, t_f].$$

P_i is bounded, hence $\tilde{P}_i := \lim_{t \to -\infty} P_i(t)$, $1 \leq i \leq N$, exists. If we assume that (in addition to the stabilizability of $(\tilde{A}_i, B_i)$) that (16) holds then - according to [9, Theorem 2.1 (iv)] - K_i is bounded and $\tilde{K}_i := \lim_{t \to -\infty} K_i(t)$ exists and defines a solution of (7) with $\tilde{K}_i \geq \tilde{P}_i$, $1 \leq i \leq N$.
The matrices $\tilde{A}_i - S_i \tilde{K}_i$, $1 \leq i \leq N$, are stable if $(\sqrt{Q_i}, \tilde{A}_i)$, $1 \leq i \leq N$, is detectable.

IV. Numerical Algorithms

Using the results given above and exploiting the monotone dependence properties of the solutions of (5) and (7) on Q_i, S_i and $K_i(t_f)$, two algorithms are now proposed to solve equation (7).
In the sequel we assume that $(\tilde{A}_i, B_i)$ is stabilizable and that $(\sqrt{Q_i}, \tilde{A}_i)$ is detectable, $1 \leq i \leq N$; this condition implies in particular that each algebraic Riccati equation (17) has a unique positive semidefinite solution.

Algorithm 1:

This numerical method to solve the coupled Riccati equations (7) is based on the successive solution of decoupled equations where the constant term is modified at each iteration.

- At iteration ν compute for $1 \leq i \leq N$ the (unique) positive semidefinite solution $K_i^{\nu+1}$

$$\tilde{A}_i^T K_i^{\nu+1} + K_i^{\nu+1} \tilde{A}_i - K_i^{\nu+1} S_i K_i^{\nu+1} + Q_i^\nu = 0 \tag{17}$$

  with

$$Q_i^\nu := Q_i + \sum_{j \neq i} \pi_{ij} K_i^\nu \tag{18}$$

  and $K_i^0 := 0$, $1 \leq i \leq N$; we recall that $(\sqrt{Q_i^\nu}, \tilde{A}_i)$ is observable for $\nu \in \mathbf{N}$ (see [9, Lemma 4.1]).

From section II we infer that $0 \leq K_i^1 \leq K_i^2 \leq K_i^3 \leq \cdots$, $1 \leq i \leq N$; if the sequences $(K_i^\nu)\nu \in \mathbf{N}$, $1 \leq i \leq N$, are bounded then $K_i^\infty := \lim_{\nu \to \infty} K_i^\nu$ exists for $1 \leq i \leq N$ and defines a positive semidefinite solution of (7).

We note that the assumptions stated at the beginning of section IV do not guarantee the existence of a positive semidefinite solution K_i^∞, $1 \leq i \leq N$, of (7). Therefore we assume in the sequel (cf. Corollary 1) that the Riccati inequality (13) has positive semidefinite solutions K_{i0}, $1 \leq i \leq N$; this yields a second version of Algorithm 1:

Initializing algorithm 1 by $K_i^0 := K_{i0}$, $1 \leq i \leq N$, we infer again from section II (ii) that the corresponding sequences verify

$$K_i^0 \geq K_i^1 \geq K_i^2 \geq \ldots \geq 0,$$

consequently there exists

$$\lim_{\nu \to \infty} K_i^\nu = K_i^\infty \geq 0,$$

where K_i^∞, $1 \leq i \leq N$ are solutions of (7) and where $\tilde{A}_i - S_i K_i^\infty$ is stable for $1 \leq i \leq N$. According to [9, Theorem 2.1 (iv)] K_i^∞, $1 \leq i \leq N$, is positive definite and is the unique positive semidefinite solution of (7), if in addition $(\sqrt{Q_i}, \tilde{A}_i)$, $1 \leq i \leq N$, is observable.

Algorithm 2 gives an alternative method for the evaluation of these solutions K_i^∞; in the sequel we assume for convenience that K_i^∞, $1 \le i \le N$, is unique.

Algorithm 2:

This algorithm is based on the defect correction method [11]. Let

$$K_i^\infty = F_i + E_i \tag{19}$$

where F_i is an approximate solution of K_i^∞, and E_i is the defect ($E_i = K_i^\infty - F_i$). Introducing (19) in (7), we obtain

$$(\tilde{A}_i - S_iF_i)^T E_i + E_i(\tilde{A}_i - S_iF_i) - E_iS_iE_i + Q_i + \tilde{A}_i^T F_i + F_i\tilde{A}_i - F_iS_iF_i$$
$$+ \sum_{j=1, j\neq i}^{N} \pi_{ij}(F_j + E_j) = 0. \tag{20}$$

This equation is rewritten as

$$\hat{A}_i^T E_i + E_i\hat{A}_i - E_iS_iE_i + \hat{Q}_i^\nu = 0 \tag{21}$$

with

$$\hat{Q}_i^\nu = Q_i + \tilde{A}_i^T F_i + F_i\tilde{A}_i - F_iS_iF_i + \sum_{j=1, j\neq i}^{N} \pi_{ij}(F_j + E_j) \tag{22}$$

and $\hat{A}_i = (\tilde{A}_i - S_iF_i)$.
$\hat{Q}_i^\nu$ is the residual obtained by inserting F_i in (7). The main advantage of this method is that it operates on the error and this is numerically stable [12]. The algorithm could be summarized as follows:

- Compute the unique positive semidefinite solutions of the decoupled equations:

$$\tilde{A}_i^T F_i + F_iA_i - F_iS_iF_i + Q_i = 0 \tag{23}$$

- Set $E_i^0 = F_i$ and $F_i^0 = 0$
- While $\max \|E_i^\nu\| \ge \epsilon$ for $i = 1, 2, \ldots N$

$$\begin{aligned}
F_i^\nu &= F_i^{\nu-1} + E_i^{\nu-1} \\
\hat{Q}_i^\nu &= Q_i + \tilde{A}^T F_i^\nu + F_i^\nu \tilde{A}_i - F_i^\nu S_i F_i^\nu + \sum_{j=1, j\neq i}^{N} \pi_{ij} F_j^c \\
\hat{A}_i^\nu &= \tilde{A}_i - S_iF_i^\nu. \\
\textit{Solve} \quad & \hat{A}_i^{\nu^T} E_i^\nu + E_i^\nu \hat{A}_i^\nu - E_i^\nu S_i E_i^\nu + \hat{Q}_i^\nu = 0
\end{aligned}$$

- End While.

Since F_i is the positive semidefinite solution of the decoupled Riccati equations $F_i \leq K_i^\infty$, and from $E_i^\nu \geq 0$ it follows that

$$F_1^0 \leq F_i^1 \leq F_i^2 \ldots .$$

If the sequence $(F_i^\nu)_{\nu \in \mathbf{N}}$ is bounded then $\lim_{\nu\to\infty} F_i^\nu = K_i^\infty$.

It should be noted that in both algorithms any known method could be used to solve the decoupled Riccati equations (17), (23).

V. Example

Consider a 3^{rd} order plant having three operating modes as follows

mode 1:

$$A_1 = \begin{pmatrix} -2.5 & 0.3 & 0.8 \\ 1 & -3 & 0.2 \\ 0 & 0.5 & -2 \end{pmatrix}; \quad B_1 = \mathrm{diag}\left\{ 0.707 \quad 1 \quad 1 \right\}$$

$$R_1 = I \qquad Q_1 = \mathrm{diag}\left\{ 25 \quad 1 \quad 11 \right\}$$

mode 2:

$$A_2 = \begin{pmatrix} -2.5 & 1.2 & 0.3 \\ -0.5 & 5. & -1 \\ 0.25 & 1.2 & 5 \end{pmatrix}; \quad B_1 = \mathrm{diag}\left\{ 0.707 \quad 1 \quad 0.707 \right\}$$

$$R_2 = I \qquad Q_2 = \mathrm{diag}\left\{ 37 \quad 70 \quad 34 \right\}$$

mode 3:

$$A_3 = \begin{pmatrix} 2 & 1.5 & -0.4 \\ 2.2 & 3 & 0.7 \\ 1.1 & 0.9 & -2 \end{pmatrix}; \quad B_3 = \mathrm{diag}\left\{ 0.707 \quad 1 \quad 1 \right\}$$

$$R_3 = I \qquad Q_3 = \mathrm{diag}\left\{ 10 \quad 16 \quad 21 \right\}$$

and the transition density matrix is given by:

$$\Pi = \begin{pmatrix} -3 & 0.5 & 2.5 \\ 1 & -2 & 1 \\ 0.7 & 0.3 & -1 \end{pmatrix}.$$

Both algorithms lead to the solution

$$K_1^\infty = \begin{pmatrix} 5.0196 & 1.0085 & 0.3839 \\ 1.0085 & 2.5138 & 0.3519 \\ 0.3839 & 0.3519 & 3.0133 \end{pmatrix}$$

$$K_2^\infty = \begin{pmatrix} 5.3250 & 0.2984 & 0.5190 \\ 0.2984 & 13.9461 & 0.7282 \\ 0.5190 & 0.7282 & 19.8888 \end{pmatrix}$$

$$K_3^\infty = \begin{pmatrix} 9.6962 & 3.7653 & 0.2102 \\ 3.7653 & 8.2580 & 0.6635 \\ 0.2102 & 0.6635 & 3.4686 \end{pmatrix}.$$

For $\epsilon = 10^{-8}$ figure 1 gives the norm of K_1^ν $(i = 1, 2, 3)$ when algorithm 1 is used with K_1^0 being the solution of decoupled Riccati equations (17), and as expected an increasing sequence is obtained.
If K_i^0 are chosen such that $\dot{K}_i \geq 0$, a decreasing sequence (K_i^ν) is obtained as depicted in figure 2. Figure 3 gives the norm of defect matrices E_i for algorithm 2.

Figure 1

Figure 2

Figure 3

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