Non-symmetric Matrix Riccati Equations

G. Freiling and G. Jank
Lehrstuhl II für Mathematik, RWTH Aachen,
Templergraben 55, D-52056 Aachen, FRG

Abstract

We prove a fundamental representation formula for all solutions of the matrix Riccati differential equation (RDE) and of the corresponding algebraic Riccati equation (ARE). This formula contains the complete information on the phase portrait of (RDE) and on the structure of the set Γ of all solutions of (ARE). In particular we describe all constant, periodic and almost periodic solutions of (RDE). Further we give an application of the fundamental representation formula to the investigation of nonautonomous Riccati equations.

Key words: Matrix Riccati differential equation, algebraic Riccati equation, asymptotic properties.
AMS(MOS) subject classifications. 34A46, 34C25, 34C27, 34E10, 49J15, 93C15.

1. Introduction.

In this paper we study the matrix Riccati differential equation (RDE)(or $(RDE)_{\mathbf{R}}$)

$$\dot{W} = B_{21} + B_{22}W - WB_{11} - WB_{12}W, \ t \in \mathbf{C} \text{ (or } t \in \mathbf{R}) \tag{1.1}$$

and the corresponding algebraic Riccati equation (ARE)

$$0 = B_{21} + B_{22}W - WB_{11} - WB_{12}W, \tag{1.2}$$

where $W(t)$ (or W) is a complex $m \times n$ matrix and where $B_{11}, B_{12}, B_{21}, B_{22}$ are constant complex matrices of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively. The matrix Riccati equation - especially $(RDE)_{\mathbf{R}}$ and (ARE) for $m = n$ - plays an important role in many branches of applied mathematics, notably in variational theory and the allied areas of optimal control and filtering, invariant imbedding, spectral factorization and dynamic programming (see [5], [14], [21]); nonsquare matrix Riccati equations appear for example in Nash and Stackelberg control problems, where the properties of Riccati equations determine the existence of the optimal open-loop strategies (see [1], [18], [28] for further references). On the other hand (RDE) is also of mathematical interest, since it is the description in local coordinates of the differential equation on a Graßmann manifold, whose flow is given by the action of a one-parameter subgroup of $Gl(n+m, \mathbf{C})$ (see [10], [22],

[27]).

In section 2 of this paper we derive a fundamental representation formula (2.10) for the solutions of (RDE); this formula contains the complete information on the phase portrait of (RDE) and in particular on the structure of the set Γ of all solutions of (ARE) (see section 3 and section 4). In the last part of this note we give an application of the fundamental representation formula to the investigation of nonautonomous Riccati equations.

2. The fundamental representation formula

In this section we describe the connection between (RDE) and the corresponding linear differential equation (L) (see [21, p.11]), and we derive a detailed formula for the general solution of (RDE).

Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $Y = \begin{pmatrix} Q \\ P \end{pmatrix}$ with complex $n \times n$ and $m \times n$ matrix functions Q and P respectively; then (RDE) and the following linear system of differential equations

$$\dot{Y} = B\,Y,\ t \in \mathbf{C} \tag{L}$$

are equivalent in the following sense:

(i) Let $I_n \in \mathbf{C}^{n\times n}$ be the unit matrix, let W be a solution of (RDE). If Q is the unique solution of the initial value problem

$$\dot{Q} = (B_{11} + B_{12}W(t))Q,\ Q(t_0) = I_n, \qquad t_0 \in G \subset \mathbf{C}$$

and $P(t) = W(t)\,Q(t)$, then $Y := \binom{Q}{P}$ is a solution of (L).

(ii) If $Y = \binom{Q}{P}$ is a solution of (L) such that $Q(t)$ is regular for $t \in G \subset \mathbf{C}$, then

$$W:\ G \to \mathbf{C}^{m\times n},\ t \to P(t)Q^{-1}(t) = W(t)$$

is a solution of (RDE).

Let $V = (v_1, \ldots, v_{n+m}) \in \mathbf{C}^{(n+m)\times(n+m)}$ be the matrix defined by a Jordan basis of generalized eigenvectors of B such that

$$V^{-1}B\,V = J = \operatorname{diag}(J_1, \ldots, J_p) = \begin{pmatrix} \lambda_1 & * & 0 \\ \ddots & \ddots & * \\ 0 & \ddots & \lambda_{n+m} \end{pmatrix} \tag{2.1}$$

with $* \in \{0, 1\}$ and (without loss of generality)

$$\operatorname{Re}(\lambda_1) \le \operatorname{Re}(\lambda_2) \le \ldots \le \operatorname{Re}(\lambda_{n+m}), \tag{2.2}$$

and where J is a Jordan canonical form of B with Jordan-blocks

$$J_\nu = \begin{pmatrix} \mu_\nu & 1 & \dots & 0 \\ 0 & \mu_\nu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu_\nu \end{pmatrix} \in \mathbf{C}^{j_\nu \times j_\nu},\ \mu_\nu \in \{\lambda_1, \dots, \lambda_{n+m}\}, 1 \le \nu \le p.$$

If

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n+m,1} & \dots & c_{n+m,n} \end{pmatrix} \in \mathbf{C}^{(n+m)\times n}$$

and

$$(x_1(t), \dots, x_{n+m}(t))\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_{n+m} t}) := V\ e^{Jt}, \tag{2.3}$$

then an arbitrary $(n+m)\times n$ matrix-solution Y of (L) has the form

$$(Y(t;C) =)Y(t) = V\ e^{Jt}C = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} Q(t;C) \\ P(t;C) \end{pmatrix} \tag{2.4}$$

with $Q(t) \in \mathbf{C}^{n\times n},\ P(t) \in \mathbf{C}^{m\times n}$ and $Y(0) = VC = \binom{Q_0(C)}{P_0(C)}$.
Further, to every Jordan-block $J_\nu, 1 \le \nu \le p$, there correspond j_ν solution-vectors of the form

$$\begin{aligned} \hat{y}_{j_1+\dots+j_{\nu-1}+1}(t) &= e^{\mu_\nu t} v_{j_1+\dots+j_{\nu-1}+1} =: e^{\mu_\nu t} x_{j_1+\dots+j_{\nu-1}+1}(t), \\ &\vdots \\ \hat{y}_{j_1+\dots+j_\nu}(t) &= e^{\mu_\nu t} \sum_{i=1}^{j_\nu} \frac{t^{j_\nu - i}}{(j_\nu - i)!} v_{j_1+\dots+j_{\nu-1}+i} =: e^{\mu_\nu t} x_{j_1+\dots+j_\nu}(t), \end{aligned} \tag{2.5}$$

$(1 \le \nu \le p$ and $j_0 := 0)$.
Let $x_\nu(t) = \begin{pmatrix} x_{\nu,1}(t) \\ \vdots \\ x_{\nu,n+m}(t) \end{pmatrix}$, $1 \le \nu \le n+m$, be the polynomials defined by (2.5), $1 \le j \le n$ and $1 \le \ell \le m$, then we set

$$\tilde{x}_\nu := \begin{pmatrix} x_{\nu 1} \\ \vdots \\ x_{\nu n} \end{pmatrix},\ x_\nu(\ell, j) := (x_{\nu 1}, \dots, x_{\nu,j-1}, x_{\nu,n+\ell}, x_{\nu,j+1}, \dots, x_{\nu n})^T;$$

similarly we define $\tilde{v}_\nu$ and $v_\nu(\ell, j)$. Using these notations we infer from (2.4)

$$\begin{aligned} Q(t) &= (\sum_{\nu=1}^{n+m} c_{\nu 1} e^{\lambda_\nu t} \tilde{x}_\nu(t), \dots, \sum_{\nu=1}^{n+m} c_{\nu n} e^{\lambda_\nu t} \tilde{x}_\nu(t)) \\ &= \begin{pmatrix} \sum\limits_{k=1}^{n+m} x_{k1}(t) e^{\lambda_k t} c_k \\ \vdots \\ \sum\limits_{k=1}^{n+m} x_{kn}(t) e^{\lambda_k t} c_k \end{pmatrix} = \begin{pmatrix} Q_{11}(t) & \dots & Q_{1n}(t) \\ \vdots & & \vdots \\ Q_{n1}(t) & \dots & Q_{nn}(t) \end{pmatrix} \end{aligned} \tag{2.6}$$

and

$$P(t) = (\sum_{\nu=1}^{n+m} c_{\nu 1} e^{\lambda_\nu t} \begin{pmatrix} x_{\nu,n+1}(t) \\ \vdots \\ x_{\nu,n+m}(t) \end{pmatrix}, \ldots, \sum_{\nu=1}^{n+m} c_{\nu n} e^{\lambda_\nu t} \begin{pmatrix} x_{\nu,n+1}(t) \\ \vdots \\ x_{\nu,n+m}(t) \end{pmatrix})$$

$$= \begin{pmatrix} \sum_{k=1}^{n+m} x_{k,n+1}(t) e^{\lambda_k t} c_k \\ \vdots \\ \sum_{k=1}^{n+m} x_{k,n+m}(t) e^{\lambda_k t} c_k \end{pmatrix}. \tag{2.7}$$

Consequently we obtain, using the multilinearity of the determinant,

$$\det Q(t) = \sum_{k_1=1}^{n+m} \ldots \sum_{k_n=1}^{n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t} c_{k_1 1} \ldots c_{k_n n} |\tilde{x}_{k_1}(t), \ldots, \tilde{x}_{k_n}(t)|$$

$$= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t} |\tilde{x}_{k_1}(t), \ldots, \tilde{x}_{k_n}(t)| (\sum_{\pi \in \sigma_n} \text{sign } \pi \cdot c_{k_{\pi(1)}1} \ldots c_{k_{\pi(n)}n})$$

$$= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t} |\tilde{x}_{k_1}(t), \ldots, \tilde{x}_{k_n}(t)| \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}. \tag{2.8}$$

We recall that $|\tilde{x}_{k_1}(t), \ldots, \tilde{x}_{k_n}(t)| = |\tilde{v}_{k_1}, \ldots, \tilde{v}_{k_n}|$ is constant if B is semisimple, otherwise these determinants are polynomials with coefficients of the form $|\tilde{v}_{\nu_1}, \ldots, \tilde{v}_{\nu_n}|$.
Since $\det Q(t)$ is an exponential sum with polynomials as coefficients, the asymptotical distribution of the zeros of $\det Q(t)$ is well known (see Lemma 1, [4] and [16]).
For the evaluation of $(PQ^{-1})_{\ell\alpha}$, the element in the ℓ-th row and α-th column of PQ^{-1}, we use (2.6),(2.7) and (2.8). From

$$Q_{\alpha\beta}(t) = \sum_{k=1}^{n+m} c_{k\beta} e^{\lambda_k t} x_{k\alpha}(t) \tag{2.9}$$

and

$$Q^{-1}(t) = \frac{1}{\det Q(t)} \begin{pmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{pmatrix},$$

where $a_{ij}(t)$ is the minor of $Q_{ji}(t)$ with respect to $Q(t)$, we get for $\det Q(t) \neq 0$ (as with (2.8))

$$\det Q(t)(PQ^{-1}(t))_{\ell\alpha} = \sum_{k=1}^{n+m} e^{\lambda_k t} x_{k,n+\ell}(c_{k1}, \ldots, c_{kn}) \begin{pmatrix} a_{1\alpha} \\ \vdots \\ a_{n\alpha} \end{pmatrix}$$

$$= \sum_{k=1}^{n+m} e^{\lambda_k t} x_{k,n+\ell} \begin{vmatrix} Q_{11}(t) & \ldots & Q_{1n}(t) \\ \vdots & & \vdots \\ Q_{\alpha-1,1}(t) & \ldots & Q_{\alpha-1,n}(t) \\ c_{k1} & \ldots & c_{kn} \\ Q_{\alpha+1,1}(t) & \ldots & Q_{\alpha+1,n}(t) \\ \vdots & & \vdots \\ Q_{n1}(t) & \ldots & Q_{nn}(t) \end{vmatrix}$$

$$= \sum_{k=1}^{n+m} e^{\lambda_k t} x_{k,n+\ell} \sum_{\nu_1=1}^{n+m} \cdots \sum_{\nu_{\alpha-1}=1}^{n+m} \sum_{\nu_{\alpha+1}=1}^{n+m} \cdots \sum_{\nu_n=1}^{n+m} e^{\lambda_{\nu_1}+\ldots+\lambda_{\nu_{\alpha-1}}+\lambda_{\nu_{\alpha+1}}+\ldots+\lambda_{\nu_n}} \times$$

$$x_{\nu_1,1}(t)\ldots x_{\nu_{\alpha-1},\alpha-1}(t) x_{\nu_{\alpha+1},\alpha+1}(t)\ldots x_{\nu_n,n}(t) \begin{vmatrix} c_{\nu_1 1} & \cdots & c_{\nu_1 n} \\ \vdots & & \vdots \\ c_{\nu_{\alpha-1},1} & \cdots & c_{\nu_{\alpha-1},n} \\ c_{k1} & \cdots & c_{kn} \\ c_{\nu_{\alpha+1},1} & \cdots & c_{\nu_{\alpha+1},n} \\ \vdots & & \vdots \\ c_{\nu_n 1} & \cdots & c_{\nu_n n} \end{vmatrix}$$

$$= \sum_{1\le\nu_1<\nu_2<\ldots<\nu_n\le n+m} e^{(\lambda_{\nu_1}+\ldots+\lambda_{\nu_n})t} |x_{\nu_1}(\ell,\alpha),\ldots,x_{\nu_n}(\ell,\alpha)|(t) \begin{vmatrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{vmatrix} .$$

We summarize the preceding analysis as a theorem.

Theorem 1. (Fundamental representation formula for the solutions of (RDE))
Let $Y(\cdot;C), Q_0(C)$ *and* $P_0(C)$ *be defined by (2.4). Further let* $Q_0(C)$ *be regular and* $W_C := P_0(C)Q_0(C)^{-1}$. *Then*

$$W(\cdot;W_C) : \mathbf{C}\setminus\{t\in\mathbf{C}|\det Q(t)=0\} \to \mathbf{C}^{m\times n}, t\mapsto P(t)Q(t)^{-1} = W(t;W_C)$$

is a solution of (RDE) *with* $W(0;W_C) = W_C$ *and*

$$w_{\ell\alpha}(t;W_C) = \frac{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t} |x_{k_1}(\ell,\alpha),\ldots,x_{k_n}(\ell,\alpha)|(t) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t} |\tilde{x}_{k_1},\ldots,\tilde{x}_{k_n}|(t) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}, \qquad (2.10)$$

$1 \le \ell \le m, 1 \le \alpha \le n.$

Remark 1. (i) Obviously any solution $W(\cdot;W_C)$ defines a meromorphic matrix function with poles at most in the zeros of $\det Q(t)$. $W(\cdot;W_C)$ is called the solution corresponding to $S_0 = span(VC)$, the column space of VC.

The coefficients $\begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}$ appearing in (2.10) are the so-called *Plücker coordinates* of C (or of the n-dimensional subspace $span(C)$); these coefficients and also the coefficients

$|\tilde{x}_{k_1}, \ldots, \tilde{x}_{k_n}|(t)$ and $|x_{k_1}(\ell,\alpha), \ldots, x_{k_n}(\ell,\alpha)|(t)$ are coupled (see Lemma 1 and Lemma 3), which has strong influence on the behaviour of the solutions of (RDE) and on the structure of the set of all solutions of (ARE).
(ii) If the coefficients B_{ij} in (1.1) are functions of t and (or) of some parameter ρ and if the differential equation (L) has a $(n+m) \times (n+m)$ fundamental matrix solution of the form

$$(z_1(t,\rho), \ldots, z_{n+m}(t,\rho))\mathrm{diag}(e^{\lambda_1(t,\rho)}, \ldots, e^{\lambda_{n+m}(t,\rho)}),$$

then the statement of Theorem 1 remains true for the nonautonomous or parameter dependent matrix Riccati equation, if we replace the vector polynomials $x_k(t)$ in (2.10) by the functions $z_k(t,\rho)$ and $\lambda_k t$ by $\lambda_k(t,\rho), 1 \le k \le n+m$.
In the special case of T-periodic coefficients $B_{ij}(t)$ (L) has, according to the Floquet-Lyapunov-Theorem, a fundamental system of solutions $\hat{y}_1, \ldots, \hat{y}_{n+m}$ of the form (2.5) with T-periodic vector-functions $v_1, \ldots, v_{n+m}$ (instead of eigenvectors); if we define the polynomials $x_1, \ldots, x_{n+m}$ (with T-periodic coefficients) formally as with (2.5) then the fundamental representation formula (2.10) can also be used in the T-periodic case.
(iii) As far as we know the fundamental representation formula (2.10) for the general solution of (RDE) is in this detailed form new even in the autonomous case. Obviously (2.10) contains the complete information on the phase portrait of (RDE) and particularly the complete information on all the solutions of (ARE).
The properties of the solutions of (RDE) and (ARE) are very important for various branches of mathematics, therefore these equations have received considerable attention. Readers who are interested in a detailed description of the geometry of the solutions of the most important special cases of $(ARE), (RDE)$ and of the so-called extended Riccati-equation $(ERDE)$ (see [27]) are referred to the papers of Callier and Willems [7], Shayman [23]–[27] and Hermann and Martin [10], [11] on this topic and to the literature cited therein.
In the next section we use (2.10) to study the geometry of the solutions of (ARE) and (RDE). We focus our investigations mainly to those subcases which have not yet been considered in detail in [18], [28] and [23]–[27]) - in this sense our results complement those of Shayman and also those of Medanic [18] and Telford and Moore [28].

3. The solutions of the algebraic Riccati equation.

In this section we use (2.10) to derive a parametrization and a geometric description of the set Γ of all solutions of (ARE). For the formulation of our results we introduce the following notations.

3.1 Notations. a) For $\Lambda = (\lambda_1, \ldots, \lambda_{n+m})$ and $a = (a_1, \ldots, a_{n+m}) \in \{0,1\}^{n+m}$ we set

$$\zeta(a) =< \Lambda, a >= \sum_{\nu=1}^{n+m} a_\nu \lambda_\nu, \ R(a) = \mathrm{Re}\,\zeta(a) \quad \text{and} \quad I(a) = \mathrm{Im}\,\zeta(a).$$

b) Let $N = \binom{n+m}{n}$ and let $A = \{a^1, \dots, a^N\} \subset \{0,1\}^{n+m}$ be the set with

$$a^j = (a^j_1, \dots, a^j_{n+m}) \quad \text{and} \quad a^j_i = \begin{cases} 1 & \text{for} \qquad i \in \{\nu^j_1, \dots, \nu^j_n\}; \nu^j_1 < \dots < \nu^j_n; \\ 0 & \text{else.} \end{cases}$$

In the sequel we assume that the N elements of A are enumerated such that

$$j < k \quad \text{if and only if} \tag{3.1}$$

either $\mathrm{Re}(\zeta(a^j)e^{i\phi}) < \mathrm{Re}(\zeta(a^k)e^{i\phi})$ for $0 < \phi \leq \epsilon_0$ for some $\epsilon_0 > 0$ or $\zeta(a^j) = \zeta(a^k)$ and $(\nu^j_1, \dots, \nu^j_n)$ is lexicographically smaller than $(\nu^k_1, \dots, \nu^k_n)$.

c) For $a^j \in A, C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix} \in \mathbf{C}^{(n+m)\times n}$ and with the polynomials x_ν defined in (2.5) we set $p(t,a^j) = |\tilde{x}_{\nu^j_1}(t), \dots, \tilde{x}_{\nu^j_n}(t)|, p_{\ell\alpha}(t,a^j) = |x_{\nu^j_1}(\ell,\alpha), \dots, x_{\nu^j_n}(\ell,\alpha)|(t),\ 1 \leq \ell \leq m, 1 \leq \alpha \leq n$, and $D(a^j, C) = \begin{vmatrix} c_{\nu^j_1} \\ \vdots \\ c_{\nu^j_n} \end{vmatrix}$. With these abbreviations (2.10) can be written in the form

$$w_{\ell,\alpha}(t; W_C) = \frac{\sum\limits_{a^j \in A} e^{t\zeta(a^j)} p_{\ell\alpha}(t,a^j) D(a^j, C)}{\sum\limits_{a^j \in A} e^{t\zeta(a^j)} p(t,a^j) D(a^j, C)}. \tag{3.2}$$

The following Lemma and Lemma 3 indicate the type of coupling of the coefficients appearing in (2.10) and (3.2).

Lemma 1. *a) Let $1 \leq k_1 < k_2 < \dots < k_n \leq n+m$, $j \in \{1, \dots, n+m\} \setminus \{k_1, \dots, k_n\}$ and $|\tilde{v}_{k_1}, \cdots, \tilde{v}_{k_n}| \neq 0$. Then at least one of the determinants*

$$|\tilde{v}_j, \tilde{v}_{k_2}, \cdots, \tilde{v}_{k_n}| \text{ or } |v_j(\ell,\alpha), v_{k_2}(\ell,\alpha), \cdots, v_{k_n}(\ell,\alpha)|, \quad 1 \leq \ell \leq m,\ 1 \leq \alpha \leq n$$

is nonzero.

b) Let $a^j \in A$ and let $p(t,a^j) = c\, t^k$, $c \neq 0$, be a monomial of degree $k \geq 1$. Then at least one of the polynomials $p_{\ell\alpha}(t,a^j)$, $1 \leq \ell \leq m$, $1 \leq \alpha \leq n$, is not identically zero and not a monomial of degree k.

Proof. $D_0 = (v_j, v_{k_2}, \dots, v_{k_n})$ has rank n and by assumption $D_1 = (\tilde{v}_{k_2}, \dots, \tilde{v}_{k_n})$ has rank $n-1$. Let the first $n-1$ rows of D_1 be linearly independent – otherwise we proceed similarly. Then we can transform D_0 by elementary column operations to the form $\begin{pmatrix} 0 & F \\ b & D \end{pmatrix}$, where $b \in \mathbf{C}^{m+1} \setminus \{0\}$ and where $F \in \mathbf{C}^{(n-1)\times(n-1)}$ is regular; this proves assertion a).
b) follows with a) from the definition of the polynomials $p(t,a^j)$ and $p_{\ell\alpha}(t,a^j)$. Notice the special form of the vector-polynomials $x_k(t)$ in (2.5).

3.2 The asymptotic behaviour of $W(re^{i\varphi}; W_C)$ for $r \to \infty$. The complex plane is divided into $2h$ sectors

$$S_k = \{t \in \mathbf{C} \mid \varphi_{k-1} \leq \ \arg t \leq \varphi_k\}, \ 1 \leq k \leq 2h, \tag{3.3}$$

where

(i) $\varphi_0 \leq 0 < \varphi_1 < \varphi_2 < \ldots < \varphi_{2h} = \varphi_0 + 2\pi \leq 2\pi$;

(ii) for $1 \leq k \leq 2h$ there is a permutation σk of $\{1, \ldots, N\}$ with

$$\mathrm{Re}(t < \Lambda, a^{\sigma k(1)} >) \leq \mathrm{Re}(t < \Lambda, a^{\sigma k(2)} >) \leq \ldots \leq \mathrm{Re}(t < \Lambda, a^{\sigma k(N)} >) \text{ for } t \in S_k;$$

(iii) the sectors S_k are the maximal sectors (3.3) with the properties (i) and (ii).

We note that the permutations σk are in general not yet uniquely determined; this is not necessary. In addition here we can assume for convenience that $\sigma 1 = \mathrm{id}$.

From the results on the asymptotic behaviour of the zeros of exponential sums [16] (see also Boese [4] for further details) and from (2.10) we infer that $\lim_{r\to\infty} W(re^{i\varphi}; W_C)$ exists for $\varphi \neq \varphi_j$:

Lemma 2. *Let $\varepsilon > 0$ and let $C \in \mathbf{C}^{(n+m)\times n}$ be such that $Q_0(C)$ is regular. Then almost all zeros (i. e. except at most finitely many) of det $Q(t)$ -defined by (2.4)- are contained in the set*

$$\bigcup_{k=1}^{2h} \{t \in \mathbf{C} \mid \varphi_k - \varepsilon < \ \arg \ t < \varphi_k + \varepsilon\}.$$

In addition, for $\varphi \in [0, 2\pi] \setminus \{\varphi_1, \ldots, \varphi_{2h}\}, 1 \leq \ell \leq m$ and $1 \leq \alpha \leq n$ there exists

$$w_{\ell\alpha}(e^{i\varphi} \cdot \infty; W_C) := \lim_{r\to\infty} w_{\ell\alpha}(re^{i\varphi}; W_C) \tag{3.4}$$

with respect to the chordal metric on the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The limits are easily determined from (2.10).

3.3 Three representations of Γ. There are several possibilities to determine the set Γ of all (complex) solutions of (ARE):

(i)

$$\Gamma = \{PQ^{-1} | \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbf{C}^{(n+m)\times n}, \ \det Q \neq 0 \ \text{ and } \ \mathrm{span} \begin{pmatrix} Q \\ P \end{pmatrix} \ \text{ is } B-\text{invariant}\}$$

(see Meyer [19], Theorem 1 for an alternative formulation and a proof).

(ii) For $\varphi \notin \{\varphi_1, \varphi_2, \cdots, \varphi_{2h}\}$
$\Gamma = \mathbf{C}^{m\times n} \cap \{W(e^{i\varphi}\cdot\infty; W_C) | C \in \mathbf{C}^{(n+m)\times n}, Q_0(C) \quad \text{regular}\}$.

This is an immediate consequence of Lemma 2 and of (2.10).

(iii) $\Gamma =$ set of all constant $\mathbf{C}^{m\times n}$-valued functions $W(\cdot; W_C)$ of the form (2.10).

Using (i), (ii) and (iii) and the fundamental representation formula (2.10) we obtain a nice parametrization of Γ. For this purpose we introduce some abbreviations.

3.4 Notations. Let $V_n(\mathbf{C}^{n+m})$ be the set of all full rank $(n+m)\times n$ matrices with complex entries. For $C \in V_n(\mathbf{C}^{n+m})$ and $1 \le k \le N$ we set

$$J_0(C) = \{j | 1 \le j \le N \text{ and } D(a^j, C) \ne 0\}$$

$$G_\zeta(a^k) = \{C \in V_n(\mathbf{C}^{n+m}) | D(a^\nu, C) = 0 \text{ for } \zeta(a^\nu) \ne \zeta(a^k) \text{ and } D(a^k; C) \ne 0\}$$

and for $\quad 1 \le \ell \le m, \quad 1 \le \alpha \le n$ let

$$\begin{aligned} nom_{\ell\alpha}(C) &= \sum_{\nu\in J_0(C)} p_{\ell\alpha}(t, a^\nu) D(a^\nu, C), \\ den(C) &= \sum_{\nu\in J_0(C)} p(t, a^\nu) D(a^\nu, C), \\ d_n(C) &= \max\{\text{degree}(nom_{\ell\alpha}(C)) | 1 \le \ell \le m, \ 1 \le \alpha \le n\}, \\ d_d(C) &= \text{degree}(\text{den}(C)), \end{aligned}$$

where here and in the sequel - for technical reasons - the degree of the zero-polynomial is defined to be $-\infty$.
Notice that for $C \in G_\zeta(a^k)$ all exponential terms in (3.2) are identical to $e^{t\zeta(a^k)}$. Hence in this case $w_{\ell\alpha}(t; W_C) = \frac{nom_{\ell\alpha}(C)}{den(C)}$ if $d_d(C) \ge 0$. In particular $W(t; W_C)$ is a constant solution if $d_d(C) = 0$ and $d_n(C) \le 0$.
For the description of Γ we use the set of parameter-matrices

$$G^0_\zeta(a^k) = \{C \in G_\zeta(a^k) | d_d(C) = 0 \text{ and } d_n(C) \le 0\}$$

and the set of indices

$$J(a^k) = \bigcup_{C\in G^0_\zeta(a^k)} J_0(C).$$

3.5 Parametrization of Γ. a) To any matrix $C \in G^0_\zeta(a^k) \ne \emptyset$ there corresponds - according to (2.10) - a solution $W_C \equiv W(t; W_C) =: (w^C_{\ell\alpha})_{\substack{1\le\ell\le m \\ 1\le\alpha\le n}}$ of (ARE), where

$$w^C_{\ell\alpha} = \frac{\sum\limits_{\nu\in J_0(C)} p_{\ell\alpha}(t, a^\nu) D(a^\nu, C)}{\sum\limits_{\nu\in J_0(C)} p(t, a^\nu) D(a^\nu, C)}. \tag{3.5}$$

On the other hand it follows from 3.3 (i), (iii) and formula (3.2) that any solutions of (ARE) can be represented in the form (3.5) (see also Theorem 3 (i)).
To each family

$$F_k = \{C \in V_n(\mathbf{C}^{n+m}) | J_0(C) \subset J(a^k)\}$$

there corresponds the family $\Gamma_k = \{W_C | C \in F_k\}$ of solutions of (ARE) and $\Gamma = \bigcup\limits_{k=1}^{N} \Gamma_k$.
It is well known that the number of solutions of (ARE) corresponds to the number of n-dimensional B-invariant subspaces of $\mathbf{C}^{n+m}$ being complementary to the span of $\binom{0_{n,m}}{I_m}$ (see 3.3 (i)); alternatively the number of elements of Γ and the structure of Γ can be easily determined from (3.5) if we know $\#J_0(C)$ and $\#J(a^k)$, $1 \le k \le N$. Obviously Γ is either finite or uncountable.
The following Lemma implies in particular that $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$ if B has only simple eigenvalues.

Lemma 3. *Let $C \in G_\zeta(a^j)$ and $D(a^\alpha, C) \neq 0$. Then* $(\lambda_{\nu_1^j}, \cdots, \lambda_{\nu_n^j}) = (\lambda_{\nu_1^\alpha}, \cdots, \lambda_{\nu_n^\alpha})$.

Proof. From the definition of $G_\zeta(a^j)$ it follows that $\zeta(a^j) = \zeta(a^\alpha)$. Let $\alpha \neq j$ and let us assume that the assertion of Lemma 3 does not hold, then there is a q with $\lambda_{\nu_i^\alpha} = \lambda_{\nu_i^j}$ for $q+1 \le i \le n$ and (without loss of generality) $\mathrm{Re}(\lambda_{\nu_q^\alpha} e^{i\varphi}) < \mathrm{Re}(\lambda_{\nu_q^j} e^{i\varphi})$ for $0 < \varphi < \epsilon_0$. Since $c_{\nu_1^j}, \dots, c_{\nu_n^j}$ are linearly independent, we have

$$c_{\nu_q^\alpha} = \sum_{i=1}^{n} \gamma_i c_{\nu_i^j},$$

and from $\begin{vmatrix} c_{\nu_1^\alpha} \\ \vdots \\ c_{\nu_n^\alpha} \end{vmatrix} \neq 0$ it follows that $(\gamma_1, \dots, \gamma_q) \neq 0$ since $c_{\nu_q^\alpha}, c_{\nu_{q+1}^\alpha} = c_{\nu_{q+1}^j}, \dots, c_{\nu_n^\alpha} = c_{\nu_n^j}$ are linearly independent.

Let $\gamma_p \neq 0$ for some $p \in \{1, \dots, q\}$ then $\begin{vmatrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_{p-1}^j} \\ c_{\nu_q^\alpha} \\ c_{\nu_{p+1}^j} \\ \vdots \\ c_{\nu_n^j} \end{vmatrix} = \gamma_p \begin{vmatrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_n^j} \end{vmatrix} \neq 0$ and

$$\mathrm{Re}(e^{i\varphi}[\lambda_{\nu_1^j} + \ldots + \lambda_{\nu_{p-1}^j} + \lambda_{\nu_q^\alpha} + \lambda_{\nu_{p+1}^j} + \ldots + \lambda_{\nu_n^j}]) < \mathrm{Re}(e^{i\varphi}\zeta(a^j)) \text{ for } 0 < \varphi < \epsilon_0.$$

This contradicts $C \in G_\zeta(a^j)$ and Lemma 3 is proved.

b) From the definition of $J(a^k)$ we infer, using Lemma 1 and Lemma 3 that $\#J(a^k) \le 1$ if at most one of the eigenvalues $\lambda_{\nu_1^k}, \cdots, \lambda_{\nu_n^k}$ is a multiple eigenvalue and if this eigenvalue

has geometric multiplicity one.
If $\#J(a^k)=0$ then $\Gamma_k=\emptyset$ and if $\#J(a^k)=1$ then $\Gamma_k=\{W_k\}$ where $W_k=(w_{k,\ell\mu})_{\substack{1\le\ell\le m\\ 1\le\mu\le n}}$ with

$$w_{k,\ell\mu}=\frac{p_{\ell\mu}(t,a^k)}{p(t,a^k)}=\frac{|v_{\nu_1^k}(\ell,\mu),\cdots,v_{\nu_n^k}(\ell,\mu)|}{|\tilde v_{\nu_1^k},\cdots,\tilde v_{\nu_n^k}|}. \tag{3.6}$$

If $\#J(a^k)=\kappa>1$ then Γ_k is uncountable and there are several possibilities for the structure of Γ_k.

If there are matrices $C\in G_\zeta^0(a^k)$ with $J_0(C)=J(a^k)$ then we infer from (3.2) that Γ_k is a $(\kappa-1)$-parametric family of solutions of (ARE) of the form (3.5) - in this case we can assume without loss of generality that $D(a^k,C)=1$ if this parameter is nonzero.
Otherwise, according to Lemma 3 and formula (3.2), we can split Γ_k into two or more subsets:

$$\Gamma_k=\bigcup_{j=1}^{n_k}\Gamma_{k_j}\quad\text{where}\quad\Gamma_{k_j}\quad\text{is a}\quad p_{k_j}-\text{parametric set and}\quad\sum_{j=1}^{n_k}p_{k_j}\le\kappa-1.$$

Since every element of Γ can be written in the form (3.5) (for some k) we have obtained a parametrization of all complex solutions of (ARE) (see also Theorem 3).
A detailed description of all real symmetric solutions of (ARE) has been given by Shayman [24],[25], the set of all complex solutions of (ARE) has been determined by Meyer [19] (see 3.3(i)).

c) If the matrix B has $n+m$ simple eigenvalues, then, according to b), (ARE) has at most N different solutions; this fact has already been observed by Potter [20] and Mårtensson [17], p. 26, who considered the most important special case of (1.1), (1.2).
If to each eigenvalue of B there corresponds exactly one Jordan block (which means that the numbers μ_ν, defined at the beginning of section 2, are pairwise different), then we get similarly that (ARE) has only a finite number $N_0\le N$ of solutions which can be determined according to b); in addition N_0 can be determined from the number of the Jordan blocks of B and from its eigenvectors of rank $\le n$.
If B is cyclic, which means that $J=V^{-1}BV$ consists of exactly one Jordan block, then (ARE) has no solution if $D=|\tilde v_1,\ldots,\tilde v_n|=0$, otherwise (ARE) has exactly one solution $W_0=(w^0_{\ell\alpha})$ with

$$w^0_{\ell\alpha}=\frac{1}{D}|v_1(\ell,\alpha),\ldots,v_n(\ell,\alpha)|.$$

This corresponds to [19, Corollary 1].

d) For special cases of (1.2) there are control-theoretical conditions, ensuring that the denominators of (3.2), (3.5) or (3.6) are nonzero (see [15], [17], [18], [23] - [27], [28]); for a survey of the most important results on the symmetric and definite solutions of (ARE) in the special case $B=\begin{pmatrix}-A & B_0\\ C & A^T\end{pmatrix}$ see [15].

3.6 The parameter-matrices C and W_C. In order to parametrize the solution $W(\cdot;W_C)$

it is convenient to use the matrix C instead of the initial-matrix W_C; here we describe how C can be determined from the initial matrix.

Let $C \in V_n(\mathbf{C}^{n+m})$ and $W_C = P_0(C)Q_0(C)^{-1}$ (see (2.4)). If $K \in \mathbf{C}^{n\times n}$ is regular and $\tilde{C} = CK$ then $W_{\tilde{C}} = W_C$; hence to the class

$$[C] = \{CK | K \in Gl(n,\mathbf{C})\} \text{ with } \det Q_0(C) \neq 0$$

there corresponds a unique matrix W_C.

Let W_{C_0} be given, then we can determine a normalized representative $C \in [C_0]$ in the following way:

Version 1: Set $C := V^{-1}\begin{pmatrix} I_n \\ W_{C_0} \end{pmatrix}$.

Version 2: (i) Choose a permutation π of $\{1,\cdots,n+m\}$ and define $V_{11} \in \mathbf{C}^{n\times m}$, $V_{12} \in \mathbf{C}^{n\times n}$, $V_{21} \in \mathbf{C}^{m\times m}$ and $V_{22} \in \mathbf{C}^{m\times n}$ by

$$V_\pi := (v_{\pi(1)},\cdots,v_{\pi(n+m)}) = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

(ii) Check if the system of equations

$$(W_{C_0}V_{11} - V_{21})\hat{C} = V_{22} - W_{C_0}V_{12} \tag{3.7}$$

for $\hat{C} =: \begin{pmatrix} c_{\pi(1)} \\ \vdots \\ c_{\pi(m)} \end{pmatrix} \in \mathbf{C}^{m\times n}$ is solvable.

If a solution $\hat{C}$ of (3.7) exists with $\det(V_{11}\hat{C} + V_{12}) \neq 0$ then we set $c_{\pi(m+j)} = e_j$, $1 \leq j \leq n$, with the canonical unit vectors $e_j \in \mathbf{C}^{1\times n}$ and we set $C =: \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix}$. In this case it follows from $\hat{V}\binom{\hat{C}}{I_n} = VC$ and (3.7) that $W_{C_0} = (V_{21}\hat{C} + V_{22})(V_{11}\hat{C} + V_{12})^{-1} = W_C$. If C cannot be determined using π then we choose succesively another of the N permutations of $\{1,\cdots,n+m\}$ until - after at most N steps - C has been determined. We propose to start in step (i) with $\pi = id$ since in most applications one is mainly interested in the case where (3.7) is solvable for $\pi = id$ (see Remark 3 (ii)). Notice that the matrix $\hat{C}$ is of lower dimension than the matrix C in version 1, hence from computational point of view it is usually better to use version 2 instead of version 1.

4. The phase portrait of $(RDE)_\mathbf{R}$ and (RDE)

Let $C \in V_n(\mathbf{C}^{n+m})$ be such that $Q_0(C)$ is regular, $\epsilon > 0$ and $\varphi \in \{\varphi_1,\cdots,\varphi_{2h}\}$, then $W(\cdot;W_C)$ can have an infinite number of poles in the sector $\{t \in \mathbf{C} | \varphi - \epsilon < \arg t < \varphi + \epsilon\}$

and - in contrast to (3.4) - $\lim_{r\to\infty} w_{\ell\alpha}(re^{i\varphi}; W_C)$ may fail to exist. According to Lemma 2 the phase portrait of (RDE) is (as $t \to \infty$) very simple if we restrict t to the interior of a sector $S_k,\ 1 \le k \le 2h$; on the other hand the behaviour of $W(\cdot; W_C)$ along one of the rays $B_\varphi = \{t \in \mathbf{C} |\, \arg t = \varphi\}$ may be very complicated. Obviously it is sufficient to describe $W(\cdot; W_C)$ along one of the rays $B_{\varphi_j},\ 1 \le j \le 2h$ - without loss of generality we assume that the positive (and consequently also the negative) halfaxis is one of these rays. For this reason we confine in this section to the description of the phase portrait of $(RDE)_{\mathbf{R}}$.

For the rest of this section let $k \in \{1, \cdots, N\}$, $C \in V_n(\mathbf{C}^{n+m})$ and $t \in \mathbf{R}$. The formulation of the following theorems is rather technical since we are considering the most general case - the formulations become much simpler if B has simple eigenvalues or even if B is only diagonalizable. For a detailed discussion of the fundamental representation formula we use the following abbreviations:

$$d(a^k, C) = \text{degree}\Big(\sum_{\zeta(a^j)=\zeta(a^k)} p(t, a^j) D(a^j, C)\Big),$$

$$d_{\ell\alpha}(a^k, C) = \text{degree}\Big(\sum_{\zeta(a^j)=\zeta(a^k)} p_{\ell\alpha}(t, a^j) D(a^j, C)\Big),\ \ 1 \le \ell \le m,\ \ 1 \le \alpha \le n,$$

$$\begin{aligned}\tilde{G}_\zeta(a^k) \ = \ & \{C \in V_n(\mathbf{C}^{n+m}) | D(a^k, C) \ne 0 \ \text{ and } \ D(a^j, C) = 0 \ \text{ if} \\ & R(a^j) > R(a^k) \ \text{ or } \ (R(a^j) = R(a^k) \ \text{ and } \ \zeta(a^j) \ne \zeta(a^k))\},\end{aligned}$$

$$\tilde{G}^0_\zeta(a^k) = \{C \in \tilde{G}_\zeta(a^k) | d(a^k, C) = 0 \ \text{ and } \ \max_{\ell,\alpha} d_{\ell\alpha}(a^j, C) \le 0 \ \text{ for } \ R(a^j) = R(a^k)\}.$$

We shall see that each solution $W(\cdot; W_C)$ with $C \in \tilde{G}^0_\zeta(a^k)$ has a limit $W_{C_0} \in \Gamma$ with $C_0 \in G^0_\zeta(a^k)$ as $t \to \infty$.

The following theorem is concerned with an important special case.

Theorem 2. *Let $C \in G^0_\zeta(a^k) (\ne \emptyset)$ and $R(a^j) \ne R(a^k)$ for $j \ne k$.*

(i) $W_C := W_k$ - where W_k is defined by (3.6) - is a solution of (ARE); W_k is real if B is real.

(ii) If $C^1 \in \tilde{G}^0_\zeta(a^k)$ with $\det Q_0(C^1) \ne 0$ then there exists $t_0 \in \mathbf{R}$, $\nu_0 \in \mathbf{N}_0$ and $K_0 > 0$ such that $W(t; W_{C^1}) \in \mathbf{C}^{m\times n}$ for $t \ge t_0$ and

$$|w_{\ell\alpha}(t, W_{C^1}) - \frac{|v_{\nu_1^k}(\ell,\alpha), \cdots, v_{\nu_n^k}(\ell,\alpha)|}{|\tilde{v}_{\nu_1^k}, \cdots, \tilde{v}_{\nu_n^k}|}| \le K_0\, t^{\nu_0} e^{t(R(a^{k-1})-R(a^k))} \qquad (4.1)$$

for $1 \le \ell \le m,\ 1 \le \alpha \le n$ and $t \ge t_0$; this means that the convergence of $W(t, W_{C^1})$ for $t \to \infty$ takes place at an exponential rate. We have $\nu_0 = 0$ if B is semisimple, otherwise $0 \le \nu_0 \le \max\{\text{degree}(p_{\ell\mu}(t, a^j)) | 1 \le \mu \le n,\ 1 \le \ell \le m,\ j \in J_0(C_1)\}$.

(iii) If $C^2 \in \tilde{G}_\zeta(a^k)$ *with* $d(a^k, C^2) = -\infty < \max_{\ell,\alpha} d_{\ell\alpha}(a^k, C^2)$ *then either* $\det Q_0(C^2) = 0$ *and* W_{C^2} *is undefined or at least one element of* $W(t, W_{C^2})$ *tends to infinity for* $t \to \infty$.

Proof. Since $R(a^j) \neq R(a^k)$ for $j \neq k$, the eigenvalues $\lambda_{\nu_1^k}, \ldots, \lambda_{\nu_n^k}$ and the corresponding eigenvectors appear in conjugate complex pairs if B is real. Hence it follows easily (as with [17], Theorem 3) that W_k is real if B is real.
From the assumptions of (ii) and from (3.2) we infer

$$w_{\ell\alpha}(t; W_{C^1}) = \frac{\sum_{j \in J_0(C^1)} e^{t\zeta(a^j)} p_{\ell\alpha}(t, a^j) D(a^j, C^1)}{\sum_{j \in J_0(C^1)} e^{t\zeta(a^j)} p(t, a^j) D(a^j, C^1)}, \tag{4.2}$$

where $p(t, a^k) D(a^k, C^1) \neq 0$ and $R(a^j) < R(a^k)$ for $j \in J_0(C^1) \setminus \{k\}$.
Using (4.2) we obtain assertion (ii) of the theorem, and in the special case $C^1 = C$ obviously $W_{C^1} = W_C = W_k$.
(iii) is an immediate consequence of formula (3.2).

The next theorem generalizes Theorem 2.

Theorem 3. *(i) If* $C \in G_\zeta^0(a^k)$ *then* $W_C =: (w_{\ell\alpha})_{\substack{1 \leq \ell \leq m \\ 1 \leq \alpha \leq n}}$ *is a solution of* (ARE) *with*

$$w_{\ell\alpha} = \frac{\sum\limits_{j \in J_0(C)} p_{\ell\alpha}(t, a^j) D(a^j, C)}{\sum\limits_{j \in J_0(C)} p(t, a^j) D(a^j, C)} \tag{4.3}$$

$$= \frac{\sum\limits_{j \in J_0(C)} \left| v_{\nu_1^j}(\ell, \alpha), \cdots, v_{\nu_n^j}(\ell, \alpha) \right| \left| \begin{matrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_n^j} \end{matrix} \right|}{\sum\limits_{j \in J_0(C)} \left| \tilde{v}_{\nu_1^j}, \cdots, \tilde{v}_{\nu_n^j} \right| \left| \begin{matrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_n^j} \end{matrix} \right|}. \tag{4.4}$$

Notice that all polynomials $p_{\ell\alpha}(\cdot,\ a^j)$, $p(\cdot,\ a^j)$ *in (4.3) are constant. Any solution of* (ARE) *can be written in the form (4.4) for an adequate* $C \in G_\zeta^0(a^k)$.

(ii) *Let* $C^0 \in \tilde{G}_\zeta^0(a^k)$. *Then there exists a solution* W_0 *of* (ARE) *with* $\lim_{t \to \infty} W(t; W_{C^0}) = W_0$, *and the convergence takes place at an exponential rate.* W_0 *is obtained from the right-hand side of (4.3) by replacing therein* C *by* C^0 *and* $j \in J_0(C)$ *by* $R(a^j) = R(a^k)$.

(iii) *Let* $C^1 \in \tilde{G}_\zeta(a^k)$,

$$\Delta(t) = \sum_{R(a^j) = R(a^k)} p(t, a^j) D(a^j, C^1) \quad \textit{and} \quad \Delta_{\ell\alpha}(t) = \sum_{R(a^j) = R(a^k)} p_{\ell\alpha}(t, a^j) D(a^j, C^1).$$

If $-\infty < d =$ *degree* $\Delta(t) \geq \max\{$*degree* $\Delta_{\ell\alpha}(t) | 1 \leq \ell \leq m, 1 \leq \alpha \leq n\} = D$, *then there exists a solution* W_0 *of* (ARE) *with*

$$\lim_{t\to\infty} W(t; W_{C^1}) = W_0, \tag{4.5}$$

and W_0 *can be determined from (3.2) and the coefficients of* t^d *in the polynomials* $\Delta(t)$ *and* $\Delta_{\ell\alpha}(t)$. *In addition there exist constants* $k > 0$ *and* $t_0 > 0$ *with*

$$|w_{\ell\alpha}(t; W_{C^1}) - w_{\ell\alpha}(t; W_0)| \leq \frac{k}{t} \quad \textit{for} \quad t \geq t_0 \qquad \textit{and } 1 \leq \ell \leq m, 1 \leq \alpha \leq n;$$

in this case we say that the convergence takes place at a polynomial rate.
If $0 \leq d < D$, *then at least one element* $w_{\ell\alpha}(t; W_{C^1})$ *tends to* ∞ *as* $x \to \infty$ *and if* $\Delta(t) \equiv 0$, *then the limit in (4.5) may fail to exist.*

Proof. (i) follows from section 3.5 - the last sentence in (i) is a consequence of 3.3 (i) and (2.10). Notice that (4.4) results from (4.3) and the definition of $G^0_\zeta(a^k)$.
From the definition of $\tilde{G}^0_\zeta$ and (3.2) we infer that

$$w_{\ell\alpha}(t; W_{C^0}) = \frac{\sum_{R(a^j)\leq R(a^k)} e^{t\zeta(a^j)} p_{\ell\alpha}(t, a^j) D(a^j, C^0)}{\sum_{R(a^j)\leq R(a^k)} e^{t\zeta(a^j)} p(t, a^j) D(a^j, C^0)}, \tag{4.6}$$

where $\sum_{R(a^j)=R(a^k)} p(t, a^j) D(a^j, C^0)$ is a nonzero constant; obviously this implies assertion (ii). Part (iii) of the Theorem is proved in the same way.

Remark 2. (i) Since any initial value problem for (RDE) is solvable and since the solution can be represented in the form (2.10), it follows from (2.10) (for $t = 0$):
For an arbitrary initial matrix $W_0 \in \mathbf{C}^{m\times n}$ we can choose $k \in \{1, \cdots, N\}$ such that there exists a matrix $C \in G_\zeta(a^k)$ with $W_0 = W_C$ such that at least one of the constants d and D, defined in Theorem 3 (iii) is unequal $-\infty$.
(ii) Using (i), (2.10) and Theorem 3 we can determine the *stable set* $S(W_C)$ of an arbitrary solution $W_0 = W_C$ of (ARE). If $C \in G_\zeta(a^k)$ and if B is semisimple then it follows from (2.10)and Theorem 3 that $S(W_C) = \{W_{C_0} | C_0 \in \tilde{G}^0_\zeta(a^k)\}$.
If B is not semisimple then the situation is more involved (see Theorem 3 (iii)) - instead of trying to describe $S(W_C)$ in the general case we propose to use directly (2.10) to determine $S(W_C)$ for a fixed solution W_C of (ARE).
(iii) The set of all (almost) periodic solutions of $(RDE)_{\mathbf{R}}$ corresponds to the set of all (almost) periodic functions of the form (3.2). Using this fact and (3.2) we get the assertions of the two following theorems - the proofs are omitted since they are similar to the proofs of Theorem 1 and Theorem 2.

For the formulation of Theorem 4 we use the abbreviations

$$\begin{aligned} G_{I_{com}}(a^k) \;=\; & \{C \in V_n(\mathbf{C}^{n+m}) | D(a^k, C) \neq 0 \text{ and } D(a^j, C) = 0 \text{ if either} \\ & R(a^j) \neq R(a^k) \text{ or if } I(a^j) \text{ and } I(a^k) \text{ are incommensurable}\}, \end{aligned}$$

$$G^1_{I_{com}}(a^k) = \{C \in G_{I_{com}}(a^k) | d(a^j,C) \le 0 \ \text{ and } \ \max_{\ell,\alpha} d_{\ell\alpha}(a^j,C) \le 0 \ \text{ if } \ R(a^j) = R(a^k)\},$$

$$\begin{aligned}\tilde{G}_{I_{com}}(a^k) \ &= \ \{C \in V_n(\mathbf{C}^{n+m}) | D(a^k,C) \neq 0 \ \text{ and } \ D(a^j,C) = 0 \ \ if \ \ R(a^j) > R(a^k) \\ &\qquad \text{or if } \ R(a^j) = R(a^k) \ \text{ and } \ I(a^j) \ \text{ and } \ I(a^k) \ \text{ are incommensurable}\},\end{aligned}$$

$$\tilde{G}^1_{I_{com}}(a^k) = \{C \in \tilde{G}_{I_{com}}(a^k) | d(a^j,C) \le 0 \ \text{ and } \ \max_{\ell,\alpha} d_{\ell\alpha}(a^j,C) \le 0 \ \text{ if } \ R(a^j) = R(a^k)\},$$

and

$$\tau_{I_{com}}(a^k) = \{j | R(a^j) = R(a^k), \quad I(a^j) \ \text{ and } \ I(a^k) \ \text{ are commensurable}\}.$$

Theorem 4. *(i) Let $C \in G^1_{I_{com}}(a^k)$ with $\det Q_0(C) \neq 0$. Then $W(\cdot; W_C)$ (restricted to $\mathbf{R}$) is a periodic solution of $(RDE)_{\mathbf{R}}$ with*

$$w_{\ell\alpha}(t; W_C) = \frac{\sum_{j \in \tau_{I_{com}}(a^k)} e^{it\, I(a^j)} p_{\ell\alpha}(t,a^j) D(a^j,C)}{\sum_{j \in \tau_{I_{com}}(a^k)} e^{it\, I(a^j)} p(t,a^j) D(a^j,C)} \tag{4.7}$$

$$= \frac{\displaystyle\sum_{j \in \tau_{I_{com}}(a^k)} e^{it\, I(a^j)} |v_{\nu_1^j}(\ell,\alpha), \ldots, v_{\nu_n^j}(\ell,\alpha)| \begin{vmatrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_n^j} \end{vmatrix}}{\displaystyle\sum_{j \in \tau_{I_{com}}(a^k)} e^{it\, I(a^j)} |\tilde{v}_{\nu_1^j}, \ldots, \tilde{v}_{\nu_n^j}| \begin{vmatrix} c_{\nu_1^j} \\ \vdots \\ c_{\nu_n^j} \end{vmatrix}}. \tag{4.8}$$

$W(\cdot; W_C)$ may have real poles. Every periodic solution of $(RDE)_{\mathbf{R}}$ is of the form (4.7), (4.8) for some $k \in \{1, \ldots, N\}$ and for some matrix $C \in G^1_{I_{com}}(a^k)$ or it can be written in the form (4.10) (see Theorem 5 (i)). The set of all periodic solutions of $(RDE)_{\mathbf{R}}$ is either empty or uncountable.

(ii) Let $C^1 = \begin{pmatrix} c_1^1 \\ \vdots \\ c_{n+m}^1 \end{pmatrix} \in \tilde{G}^1_{I_{com}}(a^k)$ and let $\tilde{C}^1 = \begin{pmatrix} \tilde{c}_1^1 \\ \vdots \\ \tilde{c}_{n+m}^1 \end{pmatrix}$ be defined by

$$\tilde{c}_k^1 = \begin{cases} c_k^1 & if \qquad k \in \bigcup_{j \in \tau_{I_{com}}(a^k)} \{\nu_1^j, \ldots, \nu_n^j\} \\ 0 & else. \end{cases}$$

If $\Delta_0(t) = \sum_{j \in \tau_{I_{com}}(a^k)} e^{tI(a^j)} p(t,a^j) D(a^j, \tilde{C}^1)$ and *$|\Delta_0(t)| \ge \delta > 0$* for *$t \in \mathbf{R}$, then $W(\cdot; W_{\tilde{C}^1})$ is a periodic solution of $(RDE)_{\mathbf{R}}$ with*

$$\lim_{t \to \infty} (W(t; W_{C^1}) - W(t; W_{\tilde{C}^1})) = 0,$$

and the convergence takes place at an exponential rate.

(iii) *Let $C^2 \in \tilde{G}_{I_{com}}(a^k)$ and $\sum_{R(a^j)=R(a^k)} e^{t\zeta(a^j)} p(t,a^j) D(a^j,C^2) =: e^{t\zeta(a^j)} \Delta(t)$, where $\Delta(t) = \sum_{\zeta(a^j)=\zeta(a^k)} p(t,a^j) D(a^j,C^2)$ is a polynomial of degree $d > 0$. If $\max_{\ell,\alpha}$(degree $(\sum_{\zeta(a^\nu)=\zeta(a^j)} p_{\ell\alpha}(t,a^\nu) D(a^\nu,C^2))) \le d$ for $R(a^j) = R(a^k)$, then there exists a* *periodic solution* $W(\cdot;W_C)$ *of* $(RDE)_{\mathbf{R}}$ *with* $\lim\limits_{t\to\infty}[W(t;W_{C^2})-W(t;W_C)]=0$, *and the convergence takes place at a polynomial rate. Using (3.2),* $W(\cdot;W_C)$ *can be determined from the coefficients of* t^d *in the polynomials* $p(t,a^j)$ *and* $p_{\ell\alpha}(t,a^j)$, $\;j\in\tau_{I_{com}}(a^k)$.

For the formulation of Theorem 5 we set

$$G_R(a^k)=\{C\in V_n(\mathbf{C}^{n+m})|D(a^k,C)\neq 0 \text{ and } D(a^j,C)=0 \text{ if } R(a^j)\neq R(a^k)\},$$

$$G_R^1(a^k)=\{C\in G_R(a^k)|d(a^j,C)\leq 0 \text{ and } \max_{\ell,\alpha} d_{\ell\alpha}(a^j,C)\leq 0 \text{ for } R(a^j)=R(a^k)\},$$

$$\tilde G_R(a^k)=\{C\in V_n(\mathbf{C}^{n+m})|D(a^k,C)\neq 0 \text{ and } D(a^j,C)=0 \text{ for } R(a^j)>R(a^k)\},$$

$$\tilde G_R^1(a^k)=\{M\in \tilde G_R(a^k)|d(a^k,C)\leq 0 \text{ and } d(a^j,C)\leq 0 \text{ for } R(a^j)=R(a^k)\}.$$

Theorem 5. *(i) Let* $C\in G_R^1(a^k)$. *If* $\det Q(t,C)\not\equiv 0$ *and* $\det Q_0(C)\neq 0$ *then* $W(\cdot;W_C)$ *is an almost periodic solution of* $(RDE)_{\mathbf{R}}$ *which is bounded if* $\det Q(t;C)\neq 0$ *for* $|\mathrm{Im}\ t|\leq \epsilon>0; W(\cdot;W_C)$ *satisfies*

$$w_{\ell\alpha}(t;W_C)=\frac{\sum\limits_{R(a^j)=R(a^k)} e^{t\zeta(a^j)}p_{\ell\alpha}(t,a^j)D(a^j,C)}{\sum_{R(a^j)=R(a^k)} e^{t\zeta(a^j)}p(t,a^j)D(a^j,C)} \tag{4.9}$$

$$=\frac{\sum\limits_{R(a^j)=R(a^k)} e^{t\zeta(a^j)}\left|v_{\nu_1^j}(\ell,\alpha),\ldots,v_{\nu_n^j}(\ell,\alpha)\right|\left|\begin{array}{c} c_{\nu_1^j}\\ \vdots\\ c_{\nu_n^j}\end{array}\right|}{\sum\limits_{R(a^j)=R(a^k)} e^{t\zeta(a^j)}\left|\tilde v_{\nu_1^j},\ldots,\tilde v_{\nu_n^j}\right|\left|\begin{array}{c} c_{\nu_1^j}\\ \vdots\\ c_{\nu_n^j}\end{array}\right|}. \tag{4.10}$$

Every almost-periodic solution of $(RDE)_{\mathbf{R}}$ *is of this form for some matrix* $C\in G_R^1(a^k)$. *Notice that the functions of the form (4.9), (4.10) are periodic if the elements of the set* $\{\zeta(a^j)-\zeta(a^k)|\ R(a^j)=R(a^k)\}$ *are commensurable.*

(ii) *Let* $C^1=\begin{pmatrix} c_1^1\\ \vdots\\ c_{n+m}^1\end{pmatrix}\in\tilde G_R^1$ *and* $\tilde C^1=\begin{pmatrix} \tilde c_1^1\\ \vdots\\ \tilde c_{n+m}^1\end{pmatrix}$

with $\tilde c_k^1=\begin{cases} c_k^1 & \text{if} \quad k\in \bigcup\limits_{R(a^j)=R(a^k)}\{\nu_1^j,\ldots,\nu_n^j\}\\ 0 & \end{cases}$.

If $\Delta(t)=\sum\limits_{R(a^j)=R(a^k)} e^{t\zeta(a^j)}p(t,a^j)D(a^j,C^1)=e^{t\zeta(a^k)}\Delta_0$, *where*

$\Delta_0=\sum\limits_{R(a^j)=R(a^k)} p(t,a^j)D(a^j,C^1)$ *and* $|\Delta_0(t)|\geq\delta>0$ *for* $t\geq t_0$,

then $W(\cdot;W_{\tilde C^1})$ *is an almost periodic solution of* $(RDE)_{\mathbf{R}}$ *with*

$$\lim_{t\to\infty}(W(t;W_{C^1})-W(t;W_{\tilde C^1})=0,$$

and the convergence takes place at an exponential rate.
(iii) *The statement of Theorem 4 remains true if we replace therein $\tilde{G}_{I_{com}}(a^k)$ and periodic by $\tilde{G}_R(a^k)$ and almost-periodic respectively.*

Remark 3. (i) In the preceeding theorems we have given a description of the most important parts of the phase portrait of $(RDE)_{\mathbf{R}}$. These results show that the phase portrait of $(RDE)_{\mathbf{R}}$ and of (RDE) is rather simple if B has only simple eigenvalues. If B is semisimple (diagonalizable) then all polynomials $p(t, a^j)$ and $p_{\ell\alpha}(t, a^j)$ in (3.2) are constant and in this case the constant, periodic and almost-periodic solutions of $(RDE)_{\mathbf{R}}$ are of the form (4.4), (4.8) and (4.10) respectively - in addition we obtain from (3.2) quite easily the corresponding stable (or unstable) sets. The phase portrait of the symplectic Riccati equation with simple or semisimple spectrum has been described in detail by Shayman [27].
In the general case the phase portrait of $(RDE)_{\mathbf{R}}$ may be very complicated - on the other hand, using (3.2), it is not difficult to analyze the phase portrait of $(RDE)_{\mathbf{R}}$ if the matrix B is fixed and if its Jordan canonical form and its (generalized) eigenvectors are known. There remains only to discuss the behaviour of some specific exponential sums as described in Theorems 2-5.

(ii) An important special case is obtained if Re $\lambda_m <$ Re λ_{m+1}, $\binom{A}{B} = (v_{m+1}, \cdots, v_{m+n})$ with $A = (\tilde{v}_{m+1}, \cdots, \tilde{v}_{m+n})$ and $\det A \neq 0$. Then $W^* = BA^{-1}$ is a solution of (ARE) which is called *dichotomic solution* (see [18, Definition 2]). According to Theorem 2 any solution $W(\cdot; W_C)$ with $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix}$ and $\begin{vmatrix} c_{m+1} \\ \vdots \\ c_{m+n} \end{vmatrix} \neq 0$ converges at an exponential rate to the dichotomic solution as $t \to \infty$. Hence the domain of attraction of W^* is open and dense in $\mathbf{C}^{m \times n}$. If rank $A = n - 1$ then it follows with Lemma 1a) that at least one element of $W(\cdot; W_C)$ tends to infinity as $t \to \infty$ if $\begin{vmatrix} c_{m+1} \\ \vdots \\ c_{m+n} \end{vmatrix} \neq 0$.
In the case of the symplectic Riccati equation the dichotomity condition Re $\lambda_m <$ Re λ_{m+1} is equivalent to the condition Re $\lambda_j \neq 0$ for $1 \leq j \leq n + m (= 2n)$.

(iii) If Re $\lambda_m =$ Re λ_{m+1} then it follows from (2.10) and Theorems 2-5 that there is no stable equilibrium of (RDE) with respect to $t \to \infty$ and that there is no solution of (ARE) whose domain of attraction is open and dense (see also [1]).

(iv) For the asymptotic behaviour of the solutions of $(RDE)_{\mathbf{R}}$ for $t \to -\infty$ we obtain similarly all the corresponding results.

5. Nonautonomous Riccati equations

In this section we give an application of the fundamental representation formula (2.10) to the investigation of nonautonomous Riccati equations (1.1). In order to simplify the representation of the results we confine to the most important special case.

5.1 Assumptions. Let the coefficients B_{ij} in (1.1) be matrix polynomials in t such that

$$B(t) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}(t) = t^{r-1} \sum_{k=0}^{r-1} A_k t^{-k} \tag{5.1}$$

with $r \in \mathbf{N}$, $A_k \in \mathbf{C}^{(m+n)\times(m+n)}$ for $0 \le k \le r-1$, and where $A_0 = \begin{pmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{pmatrix}$ has $n+m$ simple eigenvalues $\lambda_1, \dots, \lambda_{n+m}$; the eigenvector of A_0 corresponding to λ_k is denoted by $v_k, 1 \le k \le n+m$.

5.2 Notations and preliminary results.

(i) The algebraic Riccati equation $(ARE)_\infty$

$$0 = A_{21}^0 + A_{22}^0 W - W A_{11}^0 - W A_{12}^0 W \tag{5.2}$$

is called the *limiting algebraic Riccati equation* corresponding to

$$\dot{W} = B_{21}(t) + B_{22}(t) W - W B_{11}(t) - W B_{12}(t) W. \tag{5.3}$$

(ii) Let $\delta > 0$ be sufficiently small and let here $(\phi_k)_{k\in\mathbf{Z}}$ with $\cdots < \phi_0 \le 0 < \phi_1 < \phi_2 \cdots$, be the sequence of the Stokes' directions (for the definition see [13]) of the differential equation

$$\dot{y} = B(t)y, \quad t \in \mathbf{C}. \tag{5.4}$$

Then we set for $N_k(\delta) = \{t \in \mathbf{C} | \phi_{k-1} + \delta \le \arg t \le \phi_{k+1} - \delta\}$. From Assumption 5.1 and from [3] and [13] we infer that for any sector $N_k(\delta)$ there exist $n+m$ linearly independent vector solutions $\hat{y}_\mu, 1 \le \mu \le n+m$, of the linear differential equation (5.4) of the form

$$\begin{aligned} \hat{y}_\mu(t) &= x_\mu(t) \exp\{\sum_{j=0}^{r-1} \alpha_j^{(\mu)} \frac{t^{r-j}}{r-j}\} \\ &=: (t^{\beta_\mu} v_\mu + t^{\beta_\mu - 1} b_\mu(t)) e^{\alpha_\mu(t)}, \end{aligned} \tag{5.5}$$

where $\alpha_0^{(\mu)} = \lambda_\mu$, $\beta_\mu \in \mathbf{C}$ and $\alpha_j^{(\mu)} \in \mathbf{C}$ for $1 \le j \le r-1, 1 \le \mu \le n+m$, and where $b_\mu(t), 1 \le \mu \le n+m$, is uniformly bounded for $t \in N_k(\delta)$. Since we shall assume that $N_k(\delta)$ is fixed, we do not indicate the dependence of $\hat{y}_\mu$ on the sector by an additional index.

(iii) We use all notations and assumptions introduced in the sections 2 and 3 with two slight modifications:
a)Instead of the sectors S_k we consider for $\delta \ge 0$ the sectors $S_k(\delta)$, defined in the following way:
Let the sequence $(\gamma_j)_{j\in\mathbf{Z}}$ with $\cdots < \gamma_0 \le 0 < \gamma_1 < \gamma_2 \cdots$, be the sequence of

all Stokes' directions of the set $\{t^r\zeta(a^\mu)|a^\mu \in A\}$, here, as in section 3.1, $\zeta(a^\mu) = \lambda_{\nu_1^\mu} + \ldots + \lambda_{\nu_n^\mu}$. $\{\gamma_j|j \in \mathbf{Z}\}$ is the set of all solutions φ of equations of the form $f_{\mu\nu}(\varphi) = \mathrm{Re}(e^{ir\varphi}[\zeta(a^\mu) - \zeta(a^\nu)]) = 0,\ a^\mu \neq a^\nu$. We set

$$S_k(\delta) = \{t \in \mathbf{C}|\gamma_{k-1} + \delta \leq \arg t \leq \gamma_k - \delta\}.$$

Notice that for $r = 1\ \ \gamma_k = \varphi_k,\ \ 1 \leq k \leq 2h$, where φ_k is defined as in section 3.2.
b) The functions $x_\mu, 1 \leq \mu \leq n+m$, defined by (2.5), have to be substituted here by the functions x_μ, defined in (5.5); here and in the sequel we assume that the sector $N_k(\delta)$ is fixed.
According to 3.1 b), c) and (5.5) we have here for $a^j \in A$

$$\begin{aligned} p(t, a^j) &= |\tilde{x}_{\nu_1^j}(t), \ldots, \tilde{x}_{\nu_n^j}(t)| \\ &= t^{\beta_{\nu_1^j} + \ldots + \beta_{\nu_n^j}}(|\tilde{v}_{\nu_1^j}, \ldots, \tilde{v}_{\nu_n^j}| + \frac{O(1)}{t}) \end{aligned}$$

and

$$p_{\ell\alpha}(t, a^j) = t^{\beta_{\nu_1^j} + \ldots + \beta_{\nu_n^j}}(|\tilde{v}_{\nu_1^j}(\ell, \alpha), \ldots, \tilde{v}_{\nu_n^j}(\ell, \alpha)| + \frac{O(1)}{t}),$$

where Landau's O-Symbol $O(1)$ denotes a function of t which is uniformly bounded for $t \in N_k(\delta),\ \delta > 0$.

(iv) For $C \in V_n(\mathbf{C}^{n+m})$ we set, generalizing the notations of section 2,

$$Y(t; C) = (\hat{y}_1(t), \ldots, \hat{y}_{n+m}(t))C = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} \tag{5.6}$$

with $Q(t) \in \mathbf{C}^{n\times n}$ and $P(t) \in \mathbf{C}^{m\times n}$; further we set for $t \in N_k(\delta)$ with $\det Q(t) \neq 0$

$$W_C(t) = P(t)Q(t)^{-1} = (w_{\ell\alpha}^C(t))_{\substack{1\leq \ell \leq m \\ 1 \leq \alpha \leq n}}. \tag{5.7}$$

(v) We recall that Assumption 5.1 implies that the number of solutions of $(ARE)_\infty$ is $\leq \binom{n+m}{n}$.

Using the preceding assumptions and notations we infer from the proof of Theorem 1 (see also Remark 1 (ii) and (3.2)):

Corollary 1. (Fundamental representation formula for the solutions of (5.3))

$$W_C : N_k(\delta) \setminus \{t \in N_k(\delta)|\ \det Q(t) = 0\} \to \mathbf{C}$$

is a solution of the matrix Riccati equation (5.3) with $w^C_{\ell\alpha}(t) =$

$$\frac{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{\alpha_{k_1}(t)+\ldots+\alpha_{k_n}(t)} t^{\beta_{k_1}+\ldots+\beta_{k_n}} \left(|v_{k_1}(\ell,\alpha),\ldots,v_{k_n}(\ell,\alpha)| + \frac{O(1)}{t}\right) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{\alpha_{k_1}(t)+\ldots+\alpha_{k_n}(t)} t^{\beta_{k_1}+\ldots+\beta_{k_n}} \left(|\tilde v_{k_1},\ldots,\tilde v_{k_n}| + \frac{O(1)}{t}\right) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}},$$

where $\alpha_j(t)$ *is defined by (5.5).*

The following Corollary generalizes Lemma 2; for simplicity we formulate the statements of the Corollary for the sector $S_0(0) \cup S_1(0)$ containing the positive real axis. For the remaining sectors the corresponding results hold true.

Corollary 2.*Let* $\delta > 0$ *and* $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix} \in V_n(\mathbf{C}^{n+m})$ *with*

(i) $D(a^k, C) \neq 0$ *and* $D(a^j, C) = 0$ *for* $\mathrm{Re}\,\zeta(a^j) > \mathrm{Re}\,\zeta(a^k)$,

(ii) $D(a^\mu, C) \neq 0$ *and* $D(a^j, C) = 0$ *if* $\mathrm{Re}(e^{i\varphi}\zeta(a^j)) > \mathrm{Re}\,(e^{i\varphi}\zeta(a^\mu))$ *for* $\gamma_{-1} < \varphi < \gamma_0$.

a) *If* $|\tilde v_{\nu_1^k},\ldots,\tilde v_{\nu_n^k}| \neq 0$, *then the solution* W_C, *defined by (5.6), (5.7), has at most a finite number of poles in* $S_1(\delta)$,. *Further*

$$\lim_{S_1(\delta)\ni t\to\infty} W_C(t) = W_0 = (w^0_{\ell\alpha})_{\substack{1\le\ell\le m\\ 1\le\alpha\le n}} \tag{5.8}$$

exists and is a solution of $(ARE)_\infty$ *with*

$$w^0_{\ell\alpha} = \frac{|v_{\nu_1^j}(\ell,\alpha),\ldots,v^j_{\nu_n}(\ell,\alpha)|}{|\tilde v_{\nu_1^j},\ldots,\tilde v_{\nu_n^j}|}, \tag{5.9}$$

where the convergence takes place at least at a polynomial rate:

$$|w^C_{\ell\alpha}(t) - w^0_{\ell\alpha}| \le \frac{\text{const.}}{|t|} \text{ for } t \in S_1(\delta) \text{ and } |t| \ge t_0. \tag{5.10}$$

If in addition $\mu = k$, *then* W_C *has at most a finite number of poles in* $\tilde S(\delta) = \{t|\gamma_{-1} + \delta \le \arg t \le \gamma_1 - \delta\}$, *and the limit (5.8) exists for* $\tilde S(\delta) \ni t \to \infty$.

b) *If* $\mu \neq k$ *and* $|\tilde v_{\nu_1^\mu},\ldots,\tilde v_{\nu_n^\mu}| \neq 0 \neq |\tilde v_{\nu_1^k},\ldots,\tilde v_{\nu_n^k}|$, *then* W_C *has an infinite number of poles in* $B^\delta = \{t \in \mathbf{C}|\gamma_0 - \delta \le \arg t \le \gamma_0 + \delta\}$ *and at most a finite number of poles in* $S_0(\delta) \cup S_1(\delta)$; *further*

$$\lim_{S_0(\delta)\ni t\to\infty} W_C(t) = W_1 = (w^1_{\ell\alpha})_{\substack{1\le\ell\le m\\ 1\le\alpha\le n}} \tag{5.11}$$

with

$$w^1_{\ell\alpha} = \frac{|v_{\nu_1^\mu}(\ell,\alpha),\dots,v_{\nu_n^\mu}(\ell,\alpha)|}{|\tilde{v}_{\nu_1^\mu},\dots,\tilde{v}_{\nu_n^\mu}|}. \tag{5.12}$$

c) If $|\tilde{v}_{\nu_1^k},\dots,\tilde{v}_{\nu_n^k}| = 0$ *and* $|v_{\nu_1^k}(\ell,\alpha),\dots,v_{\nu_n^k}(\ell,\alpha)| \neq 0$*, then* $\lim_{S_1(\delta)\ni t\to\infty} w^C_{\ell\alpha}(t) = \infty$*; for the investigation of the behaviour of the remaining elements* $w^C_{i\nu}(t)$ *one needs representations of the solutions* $\hat{y}_\mu$ *which are more precise than those given by (5.5).*

Proof a) Since the eigenvalues $\lambda_1,\dots,\lambda_{n+m}$ are simple it follows as with the proof of Lemma 3 that

$$D(a^\nu, C) = 0 \text{ if } a^\nu \notin \{a^\mu, a^k\} \text{ and } \zeta(a^\nu) \in \{\zeta(a^\mu), \zeta(a^k)\} \tag{5.13}$$

Let $t \in S_1(\delta)$ and $|\tilde{v}_{\nu_1^k},\dots,\tilde{v}_{\nu_n^k}| \neq 0$. Dividing nominator and denominator of the representation formula for $w^C_{\ell\alpha}(t)$ (see Corollary 1) by $e^{\alpha_{\nu_1^k}(t)+\dots+\alpha_{\nu_n^k}(t)} t^{\beta_{k_1}+\dots+\beta_{k_n}} \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}$ we infer, using (5.13), that there exist $t_0 > 0$ and $\epsilon_0(\delta) > 0$ with

$$w^C_{\ell\alpha}(t) = \frac{|v_{\nu_1^k}(\ell,\alpha),\dots,v_{\nu_n^k}(\ell,\alpha)| + \frac{O(1)}{t} + O(e^{-\epsilon_0(\delta)})}{|\tilde{v}_{\nu_1^k},\dots,\tilde{v}_{\nu_n^k}| + \frac{O(1)}{t} + O(e^{-\epsilon_0(\delta)})} \tag{5.14}$$

for $t \geq t_0$. This implies (5.8) - (5.10).
If $\mu = k$ then (5.14) is valid for $\gamma_{-1} + \delta \leq \arg t \leq \gamma_1 - \delta$, which proves the last statement of a). If $\mu \neq k$ then (under the assumptions of b)) for $t \in S_0(\delta)$ the terms containing the factor $e^{\alpha_{\nu_1^\mu}(t)+\dots+\alpha_{\nu_n^\mu}(t)}$ are the dominant terms in the representation formula for $W_C(t)$. Hence we infer from the properties of asymptotic exponential polynomials (see [4], [16]) that W_C has an infinite number of poles in B^δ.
The remaining assertions of Corollary 2 are obtained similarly from Corollary 1.

Remark 4 (i) Obviously any solution of $(ARE)_\infty$ can be represented as a limit of the form (5.8), and from section 3.5 it follows that $(ARE)_\infty$ has at most $\binom{n+m}{n}$ solutions.

(ii) If A_0 has multiple eigenvalues, then the behaviour of the solutions of (5.3) may be different from the behaviour described in Corollary 2; in particular there may be solutions of (5.3) having an infinite number of poles in the sector $S_k(\delta)$. These poles have at most logarithmic density, which means that for $\delta > 0$ the number of the poles of any solution of (5.3) in $S_k(\delta) \cap \{t \in \mathbf{C} \mid |t| \leq R\}$ is at most of the order $O(\log R)$ for $R \to \infty$ (see Jank [12] for further details).

(iii) If $\gamma_0 < 0 < \gamma_1$, then Corollary 2 gives a description of the solutions of (5.3) for $t > 0$. If $\gamma_0 = 0$ then it is possible to generalize some of the results obtained in section 4 for the autonomous Riccati equation $(RDE)_{\mathbf{R}}$ to the nonautonomous case. We give the following example which corresponds to Remark 3(ii) and is an easy consequence of Corollary 1.

Corollary 3.
Let Re $\lambda_m <$ *Re* λ_{m+1}, $\binom{A}{B} = (v_{m+1},\dots,v_{m+n})$ *with* $A = (\tilde{v}_{m+1},\dots,\tilde{v}_{m+n})$ *and* $|A| \neq 0$.

Then $W^* = BA^{-1}$ *is the dichotomic solution of (5.2) and any solution* W_C *(defined by (5.6), (5.7) for k=1) with* $\begin{vmatrix} c_{m+1} \\ \vdots \\ c_{m+n} \end{vmatrix} \neq 0$ *converges for* $0 < t \to \infty$ *at least at a polynomial rate to* W^*.

References

[1] H. Abou-Kandil, G. Freiling and G. Jank, *Necessary conditions for constant solutions of coupled Riccati equations in Nash games*, Syst. Control Letters **21** (1993), 295-306.

[2] B.D.O. Anderson and D.J. Clements, *Polynomial factorization via the Riccati equation*, SIAM J.Appl.Math. **31** (1976), 179-205.

[3] W. Balser, W.B. Jurkat and D. A. Lutz, *Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations*, J. Math. Anal. Appl. **71**(1979), 48-94.

[4] F. G. Boese, *Zur Lage der Nullstellen von Exponentialpolynomen und zur Stabilität von Exponentialpolynomen*, Habilitationsschrift, Hamburg 1989.

[5] R.W. Brockett, *Finite dimensional linear systems*, John Wiley, New York 1970.

[6] R.S. Bucy, *Structural stability for the Riccati equation*, SIAM J. Control Opt. **13** (1975), 749-753.

[7] F.M. Callier and J.L. Willems, *Criterion for the convergence of the solution of the Riccati differential equation*, IEEE Trans. Aut. Control **AC-26** (1981), 1232-1242.

[8] G. Freiling and G. Jank, *Matrix Riccati Equations*, Schriftenreihe des FB Mathematik der Universität Duisburg **SM-DU 198**, 1991.

[9] P. Griffiths and J. Harris,*Principles of Algebraic Geometry*, John Wiley, New York, 1978.

[10] R. Hermann and C. Martin, *Lie and Morse theory for periodic orbits of vector fields and matrix Riccati equations, I: General Lie-theoretic methods*, Math. Systems Theory **15** (1982), 277-284.

[11] R. Hermann and C. Martin, *Lie and Morse Theory for periodic orbits of vector fields and matrix Riccati equations,II*, Math. Systems Theory **16**, (1983),297-306.

[12] G. Jank, *Asymptotic distribution of singularities of solutions of matrix-Riccati-differential equations*, J. Austral. Math. Soc., Ser. B **34** (1992), 112-131.

[13] W. B. Jurkat, *Meromorphe Differentialgleichungen*, Springer, Berlin, 1978.

[14] H. W. Knobloch and H. Kwakernaak, *Lineare Kontrolltheorie*, Springer, Berlin, 1988.

[15] V. Kuĉera, *Algebraic Riccati equation: Symmetric and definite solutions*, in: Lect. Notes of the Workshop on "The Riccati Equation in Control, Systems and Signal", S. Bittanti, Ed., Pitagora Editrice, Bologna,73-77, 1989.

[16] R. E. Langer, *On the zeros of exponential sums and integrals*, Bull. Am. Math. Soc. **37** (1931), 213-239.

[17] K. Mårtensson, *On the matrix Riccati equation*, Inform. Sci. **3** (1971), 17-49.

[18] J. Medanic, *Geometric properties and invariant manifolds of the Riccati equation*, IEEE Trans. Automat. Control **AC-27** (1982), 670-677.

[19] H. B. Meyer, *The matrix equation* $AZ + B - ZCZ - ZD = 0$, SIAM J. Appl. Math. **30** (1976), 136-142.

[20] J. E. Potter, *Matrix quadratic solutions*, SIAM J. Appl. Math. **14** (1966), 496-501.

[21] W. T. Reid, *Riccati differential equations*, Academic Press, New York, 1972.

[22] C.R. Schneider, *Global aspects of the matrix Riccati equation*, Math. Systems Theory **7** (1973), 281-286.

[23] M.A. Shayman, *On the variety of invariant subspaces of a finite-dimensional linear operator*, Trans. Amer. Math. Soc. **274** (1982), 721-747.

[24] M.A. Shayman, *Geometry of the algebraic Riccati equation, I*, SIAM J. Control Opt. **21** (1983), 375-394.

[25] M.A. Shayman, *Geometry of the algebraic Riccati equation, II,* SIAM J. Control Opt. **21** (1983), 395-409.

[26] M.A. Shayman, *On the periodic solutions of the matrix Riccati equation, Math. Systems Theory* **16** (1983), 267-287.

[27] M.A. Shayman, *Phase portrait of the matrix Riccati equation*, SIAM J. Control and Opt. **24** (1986), 1-65.

[28] A. J. Telford and J. B. Moore, *On the existence of solutions to nonsymmetric algebraic Riccati equations*, in: Lect. Notes of the Workshop on "The Riccati Equation in Control, Systems and Signal", S. Bittanti, Ed., Pitagora Editrice, Bologna,83-86, 1989.