ON THE SOLUTION OF DISCRETE-TIME MARKOVIAN JUMP LINEAR QUADRATIC CONTROL PROBLEMS *

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Abstract:

A necessary and sufficient condition for the existence of a positive semidefinite solution for the coupled algebraic discrete-time Riccati-like equations occuring in markovian jump control problems is derived. By verifying a simple matrix inequality, it is shown that such solution exists and can be obtained as a limit of a monotonic sequence. This leads to a straightforward numerical algorithm for the computation of the solution. An example is given to illustrate the proposed method.

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I. Introduction

We consider the jump-linear system described by

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k; \quad 0 \le k \le N,$$

$$y_k = C(r_k)x_k,$$

with initial state $x(0) = x_0$, $r(0) = r_0$, where $x_k \in \mathbf{R}^n$, is the plant state, $u_k \in \mathbf{R}^m$ is the control vector and $y_k \in \mathbf{R}^q$ is the process output.
Here k is the time index, r_k is the form process taking values in the set $\mathbf{M} = \{1, 2, ..., M\}$ and being a finite state discrete time Markov chain with transition probabilities

$$\mathbf{Pr}\{r_{k+1} = j | r_k = i\} = p_{ij} \ , \ 1 \le i, j \le M, \text{ with } \ p_{ii} > 0.$$

The cost criterion to be minimized is given by

$$J(x_0, r_0) = \lim_{N \to \infty} \mathbf{E} \left[\sum_{k=0}^{N-1} (x_k^T Q(r_k) x_k + u_k^T R(r_k) u_k) + x_N^T K_T(r_N) x_N \right], \tag{1}$$

where $Q(r_k) \ge 0$, $R(r_k) > 0$, $K_T(r_N) \ge 0$ for all k.
Discrete time markovian jump-linear quadratic optimal control problems have been studied in several papers. In (Blair and Sworder, 1975) the finite-time horizon case is solved. In (Chizeck et al., 1986) necessary and sufficient conditions for the existence of a steady-state solution are derived; however, these conditions are not easy to test. Ji and Chizeck (1988) introduced new and refined definitions of the controllability and observability of jump-linear systems leading to relatively simple algebraic tests to determine the existence of a steady state solution.
In the aforementioned references, the feedback law to minimize (1) is obtained by solving the system of coupled algebraic Riccati-like equations

$$K(r_k) = A^T(r_k) G_k A(r_k) + Q(r_k)$$

$$-A^T(r_k) G_k B(r_k) \{R(r_k) + B^T(r_k) G_k B(r_k)\}^{-1} B^T(r_k) G_k A(r_k) \tag{2}$$

where $G_k = \sum_{i=1}^{M} p_{r_k r_i} K(r_i)$ and $1 \le k \le M$.
According to (Chizeck et al.,1986) it is not possible to write (2) as a higher dimensional single Riccati equation. Therefore, it is clear that known results and algorithms to solve the linear-quadratic optimal control problem cannot be applied directly to jump-linear systems.
The purpose of this paper is to give necessary and sufficient conditions for the existence of a positive semidefinite solution for the set of coupled Riccati equations (2). Moreover, a simple algorithm to compute the solution sought is given.
The paper is structured as follows: in section 2 preliminary results and notations are introduced. Main theorems are given in section 3 while section 4 is dedicated to a numerical example.

2. Notations and Preliminaries

If $r_k = j$ we shall use the following notations

$$A_j := \sqrt{p_{jj}} A(r_k) \ , \quad B_j := \sqrt{p_{jj}} B(r_k) R(r_k)^{-1/2} \ ,$$

$$K_j := K(r_k) \ , \quad Q_j := Q(r_k) \ , \quad \text{and} \quad \pi_{ij} = \frac{p_{ij}}{p_{ii}},$$

$1 \leq i,j \leq M$.
Using these abbreviations we can write (2) as

$$K_j = A_j^T F_j A_j + Q_j - A_j^T F_j B_j \{I + B_j^T F_j B_j\}^{-1} B_j^T F_j A_j =: \varphi_j(F_j, Q_j) \ , 1 \leq j \leq M \ , \quad (3)$$

where $F_j = \sum_{i=1}^{M} \pi_{ji} K_i = K_j + \sum_{i \neq j} \pi_{ji} K_i$ and where φ_j is defined by (3).
To (3) there correspond M decoupled standard Riccati difference equations

$$P_j = (\varphi_j(P_j, Q_j) =) A_j^T P_j A_j + Q_j - A_j^T P_j B_j \{I + B_j^T P_j B_j\}^{-1} B_j^T P_j A_j \ , 1 \leq j \leq M. \quad (4)$$

The following properties of the strong solution of (4) are known from the literature (Chan et al.(1984), Kučera (1972), Payne and Silverman (1973), De Souza et al. (1986))
(i) The strong solution P_j^+ of (4) exists and is unique if and only if (A_j, B_j) is stabilizable.
(ii) The strong solution exists and is the only positive semidefinite solution of (4) if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes outside the unit circle.
(iii) P_j^+ exists and is stabilizing if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes on the unit circle.
(iv) P_j^+ exists and is positive definite if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes inside or on the unit circle.
(v) P_j^+ depends monotonically on the coefficients of (4), i.e. it depends monotonically on the matrix

$$\begin{pmatrix} Q_j & A_j^T \\ A_j & -B_j B_j^T \end{pmatrix}$$

(see (Wimmer,1992)).

3. Main Results

For convenience we make the following assumptions:

(H1) $B_j \ , 1 \leq j \leq M$, has full rank, i.e. $B_j^T B_j$ is regular;
(H2) $(A_j, B_j) \ , 1 \leq j \leq M$, is stabilizable.

Let $K_{j0} \geq 0 \ , 1 \leq j \leq M$, then we define the sequences $(K_j(\nu))_{\nu \in \mathbf{N_0}}$ and $(P_j(\nu))_{\nu \in \mathbf{N_0}}$,

$1 \le j \le M$ by

$$K_j(0) = P_j(0) = K_{j0}$$

and

$$K_j(\nu+1) = \varphi_j(K_j(\nu) + \sum_{i \neq j} \pi_{ji} K_i(\nu), Q_j) \ \ , P_j(\nu+1) = \varphi_j(P_j(\nu), Q_j) \tag{5}$$

$1 \le j \le M, \nu \in \mathbf{N_0}$.
In order to get further information on the behaviour of these sequences we use the following result (see (Bitmead et al.,1985; formulae (3.1),(3.2)), and (de Souza, 1989, Lemma 3.1)):

Lemma 1.
a) For $1 \le j \le M$ let $0 \le \tilde{P}_j(\nu) < \hat{P}_j(\nu)$,

$$\hat{P}_j(\nu+1) = \varphi_j(\hat{P}_j(\nu), \hat{Q}) \quad \text{and} \quad \tilde{P}_j(\nu+1) = \varphi_j(\tilde{P}_j(\nu), \tilde{Q}).$$

Then $\hat{P}_j(\nu) - \tilde{P}_j(\nu)$ satisfies the Riccati difference equation

$$\hat{P}_j(\nu+1) - \tilde{P}_j(\nu+1) = \hat{Q} - \tilde{Q} \tag{6}$$

$$+A_j^T[I - \tilde{P}_j(\nu)B_j(I + B_j^T\tilde{P}_j(\nu)B_j)^{-1}]g(\hat{P}_j(\nu) - \tilde{P}_j(\nu))[I - \tilde{P}_j(\nu)B_j(I + B_j^T\tilde{P}_j(\nu)B_j)^{-1}]^T A \ ,$$

where

$$g(\hat{P}_j(\nu) - \tilde{P}_j(\nu)) = (\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j[B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j$$
$$+B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j(B_j\tilde{P}_j(\nu)B_j + I)^{-1}B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j]^{-1}B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))$$
$$+\{I - (\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j[B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j]^{-1}B_j^T\}(\hat{P}_j(\nu) - \tilde{P}_j(\nu)) \times$$
$$\times\{I - (\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j[B_j^T(\hat{P}_j(\nu) - \tilde{P}_j(\nu))B_j]^{-1}B_j^T\}^T.$$

b) If $0 \le \tilde{P}_j(\nu) \le \hat{P}_j(\nu)$ and $\hat{Q} - \tilde{Q} \ge 0$ then

$$\hat{P}_j(\nu+1) - \tilde{P}_j(\nu+1) \ge 0.$$

The inverses in part a) of the preceding lemma exist since B_j has full rank. Hence in this case $\hat{P}_j(\nu+1) - \tilde{P}_j(\nu+1)$ is the sum of nonnegative definite matrices if $\hat{P}_j(\nu) - \tilde{P}_j(\nu) \ge 0$ and $\hat{Q} - \tilde{Q} \ge 0$. The assertion of Lemma 1 b) is obtained from part a) by a continuity argument (see (Bitmead et al., 1985).

The next lemma shows that the sequences $(K_j(\nu))$ and $(P_j(\nu))$ defined in (5) are bounded from below for $K_{j0} \ge P_j^+$.

Lemma 2.
Let $K_{j0} \ge P_j^+$ then $K_j(\nu) \ge P_j(\nu) \ge P_j^+ \ge 0$ for $1 \le j \le M, \nu \in \mathbf{N_0}$.

Proof. From $P_j^+ \le K_j(0) = P_j(0) \le K_j(0) + \sum_{i\neq j} \pi_{ji} K_i(0)$ and $P_j^+ = \varphi_j(P_j^+, Q_j)$ we infer, using Lemma 1, that $K_j(1) \geq P_j(1) \geq P_j^+$, $1 \leq j \leq M$. A simple induction argument completes the proof of Lemma 2.

The following hypothesis ensures that the sequences $(K_j(\nu))$ and $(P_j(\nu))$ are decreasing (see also (Bitmead et al., 1985, Lemma 2.)).

(**H3**): There exist matrices $K_{j0} \geq 0, 1 \leq j \leq M$ such that $K_j(1) \leq K_{j0}$, $1 \leq j \leq M$.

Theorem 1.
Under hypothesis (H3) for $1 \leq j \leq M$ the limits

$$K_j^\infty := \lim_{\nu \to \infty} K_j(\nu) \ , P_j^\infty := \lim_{\nu \to \infty} P_j(\nu)$$

exist ; furthermore we have the following monotonicity properties

$$0 \leq P_j^\infty \leq P_j(\nu+1) \leq P_j(\nu) \leq K_j(\nu) \ , \nu \in \mathbf{N_0} \tag{7}$$

and

$$P_j^\infty \leq K_j^\infty \leq K_j(\nu+1) \leq K_j(\nu) \ , \nu \in \mathbf{N_0}. \tag{8}$$

Proof. The monotonicity of the sequences $(K_j(\nu))$ and $(P_j(\nu))$ is obtained from (H3) and Lemma 1 by induction. Since $P_j(0) \geq 0$ implies $P_j(\nu) \geq 0$ for $\nu \in \mathbf{N_0}$ (see (Bitmead et al., 1985, section 2) the limit $P_j^\infty \geq 0$ exists. The estimates $P_j(\nu) \leq K_j(\nu), 1 \leq j \leq M, \nu \in \mathbf{N_0}$, are also obtained by induction from $P_j(0) = K_j(0) \geq 0$ and Lemma 1. Hence K_j^∞ exists with $K_j^\infty \geq P_j^\infty$.

Notice that $P_j^\infty = P_j^+$, $1 \leq j \leq M$, if there are no $(\sqrt{Q_j}, A_j)$ unobservable modes outside the unit circle, since in this case the decoupled algebraic Riccati equations (4) have unique positive semidefinite solutions.

Corollary 1.
The coupled system (3) has a set of positive semidefinite solutions K_j^∞ , $1 \leq j \leq M$, if and only if (H3) is satisfied.

Proof . It is obvious that (H3) is necessary for the existence of solutions K_j^∞ , $1 \leq j \leq M$, of (3). The sufficiency of (H3) follows from Theorem 1.

The next lemma generalizes a well known fact for standard difference equations (see e.g. (Caines and Mayne (1970))).

Lemma 3.
The sequences $(K_j^0(\nu))_{\nu \in \mathbf{N_0}}$ defined by $K_j(0) = 0$, $K_j(\nu+1) = \varphi_j(K_j(\nu) + \sum_{i \neq j} \pi_{ji} K_i(\nu), Q_j)$ for $\nu \in \mathbf{N_0}$ are nondecreasing. These sequences are bounded if and only if (H3) is valid; in this case

$$K_j^{0,\infty} := \lim_{\nu \to \infty} K_j^0(\nu)$$

exists and defines a solution of the coupled system (3).

Proof. Obviously $K_j(1) \geq 0$, $1 \leq j \leq M$, hence the monotonicity of the sequences $(K_j^0(\nu))$ is obtained by induction from Lemma 1 ; moreover it follows from Corollary 1 and Lemma 1 that these sequences are bounded and convergent if and only if (H3) is satisfied.

Remarks

(i) Hypothesis (H3) is important from theoretical point of view since it ensures the existence of positive semidefinite solutions of (3). However, in practical applications one can always start with the zero matrix since (H3) is automatically fulfilled if the sequences $(K_j^0(\nu))$ turn out to be bounded.

(ii) Let $K_j^{0,\infty}$ exist and $0 \leq \tilde{K}_{j0} \leq K_{j0}$, $1 \leq j \leq M$. If $K_j^{0,\infty} = K_j^\infty$, $1 \leq j \leq M$, where K_j^∞ is defined as in Theorem 1, then $\lim_{\nu\to\infty} \tilde{K}_j(\nu) = K_j^{0,\infty}, 1 \leq j \leq M$, for $\tilde{K}_j(\nu+1) = \varphi_j(\tilde{K}_j(\nu) + \sum_{i\neq j} \pi_{ji}\tilde{K}_i(\nu), Q_j)$, $1 \leq j \leq M, \nu \in \mathbf{N_0}, \tilde{K}_j(0) = \tilde{K}_{j0}$. Notice that (H3) does not imply that equation (3) has a unique positive semidefinite solution.

(iii) Let $K_j^\infty \geq 0$, $1 \leq j \leq M$, be a solution of equation (3) and $F_j^\infty = \sum_{i=1}^M \pi_{ji}K_i^\infty = K_j^\infty + \sum_{i\neq j} \pi_{ji}K_i^\infty$, $1 \leq j \leq M$, then

$$F_j^\infty = A_j^T F_j^\infty A_j + Q_j^\infty - A_j^T F_j^\infty B_j\{I + B_j^T F_j^\infty B_j\}^{-1} B_j^T F_j^\infty A_j \ , 1 \leq j \leq M \ , \quad (9)$$

where $Q_j^\infty := Q_j + \sum_{i\neq j} \pi_{ji}K_i^\infty$, $1 \leq j \leq M$. Since (9) can be written as

$$F_j^\infty = (A_j + B_j H_j^\infty)^T F_j^\infty (A_j + B_j H_j^\infty) + H_j^{\infty T} H_j^\infty + Q_j^\infty$$

with $H_j^\infty = -(I + B_j^T F_j^\infty B_j)^{-1} B_j^T F_j^\infty A_j$, a direct application of Theorem 10.14 in (Bitmead and Gevers, 1991) yields that the closed loop matrices

$$A_j^\infty := A_j + B_j H_j^\infty \ , 1 \leq j \leq M,$$

are stable if $(\sqrt{Q_j^\infty}, A_j)$ is detectable; obviously this implicit detectability condition is fulfilled if $(\sqrt{Q_j}, A_j)$ is detectable.

(iv) The stability of the matrices $A_j + B_j H_j^\infty$, $1 \leq j \leq M$, does not imply the stability of the closed loop matrices

$$A_j^{cl} := A(j) - B(j)[I + B(j)^T(\sum_{i=1}^M p_{ji}K_i^\infty)B(j)]^{-1}B(j)^T(\sum_{i=1}^M p_{ji}K_i^\infty)A(j)$$

of the original system since $A_j + B_j H_j^\infty = \sqrt{p_{jj}}A_j^{cl}$ and $A_j = \sqrt{p_{jj}}A(j)$.
To ensure the stability of the matrices A_j^{cl} the cost criteria (1) should be modified as follows (see (Bourlès et al., 1990))

$$J(x_0, r_0) = \lim_{N\to\infty} \mathbf{E} \ [\sum_{k=0}^{N-1} \rho^{2k}(x_k^T Q(r_k)x_k + u_k^T R(r_k)u_k) + x_N^T K_T(r_N)x_N],$$

where $\rho^{-1} := \min\{\sqrt{p_{jj}} | 1 \leq j \leq M\}$.

(v) The monotonicity results of Wimmer (see part (v) of section 2) can be generalized to coupled systems of the form (3); details will be presented elsewhere.

4. Example

We consider the following 3 mode discrete-time jump-linear system (Blair and Sworder, 1975).

Mode 1 :

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2.5 & 3.2 \end{pmatrix}; \qquad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$Q_1 = \begin{pmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{pmatrix}; \qquad R_1 = 2.6 \; .$$

Mode 2 :

$$A_2 = \begin{pmatrix} 0 & 1 \\ -4.3 & 4.5 \end{pmatrix}; \qquad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$Q_2 = \begin{pmatrix} 10 & -3 \\ -3 & 8 \end{pmatrix}; \qquad R_2 = 1.165 \; .$$

Mode 3 :

$$A_3 = \begin{pmatrix} 0 & 1 \\ 5.3 & -5.2 \end{pmatrix}; \qquad B_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$Q_3 = \begin{pmatrix} 5 & -4.5 \\ -4.5 & 4.5 \end{pmatrix}; \qquad R_3 = 1.111 \; .$$

The discrete state transition probability matrix is

$$\begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{pmatrix} .$$

The set of coupled Riccati-like equations is integrated as follows:

$$K_j(\nu+1) = (A_j + B_j H_j(\nu))^T F_j(\nu)(A_j + B_j H_j(\nu)) + H_j(\nu)^T H_j(\nu) + Q_j$$
$$=: \Phi_j(K_j(\nu) + \sum_{i\neq j} \pi_{ji} K_i(\nu), Q_j))$$

with $H_j(\nu) = -(I + B_j^T F_j(\nu) B_j)^{-1} B_j^T F_j(\nu) A_j$ and $F_j(\nu) = K_j(\nu) + \sum_{i\neq j} \pi_{ji} K_i(\nu)$, $1 \leq j \leq 3$. The iterative procedure is stopped if $\max \|E_j(\nu)\| \leq \epsilon$, $1 \leq j \leq 3$, where $\|\cdot\|$ is the spectral norm and

$$E_j(\nu) = K_j(\nu+1) - \Phi_j(K_j(\nu) + \sum_{i\neq j} \pi_{ji} K_i(\nu), Q_j)).$$

For $\epsilon = 10^{-6}$ and the initial values $K_j(0) = K_{j0} = \begin{pmatrix} 50 & -50 \\ -50 & 100 \end{pmatrix}$, $j = 1,2,3$, (H3) is satisfied and as expected decreasing sequences are abtained as depicted in figure 1. For the initial values $K_j(0) = K_{j0} = 0$, the monotonic evolution of the norm of the solution matrices is shown in figure 2. In both cases the algorithm converges to the following solution

$$K_1^{\infty} = \begin{pmatrix} 18.6616 & -18.9560 \\ -18.9560 & 28.1086 \end{pmatrix},$$

$$K_2^{\infty} = \begin{pmatrix} 30.8818 & -21.6010 \\ -21.6010 & 36.2739 \end{pmatrix},$$

$$K_3^{\infty} = \begin{pmatrix} 35.4175 & -38.6129 \\ -38.6129 & 49.7079 \end{pmatrix}.$$

Figure 1

Figure 2

5. Conclusions

The solution of coupled discrete-time Riccati-like equations appearing in markovian jump-linear control problems is discussed in this paper. A necessary and sufficient condition for the existence of a set of positive semidefinite solutions to these equations is obtained. It is shown that this condition ensures the existence of monotonic sequences converging to the solution of the Riccati-like equations; this leads to a straightforward numerical algorithm.

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