On the Global Existence of Solutions for Game Riccati Differential Equations [*]

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Keywords: game Riccati equation, Riccati inequalities, symplectic transformations, global existence of solutions, H_∞ control.

Abstract

For the symplectic H_∞-type matrix Riccati differential equation we derive sufficient conditions ensuring the global existence of the solutions of the corresponding initial and/or terminal value problems.

1 INTRODUCTION

We consider in this paper game Riccati differential equations of the type

$$\dot{W} = -A^TW - WA - Q + W(S_1 - \frac{1}{\gamma^2}S_2)W = Ric(H,W) \qquad (RDE)$$

on the interval $[0, t_f]$; here A, Q, S_1, S_2 are $n \times n$ matrices with real entries, $Q = Q^T, S_1 = {S_1}^T, S_2 = {S_2}^T$, $\gamma \in \mathbf{R}$ and $H := \begin{pmatrix} A & -S_1 + \frac{1}{\gamma^2}S_2 \\ -Q & -A^T \end{pmatrix}$.

In the sequel we will assume S_1, S_2 to be positive semidefinite and put $S := S_1 - \frac{1}{\gamma^2}S_2$.

In particular we are interested in the existence of the solution of (RDE) satisfying an initial condition $W(0) = W_0$ or a terminal condition $W(t_f) = W_f$. Such initial or terminal value problems occur in finite time robust control, H_∞-control and estimation, [2], [5],[11].

The main difficulty in the aforementioned references is to guarantee the existence of the solution $W(t)$ on the whole interval $[0, t_f]$.

In the sequel we denote by $W(\cdot, X_0)$ the solution of (RDE) with $W(t_f, X_0) = X_0$. It is known from the literature that $W(\cdot, X_0)$ and the maximal solution W_m (if it exists) of the algebraic Riccati equation

$$-A^TW - WA - Q + W(S_1 - \frac{1}{\gamma^2}S_2)W = 0 \qquad (ARE)$$

depend monotonically on X_0 and on $\begin{pmatrix} Q & A^T \\ A & -S \end{pmatrix}$; see [10], [8], [9], [4].

It has been proved in [4] that the set Γ of all real symmetric solutions of (ARE) is nonempty if and only if there exist symmetric matrices W_1, W_2 with

$$W_1 \leq W_2, Ric(H, W_1) \leq 0 \quad \text{and} \quad Ric(H, W_2) \geq 0.$$

Notice that up to now no conditions on the definiteness of Q or S have been required. In most applications we have $Q \geq 0$ and in this case $Ric(H, 0) \leq 0$; hence (ARE) has a positive semidefinite solution if and only if there exists $W_2 \geq 0$ such that $Ric(H, W_2) \geq 0$.

If S is not positive semidefinite (ARE) may still have a positive semidefinite solution W_+ but it turns out that it could be difficult to check if there is a $W_2 \geq 0$ with $Ric(H, W_2) \geq 0$ guaranteeing the existence of W_+.

If there exists a solution $W_+ \geq 0$ of (ARE) and if $Q \geq 0$ then it follows from comparison theorems (see [7], Satz 10.2) that $W(t, X_0)$ exists for $t \leq t_f$ while $0 \leq X_0 \leq W_+$. On the other hand, it is well known that $W(\cdot, X_0)$ may have finite escape time for $X_0 \geq W_+$.

Since in many applications one has to check the existence of $W(t, X_0)$ for $t \leq t_f$ and for different terminal (or initial) values X_0, we propose in the following section an alternative method which can be used without any definiteness assumptions on Q and/or S.

2 EXISTENCE RESULTS

In this section we mainly use properties of hamiltonian systems and the fact that there exists a transformation of (RDE) to a higher dimensional linear system.

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[*] The research described in this paper was supported by the French-German program PROCOPE (Grant 92213) and by DAAD (Grant 322-AI-p-dr)

Recall that the matrix

$$H = \begin{pmatrix} A & -S \\ -Q & -A^T \end{pmatrix} \in \mathbf{R}^{2n,2n} \tag{2.1}$$

defined in the introduction is hamiltonian, consequently there exists a symplectic matrix

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{C}^{2n,2n}, \quad \text{where } a, b, c, d, \in \mathbf{C}^{n,n},$$

such that

$$V^{-1}HV = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1^* \end{pmatrix}.$$

For simplicity, we assume in the sequel that J_1 is diagonal and has all its eigenvalues in the open left half plane, i. e. $\sigma(J_1) \subset \mathbf{C}^-$. This describes the generic case and leads to the dichotomic solution of the associated algebraic Riccati equation (ARE) (see [3]).
Let $W_0 \in \mathbf{R}^{n,n}$ with

$$\det Z_1 \neq 0, \quad \text{where } \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := V^{-1} \begin{pmatrix} I_n \\ W_0 \end{pmatrix}; \tag{2.2}$$

notice that (see [3]) the set $D^- = \{W_0 \in \mathbf{R}^{n,n} | W_0$ *has property* (2.2)$\}$ is open and dense in $\mathbf{R}^{n,n}$. Then we set $Z_0 = Z_2 Z_1^{-1}$ and introduce the Möbius-type transformation (see [6])

$$W_0 = (c + dZ_0)(a + bZ_0)^{-1}.$$

Furthermore, let Y denote a solution of $\dot{Y} = HY$, $Y(t) \in \mathbf{R}^{2n,n}$, with initial value

$$Y(0) = V \begin{pmatrix} I_n \\ Z_0 \end{pmatrix} = \begin{pmatrix} a + bZ_0 \\ c + dZ_0 \end{pmatrix} = \begin{pmatrix} I_n \\ W_0 \end{pmatrix} Z_1^{-1};$$

Y has the representation

$$Y(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{J_1 t} & 0 \\ 0 & e^{-J_1^* t} \end{pmatrix} \begin{pmatrix} I_n \\ Z_0 \end{pmatrix} = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix},$$

where $Q(t), P(t) \in \mathbf{R}^{n,n}$.
From (2.2) it follows that $(a + bZ_0)^{-1}(= Z_1)$ exists; using the fact that Z_0 is uniquely determined by W_0 we infer that $(-W_0 b + d)^{-1}$ exists as well with

$$Z_0 = (-W_0 b + d)^{-1}(W_0 a - c).$$

To Y there corresponds a solution W of (RDE) defined by (see [3] for further details)

$$\begin{aligned} W(t) &:= P(t)Q(t)^{-1} \\ &= (ce^{J_1 t} + de^{-J_1^* t} Z_0)(ae^{J_1 t} + be^{-J_1^* t} Z_0)^{-1} \\ &= (c + de^{-J_1^* t} Z_0 e^{-J_1 t})(a + be^{-J_1^* t} Z_0 e^{-J_1 t})^{-1} \\ &= (c + dZ(t))(a + bZ(t))^{-1} \end{aligned} \tag{2.3}$$

as long as $(a + bZ(t))^{-1}$ exists. Here we used $Z(t) := e^{-J_1^* t} Z_0 e^{-J_1 t}$; hence $Z(0) = Z_0$ and therefore $W(0) = W_0$ and $Z(t) = (-W(t)b + d)^{-1}(W(t)a - c)$, while $(-W(t)b + d)^{-1}$ exists.
V is symplectic, consequently we have (see [6]) $a^*d - c^*b = I_n = ad^* - bc^*$, and the matrices a^*c, b^*d, ab^* and cd^* are hermitian. Using these properties it follows by a straightforward computation that

$$\begin{aligned} Z(t) &= (a^*W(t) - c^*)(d^* - b^*W(t))^{-1} \\ &= (-W(t)b + d)^{-1}(W(t)a - c); \end{aligned} \tag{2.4}$$

notice that here $W(t)$ is not assumed to be symmetric. Differentiating and then using the different representations (2.4) of $Z(t)$ we get

$$\begin{aligned} \dot{Z} &= a^*\dot{W}(t)(d^* - b^*W(t))^{-1} \\ &+ (a^*W(t) - c^*)(d^* - b^*W(t))^{-1}(-b^*\dot{W}(t))(d^* - b^*W(t))^{-1} \\ &= a^*\dot{W}(t)(d^* - b^*W(t))^{-1} \\ &+ Z(t)(-b^*\dot{W}(t))(d^* - b^*W(t))^{-1} \\ &= a^*\dot{W}(t)(d^* - b^*W(t))^{-1} \\ &+ (d - W(t)b)^{-1}(W(t)a - c)b^*\dot{W}(t)(d^* - b^*W(t))^{-1}. \end{aligned}$$

On the other hand Z is a solution of $\dot{Z}(t) = -J_1^* Z(t) - Z(t)J_1$, hence

$$\begin{aligned} &(d - W(t)b)\dot{Z}(t)(d^* - b^*W(t)) \\ &= (d - W(t)b)(-J_1^* Z(t) - Z(t)J_1)(d^* - b^*W(t)) = \\ &(d - W(t)b)a^*\dot{W}(t) + (W(t)a - c)b^*\dot{W}(t) \\ &= (da^* - cb^*)\dot{W}(t) - W(t)(ba^* - ab^*)\dot{W}(t) = \dot{W}(t). \end{aligned}$$

Now we replace here $Z(t)$ by its representations (2.4), this yields

$$\begin{aligned} \dot{W} &= dJ_1^*c^* + cJ_1 d^* - W[bJ_1^*c^* + aJ_1 d^*] \\ &- [dJ_1^*a^* + cJ_1 b^*]W + W[bJ_1^*a^* + aJ_1 b^*]W. \end{aligned}$$

By definition W is a solution of

$$\dot{W} = -Q - WA - A^*W + WSW, \quad W(0) = W_0$$

as well and since $W_0 \in D^-$ is arbitrary it follows that

$$\begin{aligned} S &= aJ_1 b^* + bJ_1^* a^* \\ A &= aJ_1 d^* + bJ_1^* c^* \\ Q &= -cJ_1 d^* - dJ_1^* c^*. \end{aligned} \tag{2.5}$$

After these preparations and under assumption (2.2) we get the following results, where we denote by W the solution of (RDE) with $W(0) = W_0$ (as represented in (2.3)).

THEOREM 2.1. *For $\rho \in \mathbf{R}$ and $W_0 \in D^-$ we set*

$$\begin{aligned} R(\rho, W_0) &= W_0(S + \rho bb^*)W_0^* - W_0(A + \rho bd^*) \\ &- (A^* + \rho db^*)W_0^* - Q + \rho dd^*; \end{aligned}$$

with these abbreviations we have:
a) If $R(0,W_0)\geq 0$ *(or* ≤ 0*) then* $R(0,W(t))\geq 0$ *(or* ≤ 0*, respectively) while* $W(t)$ *exists.*
b) If sign(ρ*)*$R(\rho,W_0)\geq 0$ *(or* ≤ 0*) then sign(*ρ*)*$R(\rho,W(t))\geq 0$ *(or* ≤ 0*) for* $t\leq 0$ *(or* $t\geq 0$*, respectively,) while* $W(t)$ *exists.*
c) If sign(ρ*)* $R(\rho,W_0)\leq 0$ *then sign(*ρ*)* $R(0,W(t))\leq 0$ *for* $t\leq 0$ *while* $W(t)$ *exists; notice that here we do not claim that sign(*ρ*)* $R(\rho,W(t))\leq 0$.

Proof. We give a detailed proof for $\rho\geq 0$ and $R(\rho,W_0)\geq 0$; for all remaining cases the proof is similar. Defining $\phi(\rho):=-Z_0J_1-J_1^*Z_0^*+\rho I_n$ we get together with (2.5) by an elementary calculation

$$(d-W_0b)\phi(\rho)(d^*-b^*W_0^T)=W_0(S+\rho bb^*)W_0^T$$
$$+W_0(-A-\rho bd^*)+(-A^*-\rho db^*)W_0^T-Q+\rho dd^*=R(\rho,W_0). \tag{2.6}$$

From this identity and $R(\rho,W_0)\geq 0$ we infer that $\phi(\rho)\geq 0$. Multiplying this from the left by $e^{-J_1^*t}$ and from the right by e^{-J_1t} gives

$$-Z(t)J_1-J_1^*Z^*(t)\geq -\rho e^{-(J_1+J_1^*)t}$$
$$=-\rho e^{-2\,\mathrm{diag}(\mathrm{Re}\lambda_1,\cdots,\mathrm{Re}\lambda_n)t}I_n.$$

By our assumption Re $\lambda_1<0,\ldots,$ Re $\lambda_n<0$ we obtain

$$-Z(t)J_1-J_1^*Z^*(t)\geq -\rho I_n \quad\text{for}\quad t\leq 0. \tag{2.7}$$

From the proof of (2.6) it follows that (2.6) also holds if therein W_0 and $\phi(\rho)$ are replaced by $W(t)$ and $-Z(t)J_1-J_1^*Z(t)^*+\rho I_n$, respectively. Therefore we obtain

$$W(t)(S+\rho bb^*)W^T(t)^{-}W(t)(A+\rho bd^*)$$
$$-(A^*+\rho db^*)W^T(t)-Q+\rho dd^*\geq 0.$$

This holds for $t\leq 0$ as long as $W(t)$ exists. ■

REMARK 2.2. (i) For $\rho=0$ and symmetric matrices W_0 the assertion of Theorem 2.1 a) is well known and follows from properties of Ljapunov equations (see [7], Hilfssatz 10.4).
(ii) Theorem 2.1 is based on the assumption $W_0\in D^-$ (i. e. $\det Z_1\neq 0$), which has been used to ensure that W_0 is in the basin of attraction of the dichotomic solution as $t\to-\infty$ (see Remark 2.5).
If we assume instead of $\det Z_1\neq 0$ that W_0 belongs to

$$D^+=\{W_0\in\mathbf{C}^{n,n}|\det Z_2\neq 0 \textit{ in } (2.2)\},$$

then we can prove as before a modified version of Theorem 2.1. From $\tilde{Z}_0=Z_1Z_2^{-1}$ it follows that $a\tilde{Z}_0+b$ is regular with $W_0=(c\tilde{Z}_0+d)(a\tilde{Z}_0+b)^{-1}$; moreover W_0a-c is regular and $\tilde{Z}_0=(W_0a-c)^{-1}(d-W_0b)$.
Using $\tilde{Z}(t):=e^{J_1t}\tilde{Z}_0e^{J_1^*t}$ and $\tilde{\phi}:=-\tilde{Z}_0J_1-J_1^*\tilde{Z}_0+\rho I_n$ (instead of $Z(t)$ and $\phi(\rho)$), we obtain as before and with the abbreviation

$$\tilde{R}(\rho,W_0)=W_0(S+\rho aa^*)W_0^*-W_0(A+\rho ac^*)$$
$$-(A^*+\rho ca^*)W_0^*-Q+\rho cc^*$$

the following Lemma.

LEMMA 2.3 *a) If* $\tilde{R}(0,W_0)\geq 0$ *(or* ≤ 0*) then* $\tilde{R}(0,W(t))\geq 0$ *(or* ≤ 0*, respectively,) while* $W(t)$ *exists.*
b) If sign(ρ*)*$\tilde{R}(\rho,W_0)\geq 0$ *(or* ≤ 0*) then sign(*ρ*)*$\tilde{R}(\rho,W(t))\geq 0$ *(or* ≤ 0*) for* $t\geq 0$ *(or* $t\leq 0$*, respectively,) while* $W(t)$ *exists.*
c) If sign(ρ*)* $\tilde{R}(\rho,W_0)\leq 0$ *then sign(*ρ*)* $\tilde{R}(0,W(t))\leq 0$ *for* $t\geq 0$ *while* $W(t)$ *exists; notice that here we do not claim that sign(*ρ*)* $R(\rho,W(t))\geq 0$.

From the proof of Theorem 2.1 it follows that (2.7) and consequently also the assertions of Theorem 2.1 b), c) and Lemma 2.3 b), c) could be sharpened; we omit details.

Theorem 2.1 and Lemma 2.3 can be used to obtain *global* existence theorems; without loss of generality let in the sequel $t_f=0$.

THEOREM 2.4. *Let* $W_0\in D^-$ *such that* $R(\rho,W_0)\leq 0$ *holds for some* $\rho\leq 0$*. If in addition* $S+\rho bb^*>0$*, then the solution* $W=W(.,W_0)$ *of (*RDE*) with* $W(0)=W_0$ *remains bounded (with respect to some norm on* $\mathbf{R}^{n,n}$*) on each bounded subinterval of* $(-\infty,0]$*,*
i. e. $W(\cdot,W_0)$ *has no poles on the negative real axis.*

Proof. Notice that from representation (2.3) it follows that the singularities of W are poles. Given any pole of W at $t_0\in(-\infty,0)$ there is a local series representation

$$W(t)=W_{-p}\frac{1}{(t-t_0)^p}+\cdots,\qquad W_{-p}\in\mathbf{C}^{n,n}\setminus\{0\}.$$

From $R(\rho,W(t))\leq 0$ we infer (by taking the limit as $t\to t_0$)

$$W_{-p}(S+\rho bb^*)W_{-p}^T\leq 0.$$

Since $S+\rho bb^*>0$ we infer that $W_{-p}=0$, hence poles do not exist in $(-\infty,0]$. ■

REMARK 2.5. (i) From the representation (2.3) we infer that $\lim_{t\to-\infty}W(t)=:W^-$ exists if (2.2) holds and a^{-1} exists; we obtain in this case $W^-=ca^{-1}$, which obviously is a solution of (ARE). Notice that the *dichotomic solution* W^- of (ARE) exits (under our standing assumption $\sigma(J_1)\subset\mathbf{C}^-$) if and only if a^{-1} exists; for $S\geq 0$ it is well known that a^{-1} exists if and only if $(A,\sqrt{S})$ is stabilizable.
(ii) Let a^{-1} exist. It is known (see [3]) that

$$\mathbf{A}\hat{\mathbf{R}}(W^-)=\{W_0\in\mathbf{R}^{n,n}|\lim_{\mathbf{R}\ni t\to-\infty}W(t,W_0)=W^-\},$$

the *(generalized) basin of attraction* of W^-, is open and dense in $\mathbf{R}^{n,n}$ (here we used the limit with respect to the

chordal metric.)
Moreover it is known that

$$\mathbf{A}\hat{\mathbf{R}}(W^-) = D^- = \{W_0 \in \mathbf{R}^{n,n} | W_0 \text{ has property (2.2)}\}$$

and that $W(\cdot, W_0)$ has at most a finite number of poles on $(-\infty, 0]$ if $W_0 \in \mathbf{A}\hat{\mathbf{R}}(W^-)$.
In general the form of

$\mathbf{AR}(W^-) = \{W_0 \in \mathbf{R}^{n,n} | \lim_{\mathbf{R} \ni t \to -\infty} W(t, W_0) = W^-$ with respect to the euclidean metric and such that $W(\cdot, W_0)$ has no poles in $(-\infty, 0]\}$,

which is the *(restricted) basin of attraction* of W^-, cannot be described completely.
Under the assumptions of Theorem 2.4 it follows that $W_0 \in \mathbf{A}\hat{\mathbf{R}}(W^-)$ implies $W_0 \in \mathbf{AR}(W^-)$.
(iii) If $S + \rho bb^* > 0$ and $-Q + \rho dd^* \leq 0$ for some $\rho \leq 0$ then obviously $\|W_0\|$ can be chosen small enough such that the assumptions of Theorem 2.4 are fulfilled. In this case for example $W_0 = 0$ is an appropriate initial value for a solution $W(t)$ without singularities on $(-\infty, 0]$ if $0 \in D^-$.

Analogously to Theorem 2.4 we get the following existence results:

COROLLARY 2.6. *The solution W of (RDE) with $W(0) = W_0$ has no pole on the interval I if $\rho \in \mathbf{R}, W_0 \in \mathbf{R}^{n,n}$ and $I \subset \mathbf{R}$ satisfy at least one of the following 12 conditions:*

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S + \rho bb^* < 0, \rho \geq 0, I = (-\infty, 0] \tag{2.8}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S + \rho bb^* < 0, \rho \leq 0, I = [0, \infty) \tag{2.9}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S < 0, \rho \leq 0, I = (-\infty, 0] \tag{2.10}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S + \rho bb^* > 0, \rho \leq 0, I = (-\infty, 0] \tag{2.11}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S + \rho bb^* > 0, \rho \geq 0, I = [0, \infty) \tag{2.12}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S > 0, \rho \geq 0, I = (-\infty, 0] \tag{2.13}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0,\ S + \rho aa^* < 0, \rho \geq 0, I = [0, \infty) \tag{2.14}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0, S + \rho aa^* < 0, \rho \leq 0, I = (-\infty, 0] \tag{2.15}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0,\ S < 0, \rho \leq 0, I = [0, \infty) \tag{2.16}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S + \rho aa^* > 0, \rho \leq 0, I = [0, \infty) \tag{2.17}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S + \rho aa^* > 0, \rho \geq 0, I = (-\infty, 0] \tag{2.18}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S > 0, \rho \geq 0, I = [0, \infty). \tag{2.19}$$

Corollary 2.6 yields twelve different sufficient conditions for the global existence of $W(t)$, for $t \in I$; here condition (2.11) corresponds to Theorem 2.4.

EXAMPLES 2.7. Using a computer-algebra system like MAPLE it is very easy to produce nontrivial examples verifying one of the hypotheses of Corollary 2.6. Here we reproduce two of these examples:

Let

$$H = \begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 1 & 2 & 0 & -4 \\ -4 & -1 & -1 & -1 \\ -1 & -2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} A & -S \\ -Q & -A^T \end{pmatrix},$$

then it follows with

$$V := \begin{pmatrix} -\frac{7\sqrt{5}}{11} + \frac{16}{11} & 0 & 7/4 + \frac{4\sqrt{5}}{5} & -7/2 \\ -\frac{21}{11} + \frac{14\sqrt{5}}{11} & 1 & -\frac{21\sqrt{5}}{20} - 7/2 & \frac{41}{8} \\ -4/11 + \frac{10\sqrt{5}}{11} & 2 & -\frac{\sqrt{5}}{5} - 5/2 & \frac{13}{4} \\ 1 & 1 & \frac{11\sqrt{5}}{20} & -\frac{7}{8} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

that

$$V^{-1}HV = \text{diag}\,(-\sqrt{5}, -2, \sqrt{5}, 2).$$

Moreover for example $W_0 = \begin{pmatrix} -7 & 2 \\ 2 & 0.999 \end{pmatrix} \in D^+ \cap D^-$,

$$S + 1900aa^*$$

$$= \begin{pmatrix} \frac{1903679}{242} - \frac{425600\sqrt{5}}{121} & \frac{704900\sqrt{5}}{121} - \frac{1569400}{121} \\ \frac{704900\sqrt{5}}{121} - \frac{1569400}{121} & \frac{2930284}{121} - \frac{1117200\sqrt{5}}{121} \end{pmatrix} > 0,$$

and

$$\tilde{R}(1900, W_0)$$

$$= \begin{pmatrix} \frac{170790395}{242} - \frac{38190000\sqrt{5}}{121} & -\frac{14896847}{121000} + \frac{66633\sqrt{5}}{1210} \\ -\frac{14896847}{121000} + \frac{66633\sqrt{5}}{1210} & \frac{611571}{30250000} - \frac{2793\sqrt{5}}{302500} \end{pmatrix} < 0.$$

Obviously here (2.18) is fulfilled; consequently $W(t, W_0)$ with $W(0, W_0) = W_0$ exists for $t \leq 0$ and $\lim_{t\to-\infty} W(t, W_0) = W^- = \begin{pmatrix} -2 - 2\sqrt{5} & 2 \\ 2 & 1 \end{pmatrix}$.
Notice that here S is indefinite (like for example in H^∞ control problems), moreover the initial matrix W_0 and the stabilizing solution W^- of the algebraic Riccati equation are indefinite as well. Examples of this and of similar type show how Corollary 2.6 can be used to ensure the global existence of solutions of initial or terminal value problems for Riccati differential equations. Notice that we can use corollary 2.6 in combination with the monotonic dependence of the solutions of (RDE) on the terminal values in order to approximate $\mathbf{AR}(W^-)$; we omitt details.
The matrix S appearing in our example can be written in the form $S = S_1 - \frac{1}{\gamma^2} S_2$, where

$$S_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{pmatrix}, \; S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and } \gamma = 1.$$

The dependence of the stabilizing solution of H^∞-type algebraic Riccati equations on the parameter γ has been studied in detail by Gahinet [5]. In our lecture we will discuss the dependence of all solutions of the algebraic Riccati equation and of the phase portrait of the differential equation on the parameter γ. In particular we will give an explanation of the significance of the bifurcation points; further details on this topic can be found in [1].

For another example where we study (RDE) for different terminal values W_f, let the corresponding hamiltonian matrix be

$$H = \begin{pmatrix} -5 & 1 & \frac{1}{98} & 0 \\ 0 & -3 & 0 & -\frac{171}{49} \\ 0 & 0 & 5 & 0 \\ 0 & -2 & -1 & 3 \end{pmatrix}.$$

With

$$V = \begin{pmatrix} 1 & \frac{42\sqrt{87}}{221} + \frac{759}{442} & -\frac{42\sqrt{87}}{221} + \frac{759}{442} & 1 \\ 0 & \frac{3\sqrt{87}}{14} + 3/2 & -\frac{3\sqrt{87}}{14} + 3/2 & \frac{8379}{860} \\ 0 & 0 & 0 & \frac{10829}{430} \\ 0 & 1 & 1 & -\frac{4802}{215} \end{pmatrix}$$

we have

$$V^{-1}HV = \operatorname{diag}\,(-5, -\frac{3\sqrt{87}}{7}, \frac{3\sqrt{87}}{7}, 5).$$

Therefore (ARE) has the following four real symmetric solutions:

$$W_s = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{49}{57} + \frac{7\sqrt{87}}{57} \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{49}{57} - \frac{7\sqrt{87}}{57} \end{pmatrix}.$$

$$W_2 = \begin{pmatrix} -2.8886298 & 2.88127719 \\ 2.88127719 & -2.588124882 \end{pmatrix}$$

,

$$W_a = \begin{pmatrix} -303.9120092 & 33.77757823 \\ 33.77757823 & -5.759245408 \end{pmatrix}.$$

Notice that here the stabilizing solution $W_s \geq 0$ is the maximal solution and the antistabilizing solution $W_a < 0$ is the minmal solution; moreover it is known from comparison results that $W(t, W_f)$ exists for $t \leq t_f$ and $W_a \leq W_f \leq W_s$.
Next we consider $W(t, W_{fj})$ for $W_{fj} = W_s + P_{fj}$ and 5 different values of $P_{fj} \geq 0 \quad 1 \leq j \leq 5$. For

$$P_{f1} = \begin{pmatrix} 1/20 & 0 \\ 0 & 1/10 \end{pmatrix} \text{ and } P_{f2} = \begin{pmatrix} 1/19 & \frac{1}{95} \\ \frac{1}{95} & 1/10 \end{pmatrix}$$

we have

$$W_{f1} = \begin{pmatrix} 1/20 & 0 \\ 0 & -\frac{433}{570} + \frac{7\sqrt{87}}{57} \end{pmatrix} \text{ and}$$

$$W_{f2} = \begin{pmatrix} 1/19 & \frac{1}{95} \\ \frac{1}{95} & -\frac{433}{570} + \frac{7\sqrt{87}}{57} \end{pmatrix}.$$

This yields with $\rho = -\frac{19}{5}$ that

$$S - \frac{19}{5} aa^* = -\begin{pmatrix} 50.08977392 & 46.39766765 \\ 46.39766765 & 43.02627088 \end{pmatrix} < 0,$$

$$\tilde{R}(W_{f1}, \rho) = \begin{pmatrix} 0.3747755653 & -0.2819883380 \\ -0.2819883380 & 0.369226920 \end{pmatrix} \geq 0$$

and

$$\tilde{R}(W_{f2}, \rho) = \begin{pmatrix} 0.3313853413 & -0.2803022434 \\ -0.2803022434 & 0.2449448791 \end{pmatrix} \geq 0.$$

Hence, according to Corollary 2.6 (2.15), $W(t, W_{f1})$ and $W(t, W_{f2})$ exist for $t \leq t_f$.
Since the solutions of (RDE) are monotonically dependent on the terminal value (see [4], Theorem 2.2) the existence of $W(t, W_{f1})$ and $W(t, W_{f2})$ implies that $W(t, W_f)$ exists for $t \leq t_f$ if

$$W_a \leq W_f \leq W_{f1} \text{ or } W_a \leq W_f \leq W_{f2}.$$

On the other hand it can be checked easily (for example using MAPLE) that the solutions $W(\cdot, W_s + P_{fj}), \;\; 3 \leq j \leq 5$, with

$$P_{f3} = \begin{pmatrix} 2/19 & \frac{1}{95} \\ \frac{1}{95} & 1/10 \end{pmatrix}, \; P_{f4} = \begin{pmatrix} 1/19 & 0 \\ 0 & 1/5 \end{pmatrix}$$

$$\text{and } P_{f5} = \begin{pmatrix} 1/50 & 0 \\ 0 & 3/10 \end{pmatrix}$$

have poles for $t - t_f = -0.1795267853$, $t - t_f = -0.1560596674$ and $t - t_f = -0.1070537472$ respectively. Using again that the solutions of (RDE) are monotonically dependent on the terminal value, we infer that $W(t, W_f)$ has at least one pole in $(-\infty, t_f]$ if $W_{fj} \leq W_f$ for at least one $j \in \{3, 4, 5\}$.

This example shows how Corollary 2.6 can be used to check the existence of $W(t, W_f)$ for different terminal values W_f; moreover one can use Corollary 2.6 to approximate the basin of attraction of W_s.

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