On the Dependence of the Solutions of Algebraic and Differential Game Riccati Equations on the Parameter μ *

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ABSTRACT: We are investigating the dependence of the phase portrait and in particular of the constant solutions of game (or H_∞-type) Riccati differential equations (RDE_μ) on the parameter $\mu \in \mathbf{R}$. Moreover we give sufficient conditions for the existence of the solution W_μ of (RDE_μ) with $W_\mu(t_f) = W_f$ for $t \leq t_f$ and also for the existence of at least one pole of W for $t \leq t_f$.

Keywords: game Riccati differential equations, existence of solutions, dependence on parameters, algebraic Riccati equations.

1. INTRODUCTION

We consider in this paper game Riccati differential equations of the type

$$\dot{W} = -A^T W - WA - Q + W(S_1 - \mu S_2)W = Ric(W, H_\mu) \qquad (RDE_\mu)$$

on the interval $(-\infty, t_f]$; here A, Q, S_1, S_2 are $n \times n$ matrices with real entries where $Q = C^T C, S_1 = B_1 B_1^T$ and $S_2 = B_2 {B_2}^T$, $\mu \in \mathbf{R}$ and H_μ is the corresponding hamiltonian matrix given by $H_\mu := \begin{pmatrix} A & -S_1 + \mu S_2 \\ -Q & -A^T \end{pmatrix}$. It is well known that Riccati differential equations of this form play a crucial role in H_∞-control theory.
In the sequel we use the abbreviation $S_\mu := S_1 - \mu S_2$.
In particular we are interested in the global behavior of all solutions of (RDE_μ) and in

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the dependence of the solutions on the parameter $\mu > 0$. In order to study this problem we will introduce the *dual* Riccati differential equation

$$\dot{K} = AK + KA^T - S_\mu + KQK \tag{$DRDE_\mu$}$$

and the associated algebraic Riccati equations

$$-A^TW - WA - Q + WS_\mu W = 0 \tag{ARE_μ}$$

and

$$AK + KA^T - S_\mu + KQK = 0 \qquad (DARE_\mu).$$

2. PRELIMINARIES

Let $J_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, where $I_n \in \mathbf{R}^{n\times n}$ is the unit matrix, then the hamiltonian H_μ^D corresponding to the dual Riccati equations is

$$H_\mu^D = J_0 H_\mu J_0 = \begin{pmatrix} -A^T & -Q \\ -S_\mu & A \end{pmatrix}. \tag{2.1}$$

Notice that $J_0^2 = I_n$, consequently $\sigma(H_\mu)$, the spectrum of H_μ, coincides with the spectrum of H_μ^D.
In the sequel we are going to use some well known facts on the interconnection between (RDE_μ) and $(DRDE_\mu)$; for the sake of clarity we recall the following results.

LEMMA 2.1 *Let* $P_0, X_0 \in \mathbf{R}^{n\times n}$.
(i) If $\mathrm{Im}\begin{pmatrix} P_0 \\ X_0 \end{pmatrix}$ *is a n-dimensional invariant subspace of* H_μ *then* $\mathrm{Im}\begin{pmatrix} X_0 \\ P_0 \end{pmatrix}$ *is a n-dimensional invariant subspace of* H_μ^D.
(ii) If P_0 *and* X_0 *are regular and if* $W_\mu(t)$ *denotes the solution of* (RDE_μ) *with* $W_\mu(t_f) = X_0P_0^{-1}$ *then* $W_\mu^{-1}(t)$ *defines the solution of* $(DRDE_\mu)$ *with* $W_\mu^{-1}(t_f) = P_0X_0^{-1}$ *while it exists; notice that (under these assumptions)* W_μ *is constant (and invertible) if and only if* $\mathrm{Im}\begin{pmatrix} P_0 \\ X_0 \end{pmatrix}$ *is an invariant subspace of* H_μ.

REMARK 2.2 Notice that assertion (i) of Lemma 2.1 is an immediate consequence of (2.1). The solution $W_\mu(t)$ defined in Lemma 2.1 (ii) is given as $W_\mu(t) = X(t)P(t)^{-1}$, where $\begin{pmatrix} P(t) \\ X(t) \end{pmatrix} = \exp(H_\mu t)\begin{pmatrix} P_0 \\ X_0 \end{pmatrix}$. Consequently assertion (ii) of the lemma follows from the identity

$$\exp(H_\mu^D t)\begin{pmatrix} X_0 \\ P_0 \end{pmatrix} = J_0 \exp(H_\mu t) J_0 \begin{pmatrix} X_0 \\ P_0 \end{pmatrix} = J_0 \begin{pmatrix} P(t) \\ X(t) \end{pmatrix} = \begin{pmatrix} X(t) \\ P(t) \end{pmatrix}.$$

Moreover, we infer from Lemma 2.1 that each regular solution of (ARE_μ) is inverse to a regular solution of $(DARE_\mu)$.

In the whole paper we assume

$$(A, B_1) \text{ is stabilizable}, (C, A) \text{ is detectable}. \tag{2.2}$$

By Σ_n we denote the set of all real symmetric $n \times n$ matrices; moreover we confine in this paper to the discussion of *real symmetric* solutions of Riccati equations.
Next we list some known results on H_∞-type algebraic Riccati equations (see [3], Corollary 5.2 and also [8], section V).

REMARK 2.3

(i) There exists a maximal $\mu_H > 0$ such that H_μ does not have purely imaginary eigenvalues for all $\mu \in (-\infty, \mu_H)$.

(ii) Under assumption (2.2) there are at most $k \leq n$ values $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \mu_H$ such that the stabilizing solution $W_{\mu s}$ of the algebraic Riccati equation (ARE_μ) exists for $\mu \in (-\infty, \mu_H) \setminus \{\mu_1, \ldots, \mu_k\}$.

(iii) The mapping $\mu \mapsto W_{\mu s}$ is monotonically increasing and the inertia of the matrices $W_{\mu s}$ (see [6]) are constant in each interval $(-\infty, \mu_1), (\mu_2, \mu_3), \cdots, (\mu_k, \mu_H)$. The monotonicity of $W_{\mu s}$ follows from the fact that the derivative $DW_{\mu s} = \frac{\partial}{\partial \mu} W_{\mu s}$ satisfies the Ljapunov equation

$$(A - S_\mu W_{\mu s})^T DW_{\mu s} + DW_{\mu s}(A - S_\mu W_{\mu s}) = -W_{\mu s} S_2 W_{\mu s},$$

since the closed loop matrix $A - S_\mu W_{\mu s}$ is stable.

(iv) When μ traverses some point of discontinuity $\mu_1, \mu_2, \ldots, \mu_k$, at least one positive eigenvalue of $W_{\mu s}$ escapes to $+\infty$ and reappears at $-\infty$, i.e. it becomes negative (see Example 4.1).

(v) From assumption (2.2) it follows that there exists a maximal value $\mu_P \in (0, \mu_H]$ such that the stabilizing solution $W_{\mu s}$ is the maximal solution of (ARE_μ) for $\mu \in (-\infty, \mu_P)$. Example 4.2 will show that $W_{\mu s}$ exists in this particular case for $\mu \in (-\infty, \mu_H)$; here $\mu_P < \mu_H$ (but for $\mu > \mu_P$ $W_{\mu s}$ is no longer the maximal solution of (ARE_μ)) .

(vi) For $\mu \in \mathbf{R}$ let $W_\mu(\cdot, W_f)$ denote the solution of (RDE_μ) with $W_\mu(t_f, W_f) = W_f$. For $\mu = 0$ it is well known that $W_0(t, W_f)$ exists in $(-\infty, t_f]$ at least for arbitrary terminal matrices $W_f \geq 0$, moreover, $\lim_{t \to -\infty} W_0(t, W_f) = W_{0s}$ where W_{0s} is the unique positive semidefinite solution of (ARE_0).These nice properties of the stabilizing solution W_{0s} remain no longer true for the stabilizing solution $W_{\mu s}$ of (ARE_μ) if S_μ is indefinite (see Theorem 3.3).

3. THE DEPENDENCE OF THE PHASE PORTRAIT ON μ

For sake of completeness we recall some important results on matrix Riccati equations. Let μ_M be maximal in the sense that S_μ is positive semidefinite for $\mu \leq \mu_M$.

REMARK 3.1

(i) As long as $(A, \sqrt{S_\mu})$ is stabilizable and (C, A) is detectable (which is fulfilled for $\mu < \mu_0 = \min\{\mu_M, \mu_1\}$) (ARE_μ) has a unique positive semidefinite solution, which is stabilizing (i.e. $\sigma(A - S_\mu W_{\mu s}) \subset \mathbf{C}^-$, the open left half-plane) and $\mathbf{RBA}(W_{\mu s}) = \{W_f \in \Sigma_n \mid W_\mu(t, W_f)$ exists for $t \leq t_f$ and $\lim_{t \to -\infty} W_\mu(t, W_f) = W_{\mu s}\}$, the *restricted basin of attraction of* $W_{\mu s}$ (as $t \to -\infty$), contains the cone of all positive semidefinite matrices (see [10], Theorem 2.1).

(ii) Let $\mu \in (-\infty, \mu_H) \setminus \{\mu_1, \ldots, \mu_k\}$, then it is known (see for example [1]) that

$$\mathbf{GBA}(W_{\mu s}) = \{W_f \in \Sigma_n \mid W_\mu(\cdot, W_f) \text{ has a finite number of poles in } (-\infty, t_f) \text{ and } \lim_{t \to -\infty} W_\mu(t, W_f) = W_{\mu s}\},$$

the *generalized basin of attraction of* $W_{\mu s}$ (as $t \to -\infty$), is open and dense in Σ_n. Further it follows from [3], [1] that

a) $\lim_{t \to -\infty} \|W_\mu(t, W_f)\| = \infty$ for $\mu \in \{\mu_1, \ldots, \mu_k\}$ and terminal values W_f from an open and dense subset of Σ_n. This means in the notation of [9], [1] that (RDE_μ) (more precisely: its extension $(ERDE_\mu)$ to a Grassmannian manifold) has a (dichotomic) solution at infinity.

b) For $\mu = \mu_H$ there is no dichotomic solution of (RDE_μ) (and $(ERDE_\mu)$), i.e. there is no solution having an open and dense generalized basin of attraction (as $t \to -\infty$).

(iii) The preceding remarks indicate the significance of the parameter values $\mu_1, \ldots, \mu_k$ and μ_H for the global phase portrait of (RDE_μ). Usually there are further values $\tilde{\mu}_1, \ldots, \tilde{\mu}_l < \mu_H$ where an eigenvalue of one of the other solutions of (ARE_μ) is passing ∞.
In Theorem 3.3 we will show that there is an essential change in the phase portrait when μ is passing μ_M.

In order to simplify the presentation of our results we replace in the sequel (2.2) by the stronger assumption

$$(A, B_1) \text{ is stabilizable, } (C, A) \text{ is observable.} \tag{3.2}$$

Notice that a standard coordinate transformation (like [3], (3.5)) can be used to transform (RDE_μ) to a system of coupled equations where the nontrivial (quadratic) part is a game Riccati equation satisfying (3.2).

REMARK 3.2 Under hypothesis (3.2) the following results are known from the literature (see the survey articles [5] and [4], where also weaker conditions than (3.2) are discussed):

(i) For $\mu \leq 0$ (ARE_μ) has a stabilizing solution $W_{\mu s}$ and, if in addition (A, B_1) is controllable, an antistabilizing solution $W_{\mu a}$ exists (i.e. $\sigma(A - S_\mu W_{\mu a}) \subset \mathbf{C}^+$, the open right half-plane) with

$$W_{\mu a} \leq W \leq W_{\mu s} \text{ for all solutions } W \text{ of } (ARE_\mu). \tag{3.3}$$

Moreover for $\mu \leq 0$ all (real symmetric) solutions of (ARE_μ) are regular, and $K_\mu \in \Sigma_n$ is a regular solution of $(DARE_\mu)$ if and only if K_μ^{-1} is a solution of (ARE_μ).
From Remark 2.3, (v), we infer that (3.3) cannot hold true for $\mu \geq \mu_P$.

(ii) The dual algebraic Riccati equation $(DARE_\mu)$ has for $\mu < \mu_H$ a stabilizing solution $K_{\mu s}$ (i.e. here $\sigma(-A - K_{\mu s}Q) \subset \mathbf{C}^-$) and an antistabilizing solution $K_{\mu a}$ with

$$K_{\mu a} \leq K_\mu \leq K_{\mu s} \text{ for all solutions } K_\mu \text{ of } (DARE_\mu). \tag{3.4}$$

For $\mu \in (-\infty, \mu_H)$ the functions $\mu \mapsto K_{\mu s}$ (resp. $\mu \mapsto K_{\mu a}$) is real analytic and monotonically decreasing (resp. increasing) and the generalized basin of attraction **GBA** $(K_{\mu s})$, of $K_{\mu s}$ as $t \to -\infty$, which is defined analogously to **GBA** $(W_{\mu s})$, is open and dense in Σ_n (see [1] and also Remark 3.6 (i)).

From (i) and Remark 2.3 we infer that $K_{\mu s}$ is regular for $\mu \in (-\infty, \mu_H)\backslash\{\mu_1, \ldots, \mu_k\}$ and det $K_{\mu s} = 0$ for $\mu \in \{\mu_1, \ldots, \mu_k\}$.
Notice that $(DARE_\mu)$ may have a maximal and/or minimal solution for $\mu \geq \mu_H$ but for $\sigma(H_\mu^D) \cap i\mathbf{R} \neq \emptyset$ there cannot exist a stabilizing or antistabilizing solution of $(DARE_\mu)$.

The following Theorem and its Corollary show that the cone of positive semidefinite matrices is no longer invariant under the flow induced by (RDE_μ) on Σ_n as t decreases if $\mu > \mu_M$. This fact is important for H_∞ control problems since it shows that the solutions of terminal value problems for H_∞-type Riccati equations (RDE_μ) may have finite escape-time for $\mu > \mu_M$ and specific terminal values $W(t_f) \geq 0$ even if the corresponding algebraic equation (ARE_μ) has a positive semidefinite solution. This means that in this situation it may happen that the solution of the corresponding finite horizon problem does not exist while in contrast the solution to the infinite horizon problem exists.

THEOREM 3.3 *Let* $\mu \in (\mu_M, \mu_0)$, *where* $\mu_0 = \min\{\mu_1, \mu_H\}$. *Then there exists a number* $R > 0$ *and an open and dense subset* $D \subset M_R := \{W_0 \in \Sigma_n \mid W_0 > 0 \text{ and } \|W_0^{-1}\| < \frac{1}{R}\}$ *such that* $W_\mu(\cdot, W_f)$ *has for any* $W_f \in D$ *at least two poles* $t_1 < t_2 < t_f$ *and* $W_\mu(t, W_f)$ *is indefinite for* $t \in (t_1, t_2)$.

Proof. Let $W_f \in D = M_R \bigcap \mathbf{GBA}\ (K_{\mu s})$, then the solution $K_\mu(\cdot, W_f^{-1})$ of $(DRDE_\mu)$ with $K_\mu(t_f, W_f^{-1}) = W_f^{-1}$ is holomorphic near t_f and Taylor approximation yields near t_f

$$K_\mu(t, W_f^{-1}) = W_f^{-1} + (t - t_f)\dot{K}_\mu(t_f, W_f^{-1}) + O(|t - t_f|^2)$$

$$= W_f^{-1} - (t - t_f)S_\mu + (t - t_f)[AW_f^{-1} + W_f^{-1}A^T + W_f^{-1}QW_f^{-1}] + O(|t - t_f|^2);$$

here we used the Landau symbol O.
Notice that S_μ is indefinite, hence $K_\mu(t, W_f^{-1})$ becomes indefinite on some interval

$I \subset (-\infty, t_f)$ provided $\|W_f^{-1}\|$ is sufficiently small. Since the stabilizing solution $K_{\mu s}(= W_{\mu s}^{-1}) > 0$ of $(DARE_\mu)$ exists and since its basin of attraction (as $t \to -\infty$) is open and dense in Σ_n it follows that $K_\mu(t, W_f^{-1}) > 0$ for $t < t_0$ and t_0 sufficiently small. Using $W_\mu^{-1}(t, W_f) = K_\mu(t, W_f^{-1})$ this obviously implies the assertion of Theorem 3.3. Notice that the positive definiteness of $K_{\mu s}$ (which results from the observability of (C, A)) is only needed for the proof of the existence of a second pole. □

From the proof of Theorem 3.3 we infer:

COROLLARY 3.4. *Let $\mu \in \mathbf{R}$ be such that S_μ is indefinite. Then we have without assuming (2.2) or (3.2):*
There exists a number $R > 0$ such that $W_\mu(\cdot, W_f)$ has for any $W_f \in M_R$ (as defined in Theorem 3.3) at least one pole $t_0 \in (-\infty, t_f)$.

In order to get a better understanding of the global dependence of the phase portraits of (RDE_μ) and $(DRDE_\mu)$ on μ we investigate in the next step how the constant solutions are depending on μ. For convenience we restrict ourselves to the study of *unmixed* solutions (see [4], p. 64) of $(DARE_\mu)$. Lemma 2.1 can be used to derive the corresponding results for (ARE_μ).
Following [5] we call a subset $\sigma_0(A_0)$ of the spectrum of $A_0 \in \Sigma_n$ a *c-set* if $\sigma_0(A_0)$ contains no purely imaginary eigenvalues and if $\lambda \in \sigma_0(A_0)$ implies $-\overline{\lambda} \notin \sigma_0(A_0)$. A c-set $\sigma_0(H_\mu^D)$ is called *maximal* if the dimension of the direct sum of spectral subspaces of H_μ^D associated with the eigenvalues in $\sigma_0(H_\mu^D)$ equals n.
Since (C, A) is observable, to each maximal c-set $\sigma_0(H_\mu^D)$ there corresponds a unique (unmixed) solution $K_\mu(\sigma_0) = PX^{-1}$, where span $\begin{pmatrix} X \\ P \end{pmatrix}$ is the n-dimensional spectral subspace of H_μ^D corresponding to $\sigma_0(H_\mu^D)$, and the spectrum of the closed loop matrix $-A - K_\mu(\sigma_0)Q$ coincides (counting multiplicity) with $\sigma_0(H_\mu^D)$.
Let $I \subset (-\infty, \mu_H)$ be an open interval such that $\sigma_0(H_\mu^D)$ is a maximal c-set for each $\mu \in I$, then $\mu \mapsto K_\mu(\sigma_0)$ is real analytic on I and differentiation of $(DARE_\mu)$ yields (for $K := K_\mu(\sigma_0)$) that $DK_\mu(\sigma_0) = \frac{\partial}{\partial \mu} K_\mu(\sigma_0)$ satisfies the Ljapunov equation

$$-(A + K_\mu(\sigma_0)Q)DK_\mu(\sigma_0) - DK_\mu(\sigma_0)(A + K_\mu(\sigma_0)Q)^T = S_2. \tag{3.5}$$

For $K \in \Sigma_n$ we denote by $\nu(K), \pi(K)$ and $\delta(K)$ the inertia of K, i.e. the number of eigenvalues having negative, positive and zero real part, respectively.
Summarizing we obtain:

THEOREM 3.5 *Let $I \subset (-\infty, \mu_H)$ be an open interval and let $\sigma_0(H_\mu^D), \mu \in I$, be maximal c-sets such that the mapping $K(\sigma_0) : I \to \Sigma_n, \mu \mapsto K_\mu(\sigma_0)$ is defined and real analytic. Then for $\mu \in I$*

$$\begin{aligned} \delta(DK_\mu(\sigma_0)) &= 0, \\ \pi(DK_\mu(\sigma_0)) &\leq \pi(-A - K_\mu(\sigma_0)Q), \\ \nu(DK_\mu(\sigma_0)) &\leq \nu(-A - K_\mu(\sigma_0)Q). \end{aligned}$$

If $DK_\mu(\sigma_0)$ is regular or if (A, B_2) is controllable then

$$\pi(DK_\mu(\sigma_0)) = \pi(-A - K_\mu(\sigma_0)Q) \text{ and } \nu(DK_\mu(\sigma_0)) = \nu(-A - K_\mu(\sigma_0)Q).$$

Proof. Since $DK_\mu(\sigma_0)$ is a solution of the Ljapunov equation (3.5) the assertions of the theorem are immediate consequences of [6], section 13.1, Theorems 3 and 4. $\square$

REMARK 3.6

(i) From Theorem 3.5 we easily recover the assertions of Remark 3.2 (ii) on the monotonicity of $K_{\mu s}$ and $K_{\mu a}$.
$K_{\mu s}$ is regular for $\mu \in (-\infty, \mu_H) \setminus \{\mu_1, \dots, \mu_k\}$ and when μ is crossing one of the points μ_j, $1 \le j \le k$, then at least one of the positive eigenvalues of $K_{\mu s}$ becomes negative - this is essentially the dual version of Remark 2.3 (iv) where $W_{\mu s} = K_{\mu s}^{-1}$ is considered. Obviously a similar behavior is shown by $K_{\mu a}$ and $W_{\mu a}$.

(ii) Let $\sigma_0(H_\mu^D), \mu \in I$, be a maximal c-set like in Theorem 3.5. If (A, B_2) is controllable then, according to Theorem 3.5, $K_{\mu s}$ and $K_{\mu a}$ are the only unmixed solutions of $(DARE_\mu)$ depending in I monotonically on μ. In general it follows from (3.5) that some of the eigenvalues of $K_\mu(\sigma_0)$ will be independent on μ and that the number of increasing or decreasing eigenvalues of $K_\mu(\sigma_0)$ (with respect to μ) will depend on the number of eigenvalues in $\sigma_0(H_\mu^D) \cap \mathbf{C}^-$, $\sigma_0(H_\mu^D) \cap \mathbf{C}^+$ and on S_2.
It is clear that any differentiable family K_μ, $\mu \in I$, of solutions of $(DARE_\mu)$ satisfies a Ljapunov equation (3.5) (with $K_\mu(\sigma_0)$ replaced by K_μ).
Hence the monotonicity behavior of this family is also essentially determined by the part of the spectrum (counting multiplicity) of H_μ^D which is associated with the n-dimensional H_μ^D - invariant subspace span $\begin{pmatrix} X_\mu \\ P_\mu \end{pmatrix}$ which determines $K_\mu = P_\mu X_\mu^{-1}$.

(iii) Taking into account the monotonic dependence of the solutions of terminal (or initial) value problems for Riccati differential equations on the terminal (or initial) values (see [2]) and using the results of [9] and [1] on the dimension of the (generalized) basins of attraction of the constant, periodic and almost periodic solutions of Riccati differential equations one gets a rather good information on the global phase portrait of (game) Riccati differential equations.
Of course it is very difficult to describe the most general situation.

(iv) In H_∞ control theory one is usually interested in the existence of $W_\mu(t, W_f)$ for some $W_f \ge 0$ on $(-\infty, t_f]$. Summarizing the preceding results we obtain:
As long as $\mu (=: \frac{1}{\gamma^2}) \le \mu_M$, $W_\mu(t, W_f)$ exists for $t \le t_f$ (according to [10], Theorem 2.1).
For $\mu_M < \mu < \mu_1 (\le \mu_H)$ it is known that
a) $W_\mu(t, W_f)$ exists for $t \le t_f$ if $0 \le W_f \le W_{\mu s}$ (see [7], Lemma 2),
b) $W_\mu(t, W_f)$ exists for $t \le t_f$ if W_f satisfies one of the conditions in [2], Corollary 5.5,
c) $W_\mu(\cdot, W_f)$ has under the conditions stated in Corollary 3.4 at least one pole in $(-\infty, t_f)$.

If the existence of $W_\mu(t, W_f)$ has to be checked for $\mu_M < \mu$ and different terminal values W_f we propose to use [2], Corollary 5.5; for convenience we recall one of the assertions of this corollary.

Notice that for $\mu < \mu_H$ the hamiltonian matrix H_μ has $2n$ eigenvalues $\pm\lambda_1, \cdots, \pm\lambda_n$, where $\lambda_1, \cdots, \lambda_n \in \mathbf{C}^-$. Moreover there exists a symplectic matrix

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{C}^{2n,2n}, \quad \text{where } a, b, c, d, \in \mathbf{C}^{n,n},$$

such that

$$V^{-1} H V = \text{diag}\,(\lambda_1, \cdots, \lambda_n, -\bar{\lambda}_1, \cdots, -\bar{\lambda}_n). \tag{3.6}$$

By D^- we denote the set of all matrices $W_f \in \mathbf{C}^{n,n}$ such that the matrix $Z_1 \in \mathbf{C}^{n,n}$ defined by

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := V^{-1} \begin{pmatrix} I_n \\ W_f \end{pmatrix}$$

is regular. Notice that D^- is open and dense in $\mathbf{C}^{n,n}$.

The following lemma is a consequence of [2], Corollary 5.5 and Remark 5.4; the proof of these results is rather involved and uses properties of symplectic transformations and meromorphic matrix functions (for details see [2]).

LEMMA 3.7 *With the preceding notations we have for* $\mu < \mu_H$*:*
If there exists a number $\rho \le 0$ *and a matrix* $W_f \in D^-$ *with* $S_\mu + \rho a a^* < 0$ *and*

$$\tilde{R}(\rho, W_f) := W_f(S_\mu + \rho a a^*) W_f^* - W_f(A + \rho a c^*) - (A^* + \rho c a^*) W_f^* - Q + \rho c c^* \ge 0,$$

then the solution $W_\mu(\cdot, W_f)$ *of* (RDE_μ) *with* $W_\mu(t_f, W_f) = W_f$ *has no pole on* $(-\infty, t_f]$ *and, if* $\mu \in (-\infty, \mu_H) \setminus \{\mu_1, \cdots, \mu_k\}$*, then* $\lim_{t\to-\infty} W_\mu(t, W_f) = W_{\mu s}$.

An application of Lemma 3.7 will be given in Example 4.3.

4. EXAMPLES

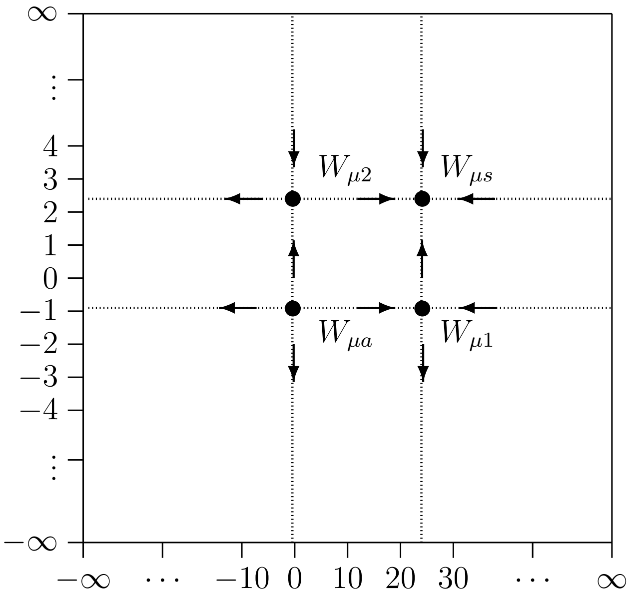
In this section we are reproducing three examples which have been constructed using the computer-algebra system MAPLE.

EXAMPLE 4.1 In our first example we consider for simplicity an algebraic Riccati equation (ARE_μ) and the corresponding (RDE_μ) where all coefficients are diagonal matrices:

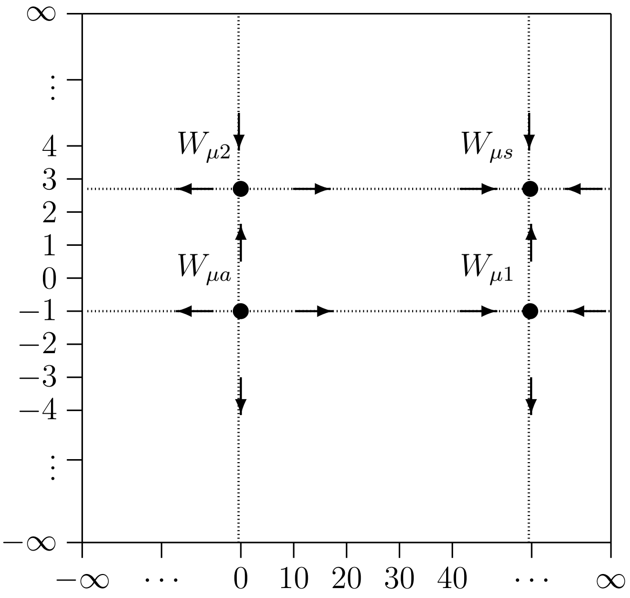
$$A := \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \; S_1 := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; S_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; Q := \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

In order to visualize the dependence of the solutions of (RDE_μ) corresponding to diagonal terminal values we present the following 8 figures, where 8 different values of μ are considered; arrows indicate the direction of the Riccati flow as t *decreases.*

$\mu = \frac{13}{20}$

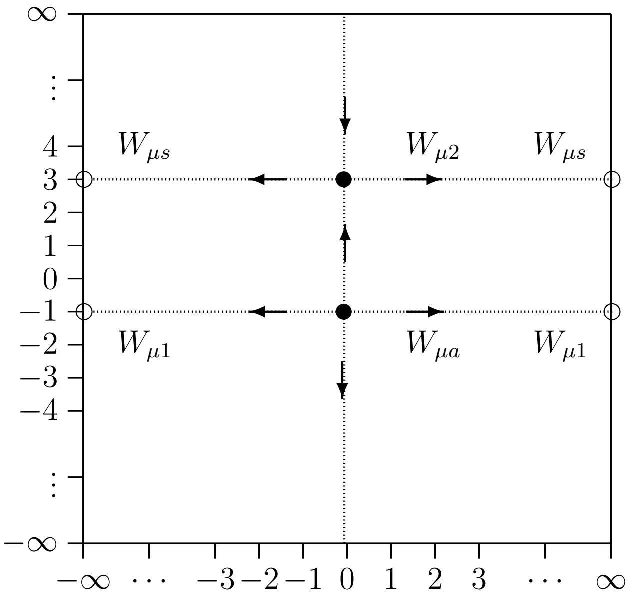


$\mu = \frac{17}{20}$

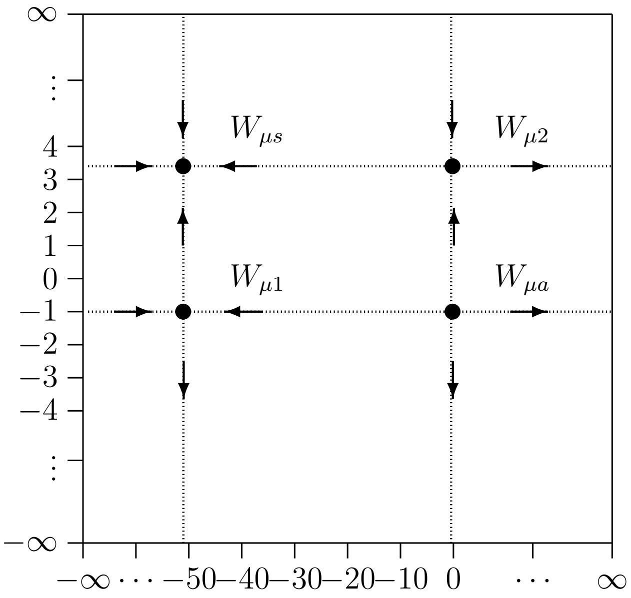


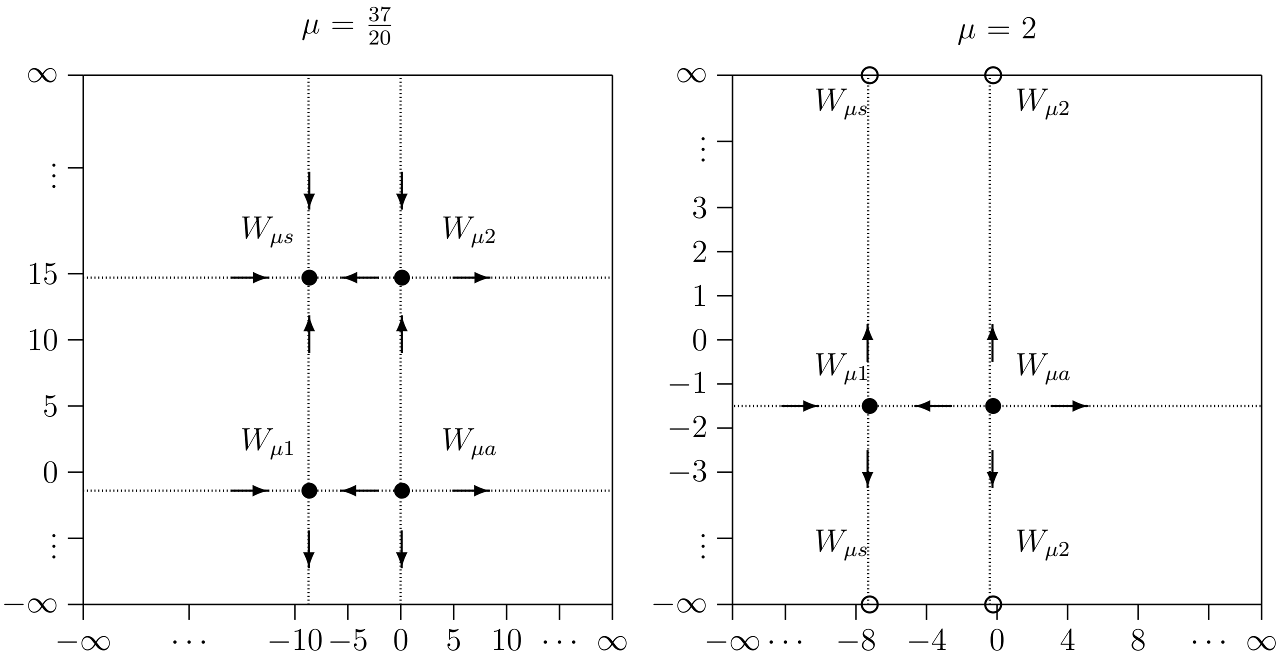
Notice that $W_{\mu s}$ is increasing and $W_{\mu a}$ is decreasing as μ is increasing.

$\mu = 1$

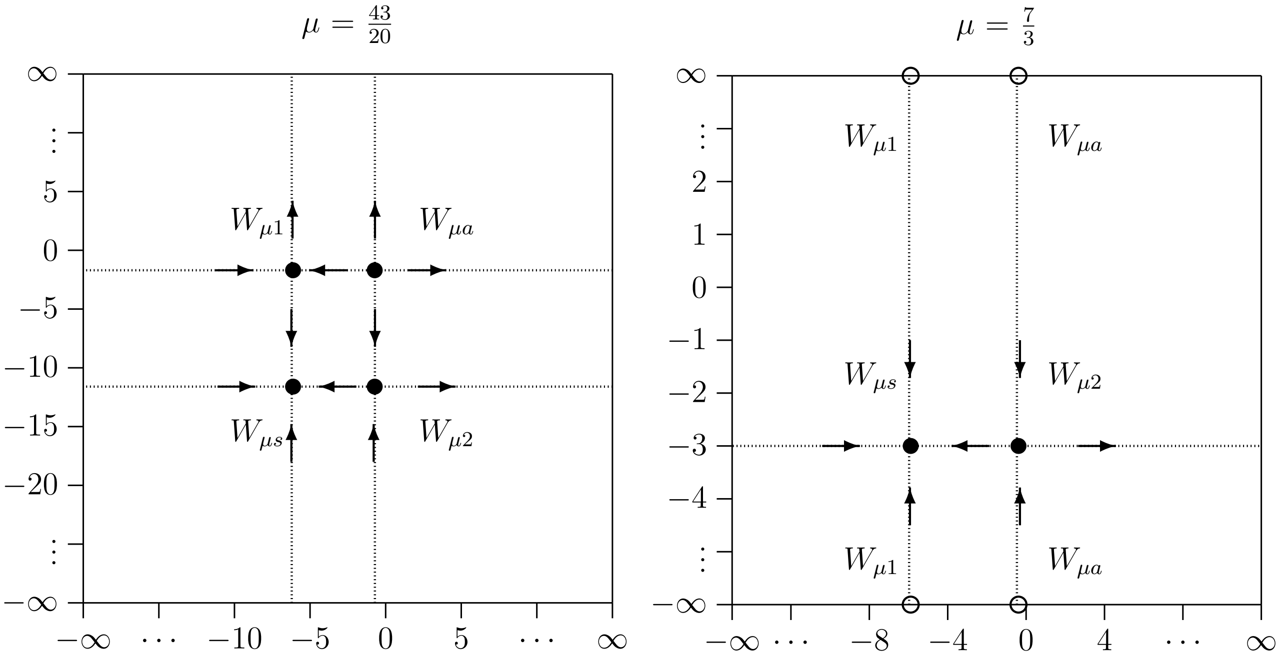


$\mu = \frac{23}{20}$





For $\mu = 1$ and $\mu = 2$ there are only two (finite, real symmetric) solutions of (ARE_μ) (indicated by $\bullet$) and two *unbounded real symmetric solutions* (indicated by $\circ$).



With the notations introduced in Remark 2.2 (i), (ii) we have in this example: $W_{\mu s} \geq 0$ for $\mu < \mu_1 = 1$, $W_{\mu s}$ is indefinite for $\mu_1 < \mu < \mu_2 = 2$ and $W_{\mu s} \leq 0$ for $\mu_2 < \mu < \mu_H = \frac{7}{3}$. Notice that H_{μ_H} has the double eigenvalue 0. For $\mu = \mu_H$ there are only two finite solutions and when μ increases and is passing $\mu = \mu_H$ then $W_{\mu s}$, $W_{\mu 2}$ respectively $W_{\mu a}$, $W_{\mu 1}$

remain or become finite but they are no longer real symmetric.

Notice that the finite and infinite solutions of (ARE_μ) are continuous with respect to the chordal metric on $(\mathbf{C} \bigcup \{\infty\})^4$.

In agreement with Remark 2.3 and Remark 3.6 both eigenvalues of $W_{\mu s}$ are monotonically increasing in $(-\infty, 1)$, $(1, 2)$ and $(2, \frac{7}{3})$ and the eigenvalues of $W_{\mu a}$ are monotonically decreasing on $(-\infty, \frac{7}{3})$. Moreover (in agreement with Theorem 3.5) in each case one eigenvalue of $W_{\mu 1}$ and $W_{\mu 2}$ is increasing and the second eigenvalue is decreasing.

In the following figures we present for completeness the eigenvalues of the solutions of (ARE_μ) as functions of μ, these figures have been computed using MAPLE.

The following sequence of values for μ has been used:

$$\begin{gathered}\mu - data : \{0.8476190476, 0.8952380952, 0.9428571429, 0.9754761905, 1.038095238,\\ 1.085714286, 1.133333333, 1.180952381, 1.228571429, 1.276190476, 1.323809524,\\ 1.371428571, 1.419047619, 1.466666667, 1.514285714, 1.561904762, 1.609523810,\\ 1.657142857, 1.704761905, 1.752380952, 1.800000000, 1.847619048, 1.895238095,\\ 1.916514691, 1.937791287, 1.959067882, 1.990344478, 2.011621074, 2.022897670,\\ 2.044174265, 2.065450861, 2.086727457, 2.108004053, 2.129280648, 2.150557244\}.\end{gathered}$$

EXAMPLE 4.2 Our second example shows that $W_{\mu a}$ can become infinite before $W_{\mu s}$ becomes infinite (i. e. when μ is increasing at least one of the eigenvalues is passing $-\infty$ and becomes positive afterwards).
We consider (ARE_μ) for

$$A = \begin{pmatrix} -3 & 1 \\ 0 & 2 \end{pmatrix}, \ S_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 4 \end{pmatrix}, \ S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}.$$

Since the solutions of (ARE_μ) are not necessarily diagonal we reproduce here for convenience the dependence of the eigenvalues of the solutions of the corresponding algebraic Riccati equation (ARE_μ) on μ.
For the subsequent figures we used the following sequence of values of μ:

$$\begin{aligned}\mu - data = \{&0.28, 0.33, 0.38, 0.43, 0.48, 0.53, 0.58, 0.63, 0.68, 0.73, 0.78, 0.83, 0.88,\\ &0.93, 0.98, 1.03, 1.08, 1.13, 1.18, 1.23, 1.28, 1.33, 1.38, 1.43, 1.48, 1.53, 1.58,\\ &1.63, 1.68, 1.73, 1.78, 1.83, 1.88, 1.93, 1.98, 2.03, 2.08, 2.13, 2.18, 2.23\}.\end{aligned}$$

Notice that the dependence of the eigenvalues is in accordance with Remark 2.3 and Remark 3.6. In the interval considered one of the eigenvalues of $W_{\mu 1}$ and of $W_{\mu a}$ is nearly constant and the second eigenvalue of these solutions of (ARE_μ) is passing ∞ as μ is passing (approximately) $\tilde{\mu} = 0.9$ and $\hat{\mu} = 0.55$, respectively.
For $\tilde{\mu} < \mu < \mu_H = 2.3871$ there are four finite, real symmetric solutions of (ARE_μ) which become complex when μ is passing μ_H. In this example we have $\mu_M = \frac{1}{2}$, this yields that, according to Corollary 3.4, $W_\mu(\cdot, W_f)$ has for $\frac{1}{2} < \mu < \mu_H$ at least one pole in $(-\infty, t_f)$ if $W_f \in M_R$ and if $R > 0$ is sufficiently large; notice that $W_{\mu s}$ is still the maximal solution of (ARE_μ) for $\frac{1}{2} < \mu < \mu_P$. This indicates the drastic change in the global phase portrait of (RDE_μ) when μ is passing μ_M.

EXAMPLE 4.3 Here we study the solutions of (RDE_μ) for fixed $\mu = \frac{25}{49}$ and different terminal values $W_\mu(t_f, W_f) = W_f$ using the coefficients

$$A = \begin{pmatrix} -5 & 1 \\ 0 & -3 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 4 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

The corresponding hamiltonian matrix is

$$H = \begin{pmatrix} -5 & 1 & \frac{1}{98} & 0 \\ 0 & -3 & 0 & -\frac{171}{49} \\ 0 & 0 & 5 & 0 \\ 0 & -2 & -1 & 3 \end{pmatrix}.$$

With

$$V := \begin{pmatrix} 1 & \frac{42\sqrt{87}}{221} + \frac{759}{442} & \frac{430}{10829} & -\frac{98}{221} + \frac{1771\sqrt{87}}{38454} \\ 0 & \frac{3\sqrt{87}}{14} + 3/2 & \frac{171}{442} & -1/2 + \frac{7\sqrt{87}}{174} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{196}{221} & \frac{7\sqrt{87}}{261} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$V^{-1}HV = \text{diag}\,(-5, -\frac{3\sqrt{87}}{7}, 5, \frac{3\sqrt{87}}{7}).$$

Therefore (ARE_μ) has the following four real symmetric solutions:

$$W_{\mu s} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{49}{57} + \frac{7\sqrt{87}}{57} \end{pmatrix}, \qquad W_{\mu 1} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{49}{57} - \frac{7\sqrt{87}}{57} \end{pmatrix}.$$

$$W_{\mu 2} = \begin{pmatrix} -2.888629731 & 2.881277189 \\ 2.881277189 & -2.588124882 \end{pmatrix}, \qquad W_{\mu a} = \begin{pmatrix} -303.9119926 & 33.77757643 \\ 33.77757643 & -5.759245120 \end{pmatrix}.$$

Notice that here the stabilizing solution $W_{\mu s} \geq 0$ is the maximal solution and the anti-stabilizing solution $W_{\mu a} < 0$ is the minimal solution; moreover it is known (see Remark 3.6, (vi), b)) that $W_\mu(t, W_f)$ exists for $t \leq t_f$ and $W_{\mu a} \leq W_f \leq W_{\mu s}$.
Next we consider $W_\mu(t, W_{fj})$ for $W_{fj} = W_{\mu s} + P_{fj}$ and 5 different values of $P_{fj} \geq 0 \quad 1 \leq j \leq 5$. For

$$P_{f1} = \begin{pmatrix} 1/20 & 0 \\ 0 & 1/10 \end{pmatrix} \text{ and } P_{f2} = \begin{pmatrix} 1/19 & \frac{1}{95} \\ \frac{1}{95} & 1/10 \end{pmatrix}$$

we have

$$W_{f1} = \begin{pmatrix} 1/20 & 0 \\ 0 & -\frac{433}{570} + \frac{7\sqrt{87}}{57} \end{pmatrix} \text{ and } W_{f2} = \begin{pmatrix} 1/19 & \frac{1}{95} \\ \frac{1}{95} & -\frac{433}{570} + \frac{7\sqrt{87}}{57} \end{pmatrix}.$$

This yields with $\rho = -\frac{19}{5}$ that

$$S_\mu - \frac{19}{5} aa^* = -\begin{pmatrix} 50.08977391 & 46.39766761 \\ 46.39766761 & 43.02627084 \end{pmatrix} < 0,$$

$$\tilde{R}(W_{f1}, \rho) = \begin{pmatrix} 0.3747755653 & -0.2819883386 \\ -0.2819883386 & 0.369226930 \end{pmatrix} \geq 0$$

and

$$\tilde{R}(W_{f2}, \rho) = \begin{pmatrix} 0.3313853413 & -0.2803022436 \\ -0.2803022436 & 0.244944882 \end{pmatrix} \geq 0.$$

Hence, according to Lemma 3.7, $W_\mu(t, W_{f1})$ and $W_\mu(t, W_{f2})$ exist for $t \leq t_f$.
Since the solutions of (RDE_μ) are monotonically dependent on the terminal value (see [2], Theorem 2.2) the existence of $W_\mu(t, W_{f1})$ and $W_\mu(t, W_{f2})$ implies that $W_\mu(t, W_f)$ exists for $t \leq t_f$ if

$$W_{\mu a} \leq W_f \leq W_{f1} \text{ or } W_{\mu a} \leq W_f \leq W_{f2}.$$

On the other hand it can be checked easily (for example using MAPLE) that the solutions $W(\cdot, W_{\mu s} + P_{fj}),\ \ 3 \leq j \leq 5$, with

$$P_{f3} = \begin{pmatrix} 2/19 & \frac{1}{95} \\ \frac{1}{95} & 1/10 \end{pmatrix},\ P_{f4} = \begin{pmatrix} 1/19 & 0 \\ 0 & 1/5 \end{pmatrix} \text{ and } P_{f5} = \begin{pmatrix} 1/50 & 0 \\ 0 & 3/10 \end{pmatrix}$$

have poles for $t - t_f = -0.1795267853$, $t - t_f = -0.1560596674$ and $t - t_f = -0.1070537472$ respectively.
Using again that the solutions of (RDE_μ) are monotonically dependent on the terminal value, we infer that $W_\mu(t, W_f)$ has at least one pole in $(-\infty, t_f]$ if $W_{fj} \leq W_f$ for at least one $j \in \{3, 4, 5\}$.
This example shows how Lemma 3.7 can be used to check the existence of $W_\mu(t, W_f)$ for different terminal values W_f; moreover one can use Lemma 3.7 to approximate the basin of attraction of $W_{\mu s}$.

5. CONCLUDING REMARKS

In this paper the existence of solutions to algebraic and differential game (or H_∞) Riccati equations is examined. In particular, the dependence of the solutions of such equations on the parameter $\mu \in \mathbf{R}$ is investigated. It is shown that several cases could occur (see Remark 3.6) and that for H_∞-type Riccati equations (RDE_μ) may have finite escape-time for $\mu > \mu_M$ and specific terminal values $W(t_f) \geq 0$ even if the corresponding algebraic equation (ARE_μ) has a positive semidefinite solution. Sufficient conditions for the existence of the solution W_μ of (RDE_μ) with $W_\mu(t_f) = W_f$ for $t \leq t_f$ and for the existence of at least one pole of W for $t \leq t_f$ are also given. Three examples are treated in details to illustrate the results obtained.

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