

# Generalized Riccati difference and differential equations \*

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## Abstract:

In this paper we investigate generalized Riccati differential and difference equations obtained from standard Riccati equations by adding a semidefinite perturbation term. For such equations we give results on the monotonic dependence of the solutions on the coefficients and initial values as well as results on convergence of solutions.

**Key words:** generalized Riccati equations, monotonicity, comparison of solutions, convergence of solutions, jump linear systems, guaranteed cost control.

## 1. Introduction

In this paper we are considering generalized differential Riccati equations of the type

$$\dot{K} = -A^T(t)K - KA(t) - Q(t) + KS(t)K - \Pi(t, K), \quad (1.1)$$

where  $A(t), Q(t), S(t), K(t) \in \mathbf{R}^{n \times n}$  and  $\Pi(t, K)$  is symmetric if  $K$  is symmetric, and the corresponding difference equation

$$K(\nu + 1) = A^T(\nu)K(\nu)A(\nu) + Q(\nu) - A^T(\nu)K(\nu)B(\nu)\{I + B^T(\nu)K(\nu)B(\nu)\}^{-1}B^T(\nu)K(\nu)A(\nu) + \Pi(\nu, K(\nu)), \quad (1.2)$$

where  $A(\nu), K(\nu), Q(\nu) \in \mathbf{R}^{n \times n}$ ,  $B(\nu) \in \mathbf{R}^{n \times m}$ .

In equation (1.1) the coefficient matrices and also  $\Pi(t, K)$  are continuous. These assumptions are sufficient for the purpose of this paper but most of the results could also be obtained for complex and Lebesgue measurable coefficients.

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Equations of type (1.1) or (1.2) occur in robust control problems using guaranteed cost approach [9], [10] or in stochastic control when dealing with jump linear systems [2], [1], [6].

It is shown that there are similar methods and results for continuous and discrete-time problems concerning convergence of solutions, monotonicity with respect to initial conditions and coefficient matrices.

The paper is divided into two main sections. First differential equations are considered, then in section 3, the corresponding generalized Riccati difference equations are investigated.

## 2. Generalized differential Riccati equations

### 2.1 A comparison theorem

It is known from the literature that the solutions of standard Riccati differential equations of the form

$$\dot{K} = -A^T(t)K - KA(t) - Q(t) + KS(t)K \quad (2.1)$$

are monotonically dependent on  $Q$  and  $-S$  (see [7], Satz 10.2 and [5]). For  $S \geq 0$  it has been shown by Coppel [4] that the solutions of (2.1) are also monotonically dependent on  $JH$ , with

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where  $I$  is the identity matrix and where

$$H = \begin{pmatrix} A & -S \\ -Q & -A^T \end{pmatrix}$$

is the hamiltonian matrix corresponding to (2.1) (see also Royden [14] and Kratz [8]). In addition it has been shown by Wimmer [16] (see also [18]) that the strong solution of the algebraic Riccati equation

$$0 = A^T K + KA + Q - KSK$$

also depends monotonically on  $JH$  if  $S \geq 0$ . Similar results have been obtained in the discrete time case (cf. [2], [3], [11], [12], [17] and section 3 of this paper).

The following theorem is a generalization of the aforementioned results for (2.1) since we do not assume that  $S(t) \geq 0$ , moreover we give a simple proof.

**Theorem 2.1** For  $i = \{1, 2\}$  let  $K_i$  be a solution of

$$\dot{K}_i = -A_i^T(t)K - KA_i(t) - Q_i(t) + KS_i(t)K$$

on some interval  $\mathcal{I}$ . If for some  $t_f \in \mathcal{I}$

$K_1(t_f) \leq K_2(t_f)$  or  $(K_1(t_f) < K_2(t_f))$  and if

$$JH_2(t) = \begin{pmatrix} Q_2 & A_2^T \\ A_2 & -S_2 \end{pmatrix}(t) \geq JH_1(t) = \begin{pmatrix} Q_1 & A_1^T \\ A_1 & -S_1 \end{pmatrix}(t) \text{ for } t \in \mathcal{I}$$

then  $K_1(t) \leq K_2(t)$  or  $(K_1(t) < K_2(t))$ , respectively, for all  $t \in \mathcal{I} \cap (-\infty, t_f]$ .

**Proof.**  $K_0 = K_2 - K_1$  satisfies  $K_0(t_f) \geq 0$  (or  $K_0(t_f) > 0$ ), respectively, and

$$\begin{aligned} \dot{K}_0 &= \dot{K}_2 - \dot{K}_1 = (K_2 - K_1)S_2(K_2 - K_1) + K_1S_2(K_2 - K_1) + (K_2 - K_1)S_2K_1 + K_1(S_2 - S_1)K_1 \\ &\quad - A_2^T(K_2 - K_1) - (K_2 - K_1)A_2 - (A_2 - A_1)^TK_1 - K_1(A_2 - A_1) - (Q_2 - Q_1) = \\ &\quad \tilde{A}^TK_0 + K_0\tilde{A} + \begin{pmatrix} I & K_1 \end{pmatrix} (JH_1 - JH_2) \begin{pmatrix} I \\ K_1 \end{pmatrix} \leq \tilde{A}^TK_0 + K_0\tilde{A} \end{aligned}$$

since

$$\begin{pmatrix} I & K_1 \end{pmatrix} (JH_1 - JH_2) \begin{pmatrix} I \\ K_1 \end{pmatrix} \leq 0$$

and where we used the abbreviation  $\tilde{A} = -A_2^T + S_2K_1 + \frac{1}{2}S_2K_0$ .

Hence, the result is an immediate consequence of [7], Hilfssatz 10.3 (and its proof).  $\square$

It is obvious that theorem 2.1 can be used for the investigation of the monotonicity of equation (1.1) with respect to the matrix

$$\begin{pmatrix} Q + \Pi(\cdot, K) & A^T \\ A & -S \end{pmatrix} (t).$$

## 2.2 Monotonicity Results

In the remaining part of this section the generalized Riccati equation (1.1) is investigated, and the following assumptions will be used:

(C1)  $A, Q, S$  are constant matrices with  $Q \geq 0$  and  $S \geq 0$ .

(C2)  $\Pi(t, K) = \Pi(K)$  is independent of  $t$  with  $\Pi(K) \geq 0$  if  $K \geq 0$ .

(C3) If  $0 \leq K_1 \leq K_2$ , then  $\Pi(K_1) \leq \Pi(K_2)$ , and  $\alpha\Pi(K) - \Pi(\alpha K) \geq 0$  for  $K \geq 0$  and  $\alpha \in [1, \infty)$ .

Assumptions (C2) and (C3) are satisfied for example if  $\Pi(K)$  is the sum of terms of the form  $C^TKC$  and/or  $k\|K\|(I + k\|K\|I)^{-1}$ , where  $k$  is a positive number.

Generalized Riccati differential equations with perturbation term  $\Pi(K)$  satisfying (C2) and (C3) were studied in [20], [13], [5] (in the nonautonomous case) and in [1] (see also the references given therein).

The following results have been obtained in [1] for differential equations (1.1) with a special perturbation term  $\Pi(K)$  satisfying (C2) and (C3); the proofs are omitted here since they are easily obtained from the corresponding proofs in [1].

**Lemma 2.2**([1], Lemma 1) *Let  $K_1, K_2$  be solutions of the coupled system (1.1) on the interval  $\mathcal{I}$  with  $0 \leq K_1(t_f) \leq K_2(t_f)$  (or  $0 \leq K_1(t_f) < K_2(t_f)$ ) for some  $t_f \in \mathcal{I}$ , then*

$K_1(t) \leq K_2(t)$  (or  $K_1(t) < K_2(t)$ ) for  $t \in \mathcal{I} \cap (-\infty, t_f]$ .

**Remark 2.3** (i) It is worth while to point out that the result of Lemma 2.2 remains true without assuming  $K_1(t_f) \geq 0$  if we assume that  $K_1 \leq K_2$  implies  $\Pi(K_1) \leq \Pi(K_2)$ ; here we do not use this version of Lemma 2.2.

(ii) The following theorem shows that the solutions of

$$\dot{K} = -A^T K - KA - Q + KSK - \Pi(K) \quad , K(t_f) = K_f \geq 0 \quad (2.2)$$

are bounded below by the corresponding solutions of the decoupled Riccati equation

$$\dot{P} = -A^T P - PA - Q + PSP, \quad P(t_f) = K_f. \quad (2.3)$$

As a consequence of Theorem 2.1 we obtain

$$0 \leq P(t) \leq K(t) \leq L(t), \quad \text{for } t \leq t_f,$$

while the solution  $L$  of

$$\dot{L} = -A^T L - LA - Q - \Pi(L) \quad , \quad L(t_f) = K_f \geq 0$$

exists. Notice that  $L(t)$  exists for  $t \in \mathbf{R}$  if  $\Pi(L)$  is linear. In this case  $P(t)$  and  $K(t)$  exist for  $t \leq t_f$ .

The next theorem gives sufficient conditions for the monotonicity of the solution  $K(t)$  of (2.2) and for the existence of positive semidefinite solutions of the corresponding algebraic Riccati equation

$$0 = -A^T K - KA - Q + KSK - \Pi(K). \quad (2.4)$$

**Theorem 2.4** *If there exists a solution  $K_0 \geq 0$  of the generalized algebraic Riccati inequality*

$$-A^T K_0 - K_0 A - Q + K_0 S K_0 - \Pi(K_0) \geq 0, \quad (2.5)$$

*then the solutions  $P(t)$  and  $K(t)$  of the initial value problems (2.3) and (2.2) with  $K_f := K_0$  exist for  $t \in (-\infty, t_f]$  with*

$$\dot{K}(t) \geq 0, \quad \dot{P}(t) \geq 0$$

*and*

$$0 \leq P(t) \leq K(t) \leq K_f$$

*for  $t \in (-\infty, t_f]$ ; in addition*

$$K^\infty := \lim_{t \rightarrow -\infty} K(t) \quad \text{and} \quad P^\infty := \lim_{t \rightarrow -\infty} P(t)$$

*exist.  $K^\infty$  is a solution of (2.4) and  $P^\infty$  is a solution of the decoupled algebraic Riccati equation*

$$0 = -A^T P - PA - Q + PSP. \quad (2.6)$$

As an immediate consequence of theorem 2.4 we get

**Corollary 2.5** *The following statements are equivalent:*

- (i) (2.4) has a solution  $K^\infty \geq 0$ .
- (ii) (2.5) has a positive semidefinite solution.

### 2.3 Boundedness and convergence of solutions

The following theorem gives a sufficient condition for the existence and convergence of the solutions of (2.2).

**Theorem 2.6** *Let (2.4) have a unique positive semidefinite solution  $K^+$ . If  $K^+ > 0$ , then for any  $K_f \geq 0$  the solution  $K$  of the initial value problem (2.2) exists for  $t \leq t_f$  and  $\lim_{t \rightarrow -\infty} K(t) = K^+$ .*

**Proof.** Let  $K^0$  and  $K^\alpha$  be the solutions of the differential equation (2.2) with  $K^0(t_f) = 0$  and  $K^\alpha(t_f) = \alpha K^+$ , ( $\alpha > 0$ ), respectively. Then  $K^0$  is decreasing (this means  $K^0$  is increasing as  $t$  decreases) with  $0 \leq K^0(t) \leq K^+$  (see Lemma 2.2), hence  $\lim_{t \rightarrow -\infty} K^0(t) = \tilde{K}$  exists.  $\tilde{K} = K^+$  since  $K^+$  is unique.

Next we choose  $\alpha \geq 1$  such that  $\alpha K^+ \geq K_f$ . We get

$$\begin{aligned} \dot{K}^\alpha(t_f) &= -\alpha A^T K^+ - \alpha K^+ A - Q + \alpha^2 K^+ S K^+ - \Pi(\alpha K^+) \\ &\quad -\alpha[A^T K^+ - K^+ A - Q + K^+ S K^+ - \Pi(K^+)] \\ &= (\alpha - 1)Q + \alpha(\alpha - 1)K^+ S K^+ + (\alpha \Pi(K^+) - \Pi(\alpha K^+)) \geq 0, \end{aligned}$$

here we used (C1) and (C3). Hence by theorem 2.4  $\dot{K}^\alpha(t) \geq 0$  for  $t \leq t_f$  and with the same argument as above we have  $\lim_{t \rightarrow -\infty} K^\alpha(t) = K^+$ . From  $K^0(t) \leq K(t) \leq K^\alpha(t)$  for  $t \leq t_f$  we infer that  $\lim_{t \rightarrow -\infty} K(t) = K^+$ .  $\square$

**Remark 2.7** (i) If (2.4) has at least one solution  $\tilde{K} > 0$ , then it follows as in the proof of theorem 2.6 that any solution of (2.2) is bounded on  $(-\infty, t_f]$  if  $K_f \geq 0$ . A numerical algorithm to compute  $K^+$  is given in [1].

(ii) Wonham [20], theorem 2.1 has obtained a sufficient condition for the existence of a unique positive definite solution of (2.4) for the case of linear perturbations  $\Pi$ ; in addition Wonham has given a sufficient condition for the existence of  $\lim_{t \rightarrow -\infty} K(t)$ . Notice that the assumptions of theorem 2.6 are weaker than the assumptions used in [20], theorem 2.1 (iv).

(iii) If (2.5) has a positive semidefinite solution then, according to theorem 2.4, the algebraic Riccati equations (2.4) and (2.6) have solutions  $\tilde{K}$  and  $P^+$  with  $\tilde{K} \geq P^+ \geq 0$ .

(iv) As a consequence of theorem 2.1 and of the preceding results the monotonic dependence of  $K$  with respect to the initial values remains true if  $S$  is only assumed to be symmetric. The same can be said about the *monotonicity part* of theorem 2.4.

### 3. Generalized difference Riccati equations

In this section we present the discrete-time versions of the results of section 2.

#### 3.1 A Comparison Theorem

For  $\nu \in \mathbf{N}$ , let  $A, \tilde{A}, A(\nu), \tilde{A}(\nu) \in \mathbf{R}^{n \times n}$ ;  $B, \tilde{B}, B(\nu), \tilde{B}(\nu) \in \mathbf{R}^{n \times m}$  and let  $Q, \tilde{Q}, Q(\nu), \tilde{Q}(\nu) \in \mathbf{R}^{n \times n}$  be positive semidefinite.

For  $K \in \mathbf{R}^{n \times n}$ , define  $\Phi$  by

$$\Phi(A, B, Q, K) = A^T K A - A^T K B (I + B^T K B)^{-1} B^T K A + Q.$$

The following comparison theorem is in essentially due to Wimmer and Pavon (see [17] and [19]).

**Theorem 3.1** (i) Let  $K$  and  $\tilde{K} \in \mathbf{R}^{n \times n}$  be symmetric with  $K \geq \tilde{K}$  and  $I + \tilde{B}^T \tilde{K} \tilde{B} > 0$ . If

$$\begin{pmatrix} Q & A^T \\ A & -BB^T \end{pmatrix} \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^T \\ \tilde{A} & -\tilde{B}\tilde{B}^T \end{pmatrix}$$

then

$$\Phi(A, B, Q, K) \geq \Phi(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{K}).$$

(ii) Let  $K_0 \geq \tilde{K}_0 \geq 0$  then the sequences  $(K(\nu))$  and  $(\tilde{K}(\nu))$  with

$$K(0) = K_0, \quad K(\nu + 1) = \Phi(A(\nu), B(\nu), Q(\nu), K(\nu))$$

and

$$\tilde{K}(0) = \tilde{K}_0, \quad \tilde{K}(\nu + 1) = \Phi(\tilde{A}(\nu), \tilde{B}(\nu), \tilde{Q}(\nu), \tilde{K}(\nu))$$

for  $\nu \geq 0$ , are well defined. If in addition

$$\begin{pmatrix} Q & A^T \\ A & -BB^T \end{pmatrix}(\nu) \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^T \\ \tilde{A} & -\tilde{B}\tilde{B}^T \end{pmatrix}(\nu) \quad \text{for } \nu \geq 0$$

then  $K(\nu) \geq \tilde{K}(\nu) \geq 0$  for  $\nu \geq 0$ .

**Proof.** (i) We have

$$\Phi(A, B, Q, K) \geq \Phi(\tilde{A}, \tilde{B}, \tilde{Q}, K) \geq \Phi(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{K});$$

the first inequality has been proved in [17], theorem 2.2, and the second inequality follows from [15] (see [3], lemma 10.1 and theorem 10.5 for further details).

(ii) corresponds to [19], Theorem 3.1; we add a short proof:

$$\begin{aligned} K(\nu + 1) &= \Phi(A(\nu), B(\nu), Q(\nu), K(\nu)) \\ &= (A(\nu) - B(\nu)F(\nu))^T K(\nu) (A(\nu) - B(\nu)F(\nu)) + F(\nu)F^T(\nu) + Q(\nu) \end{aligned}$$

with  $F(\nu) = (I + B^T(\nu)K(\nu)B(\nu))^{-1}B^T(\nu)K(\nu)A(\nu)$ .

Consequently  $K(\nu) \geq 0$  (resp.  $\tilde{K}(\nu) \geq 0$ ) implies  $K(\nu + 1) \geq 0$  (resp.  $\tilde{K}(\nu + 1) \geq 0$ ), and since  $K_0, \tilde{K}_0 \geq 0$  the sequences  $K(\nu)$  and  $\tilde{K}(\nu)$  are well defined.

Now using part (i) and  $K_0 \geq \tilde{K}_0$  we get by induction  $K(\nu) \geq \tilde{K}(\nu)$  for  $\nu \geq 0$ .  $\square$

Theorem 3.1 can be used to compare the solutions of difference equations of the type (1.2) with different perturbation terms  $\Pi(\nu, K(\nu))$ .

### 3.2 Monotonicity results

In the sequel we are studying (1.2) and the following assumptions are made for convenience:

- (D 1)  $A, B$  and  $Q$  are constant matrices with  $Q \geq 0$ .
- (D 2)  $\Pi(\nu, K) =: \Pi(K)$  is independent of  $\nu$  with  $\Pi(K) \geq 0$  for  $K \geq 0$ .
- (D 3) If  $0 \leq K \leq \tilde{K}$  then  $\Pi(K) \geq \Pi(\tilde{K})$   
and  $\alpha\Pi(K) - \Pi(\alpha K) \geq 0$  for  $K \geq 0$  and  $\alpha \geq 1$ .

We shall give sufficient conditions ensuring that the solutions of (1.2) and of the corresponding unperturbed difference equation are convergent to solutions of the algebraic equations

$$K = \Phi(A, B, Q, K) + \Pi(K) \quad (3.1)$$

and

$$P = \Phi(A, B, Q, P) \quad (3.2)$$

respectively.

- (H) There exists a matrix  $K_0 \geq 0$  with  $K(1) \leq K_0$ , where  $K(0) = K_0$  and  $K(\nu + 1) = \Phi(A, B, Q, K(\nu)) + \Pi(K(\nu))$  for  $\nu \geq 0$ .

**Theorem 3.2** *Let  $K_0$  satisfy hypothesis (H), then the limits*

$$K^\infty := \lim_{\nu \rightarrow \infty} K(\nu) \quad \text{and} \quad P^\infty := \lim_{\nu \rightarrow \infty} P(\nu)$$

*exist; here  $P(0) := K_0$  and  $P(\nu + 1) = \Phi(A, B, Q, P(\nu))$  for  $\nu \geq 0$ . Furthermore we have the following monotonicity properties*

$$0 \leq P^\infty \leq P(\nu + 1) \leq P(\nu) \leq K(\nu) \quad \text{for } \nu \geq 0$$

*and*

$$P^\infty \leq K^\infty \leq K(\nu + 1) \leq K(\nu) \quad \text{for } \nu \geq 0.$$

**Proof.** The monotonicity of the sequences  $(K(\nu))$  and  $(P(\nu))$  is obtained from theorem 3.1 by induction; furthermore it follows - on account of  $P(0) = K(0) = K_0 \geq 0$  and  $\Pi(K) \geq 0$  for  $K \geq 0$  - that  $0 \leq P(\nu) \leq K(\nu)$  for  $\nu \geq 0$ .

Hence the sequences  $(K(\nu))$  and  $(P(\nu))$  are monotonically decreasing, bounded and consequently convergent with limits  $K^\infty \geq P^\infty \geq 0$ .  $\square$

From Theorem 3.2 we infer (see also [2])

**Corollary 3.3** (i) (3.1) has a positive semidefinite solution  $K^\infty$  if and only if **(H)** is satisfied.

(ii) The sequence  $(K^0(\nu))$  defined by

$$K^0(0) = 0, \quad K^0(\nu + 1) = \Phi(A, B, Q, K(\nu)) \quad \text{for } \nu \geq 0$$

is nondecreasing; if this sequence is bounded  $\lim_{\nu \rightarrow \infty} K^0(\nu) =: K^{0,\infty} \geq 0$  exists and is a solution of the algebraic equation (3.1).

**Remark 3.4** Corollary 3.3 (i) gives a necessary and sufficient condition for the existence of a positive semidefinite solution of (3.1). Obviously this condition is satisfied if and only if the sequence  $(K^0(\nu))$  defined in corollary 3.3 (ii) is convergent; hence we propose to use this sequence to test **(H)** and to determine simultaneously  $K^{0,\infty}$  (if it exists); notice that  $K^{0,\infty}$  is the smallest positive semidefinite solution of (3.1).

### 3.3 Boundedness of Solutions and Convergence

If  $K(0) = K_0$  is not satisfying **(H)** we use the following theorem.

**Theorem 3.5.** If (3.1) has a solution  $K^+ > 0$  then the sequence  $(K(\nu))$  defined by

$$K(0) = K_0 \geq 0, \quad K(\nu + 1) = \Phi(A, B, Q, K(\nu)) \quad \text{for } \nu \geq 0$$

is bounded for any  $K_0 \geq 0$ .

If in addition  $K^+$  is the unique positive semidefinite solution of (3.1) then  $\lim_{\nu \rightarrow \infty} K(\nu) = K^+$ .

**Proof.** From theorem 3.1 we infer as before that  $K(\nu) \geq 0$  for  $\nu \geq 0$ . Now we choose  $\alpha \geq 1$  such that  $\alpha K^+ \geq K(0)$  and we consider the sequence defined by

$$K^\alpha(0) = \alpha K^+, \quad K^\alpha(\nu + 1) = \Phi(A, B, Q, K^\alpha(\nu)) \quad \text{for } \nu \geq 0.$$

Using  $K^\alpha(0) \geq K(0)$  we obtain from theorem 3.1 by induction that  $K^\alpha(\nu) \geq K(\nu) \geq 0$  for  $\nu \geq 0$ . Since  $K^+$  is a solution of (3.1) it follows from

$$\begin{aligned} K^\alpha(0) - K^\alpha(1) &= \alpha K^+ - K^\alpha(1) \\ &= \alpha K^+ - \Phi(A, B, Q, \alpha K^+) - \Pi(\alpha K^+) \\ &= \alpha[A^T K^+ A - A^T K^+ B \{I + B^T K^+ B\}^{-1} B^T K^+ A + Q + \Pi(K^+)] \\ &\quad - [A^T \alpha K^+ A - A^T \alpha K^+ B \{I + B^T \alpha K^+ B\}^{-1} B^T \alpha K^+ A + Q + \Pi(\alpha K^+)] \\ &\geq \alpha \Phi(A, B, Q, K^+) - \Phi(A, B, Q, \alpha K^+) \\ &= \alpha[\Phi(A, B, Q, K^+) - \Phi(A, \sqrt{\alpha} B, Q, K^+)] + (\alpha - 1)Q \geq 0. \end{aligned}$$

The first inequality results from (D 3), the second inequality follows from  $(\alpha - 1)Q \geq 0$  and with theorem 3.1, since

$$\begin{pmatrix} Q & A^T \\ A & -BB^T \end{pmatrix} \geq \begin{pmatrix} Q & A^T \\ A & -\alpha BB^T \end{pmatrix} \quad \text{for } \alpha \geq 1.$$

Hence the initial value  $\alpha K^+ = K^\alpha(0)$  satisfies **(H)** and therefore it follows from theorem 3.1 and theorem 3.2 that  $0 \leq K(\nu+1) \leq K^\alpha(\nu+1) \leq K^\alpha(\nu)$  for  $\nu \geq 0$ , and consequently  $(K(\nu))$  is bounded.

If  $K^+ > 0$  is the unique positive semidefinite solution of (3.1) then it follows from

$$0 \leq K^0(\nu) \leq K(\nu) \leq K^\alpha(\nu) \text{ for } \nu \geq 0$$

and  $\lim_{\nu \rightarrow \infty} K^0(\nu) = K^{0,\infty} = K^+ = \lim_{\nu \rightarrow \infty} K^\alpha(\nu)$  that  $\lim_{\nu \rightarrow \infty} K(\nu) = K^+$ . □

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