On Global Existence of Solutions to Coupled Matrix Riccati Equations in Closed Loop Nash Games *

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Abstract. We present comparison and global existence results for solutions of coupled matrix Riccati differential equations appearing in closed loop Nash games and in mixed H_2/H_∞-type problems. Convergence of solutions is established for the diagonal case. Solutions of the corresponding algebraic equations are discussed using numerical examples.

I. INTRODUCTION

Consider a two-player linear-quadratic differential game defined by

$$\dot{x} = Ax + B_1u_1 + B_2u_2 \;\; ; \quad x(0) = x^0, \;\; x \in \mathbf{R}^n, \;\; u_i \in \mathbf{R}^{r_i} \quad (i = 1, 2) \tag{1.1}$$

where the cost functionals associated with each player are

$$\begin{aligned} \mathcal{I}_1 &= \frac{1}{2}x_f^T K_{1f} x_f + \frac{1}{2}\int_0^{t_f} (x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2) dt; \\ \mathcal{I}_2 &= \frac{1}{2}x_f^T K_{2f} x_f + \frac{1}{2}\int_0^{t_f} (x^T Q_2 x + u_2^T R_{22} u_2 + u_1^T R_{21} u_1) dt \end{aligned} \tag{1.2}$$

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with $x_f := x(t_f)$.
All weighting matrices are assumed to be positive semidefinite with $R_{ii} > 0$ $(i = 1, 2)$. It is well known [15], section 3.1, and [1], Corollary A-1, that closed loop perfect state linear Nash equilibrium strategies have the form

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) x(t),$$
$$u_2^*(t) = -R_{22}^{-1} B_2^T K_2(t) x(t), \tag{1.3}$$

where $K_1(t)$ and $K_2(t)$ must satisfy the set of coupled Riccati type equations:

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2 + K_2 S_{22} K_1 - K_2 S_{12} K_2;$$
$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_{22} K_2 + K_2 S_{11} K_1 + K_1 S_{11} K_2 - K_1 S_{21} K_1; \tag{1.4}$$

with $K_1(t_f) = K_{1f}$, $K_2(t_f) = K_{2f}$ and where

$$S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \text{ for } 1 \leq i, j \leq 2.$$

As far as we know up to now there are no general results ensuring the existence of the solutions of (1.4) on $[0, t_f]$.
Lukes [8] established the uniqueness of a *linear* feedback Nash equilibrium (if it exists). Papavassilopoulos and Cruz [10], [11] obtained sufficient conditions for the (local) existence of the solutions of (1.4) on a finite interval and on the uniqueness of Nash strategies for a class of analytic differential games. In [12] Brower's fixed-point theorem is used to study the existence of positive semidefinite solutions of the algebraic Riccati equations corresponding to (1.4). Series solutions to (1.4) are proposed in [2] and [9]; this approach does not provide apriori information on the maximal interval of existence of the solutions of the coupled Riccati equations. For further information and references on coupled Riccati equations see [1].
The purpose of this paper is to derive sufficient conditions for the global existence of the solutions of (1.4) for $t \leq t_f$.
A first result on the global existence of solutions of closed loop Nash matrix Riccati equations has been established in [13] for the special case $B_1 = B_2$ and $R_{11} = R_{22} = -R_{12} = -R_{21} = I_n$, where I_n is the identity matrix; in this case $K_1 - K_2$ satisfies a linear differential equation and $K_1 + K_2$ satisfies a standard Riccati equation - hence K_1 and K_2 cannot blow up in finite intervals $[t_a, t_f]$.
In sections II and III of this paper comparison theorems are applied in order to prove existence results for the solution of (1.4) in a more general situation.
Recently new results in mixed H_2/H_∞ control were obtained in [7]. These results are based on closed loop Nash strategies and hence on the existence of solutions for coupled Riccati equations similar to (1.4); in section IV we derive sufficient conditions for the global existence of the solutions of of the corresponding initial value problem. Convergence results as $t \to -\infty$ are not known for the general case. In section V we study convergence in the case of diagonal matrices. In section VI solutions of coupled algebraic equations are discussed and two numerical examples are given.
After this paper was finished we became aware of reference [16] where in particular the phase portrait of (1.4) is discussed for $n = 1$.

II. EXISTENCE AND COMPARISON THEOREMS FOR POSITIVE SEMIDEFINITE SOLUTIONS.

In this section we derive upper and lower bounds for the solutions of (1.4) ensuring the global existence of the solutions for $t \le t_f$.

Lemma 2.1: Let K_1 and K_2 be solutions of (1.4) on the interval $[t_0, t_f]$. Then $K_1(t) \ge 0$ and $K_2(t) \ge 0$ for $t_0 \le t \le t_f$.

Proof. Let $x_1 \in \mathbf{R}^n \setminus \{0\}$ and let x be the solution of the initial value problem

$$\dot{x} = [A - S_{11}K_1(t) - S_{22}K_2(t)]x, \quad x(t_0) = x_1. \tag{2.1}$$

From (1.4) and (2.1) we infer for $i = 1, 2$

$$\frac{d}{dt}[x^T K_i x] = \dot{x}^T K_i x + x^T \dot{K}_i x + x^T K_i \dot{x}$$

$$= x^T\{(A^T K_i - K_1 S_{11} K_i - K_2 S_{22} K_i) + (K_i A - K_i S_{11} K_1 - K_i S_{22} K_2) +$$
$$-A^T K_i - K_i A - Q_i + K_i S_i K_i - \sum_{j \ne i} K_j S_{ij} K_j + \sum_{j \ne i} (K_i S_j K_j + K_j S_j K_i)\}x$$
$$= -x^T\{Q_i + K_1 S_{i1} K_1 + K_2 S_{i2} K_2\}x.$$

Hence integration from t_0 to t_f yields

$$x_1{}^T K_i(t_0) x_1 = \int_{t_0}^{t_f} x^T(\tau)\tilde{Q}_i(\tau)x(\tau)d\tau + x^T(t_f)K_{if}x(t_f) \ge 0,$$

where $\tilde{Q}_i := Q_i + K_1 S_{i1} K_1 + K_2 S_{i2} K_2$.
Since x_1 was arbitrary and $Q_i,\ S_{i1},\ S_{i2},\ K_{if} \ge 0$, this implies the assertion of the lemma. ■

Remark 2.2: Notice that the assertion of Lemma 2.1 and the subsequent results of this section remain true if we replace the assumption $S_{12},\ S_{21} \ge 0$ by the implicit condition $\tilde{Q}_1(t),\ \tilde{Q}_2(t) \ge 0$ for $t \in [t_0, t_f]$ (see also condition (6) in [12]).

In the next two theorems we are giving sufficient conditions ensuring that K_1 and K_2 do not have finite escape time; the main key for obtaining these results are comparison theorems for Riccati type differential equations (see Theorem 7.1).

Theorem 2.3: Let $Q \in \mathbf{R}^{n,n}$ be symmetric. Then the solutions K_1 and K_2 of (1.4) exist for $t < t_f$ with

$$0 \le K_1(t) + K_2(t) \le L_Q(t), \tag{2.2}$$

while

$$R_1(K_1(t), K_2(t), Q) \ge 0. \tag{2.3}$$

Here $R_1(K_1, K_2, Q) :=$

$$Q + (K_1 + K_2)(S_{11} + S_{22})(K_1 + K_2) - K_1 S_{21} K_1 - K_2 S_{12} K_2 - (K_1 S_{22} K_1 + K_2 S_{11} K_2)$$

and L_Q is the (unique) solution of the linear terminal value problem

$$\dot{L}_Q(t) = -L_Q(t)A - A^T L_Q(t) - (Q_1 + Q_2 + Q), \;\; L_Q(t_f) = K_{1f} + K_{2f}.$$

Proof. The first inequality in (2.2) results from Lemma 2.1. Since $L_Q(t)$ exists for $t < t_f$, we infer from Theorem 7.1 and

$$(\dot{K_1} + \dot{K_2}) = -A^T(K_1 + K_2) - (K_1 + K_2)A - (Q_1 + Q_2 + Q) + R_1(K_1, K_2, Q)$$

that the second inequality in (2.2) holds while $R_1(K_1(t), K_2(t), Q) \geq 0$. ■

Remark 2.4: (i) The symmetric matrix appearing in Theorem 2.3 and subsequently is used as a parameter and can be chosen arbitrarily large.
(2.3) is an implicit condition since we are using the (unknown) solutions K_1 and K_2 which might be unbounded on finite intervals.
(ii) Notice that condition (2.3) is trivially fulfilled for

$$S_{11} \geq S_{21}, \; S_{22} \geq S_{12} \text{ and } Q \geq 0$$

if either $n = 1$ or if all matrices $A, \; K_{if}, \; S_{ij}, \; Q_i, \; 1 \leq i, j \leq 2$, are diagonal matrices.

Hence in this case $K_1(t)$ and $K_2(t)$ exist for $t \in (-\infty, t_f]$ and we say that the solutions of (1.4) exist globally. In section V of this paper we give sufficient conditions for the convergence of these solutions as $t \to -\infty$.

In the next theorem we use an H_∞-type comparison equation for the formulation of a global existence result.

Theorem 2.5: Let $S_{12} = S_{21} = 0$, let $Q \in \mathbf{R}^{n,n}$ be symmetric and assume that the solution P of the H_∞-type Riccati equation

$$\dot{P_0} = -A^T P_0 - P_0 A - (Q_1 + Q) + P_0(S_{11} - S_{22})P_0, \;\; P_0(t_f) = K_{1f} \tag{2.4}$$

exists on $[t_0, t_f]$. Then the solutions K_1 and K_2 of (1.4) exist on $[t_0, t_f]$ while

$$R_2(K_1(t), K_2(t), Q) := [Q + (K_1 + K_2)S_{22}(K_1 + K_2) - K_2 S_{22} K_2](t) \geq 0. \tag{2.5}$$

Moreover for these $t \in [t_0, t_f]$

$$0 \leq K_1(t) \leq P_0(t) \text{ and } 0 \leq K_2(t) \leq L(t),$$

where L is the solution of the linear differential equation

$$\dot{L} = -\tilde{A}(t)^T L - L\tilde{A}(t) - Q_2, \;\; L(t_f) = K_{2f}, \tag{2.6}$$

with

$$\tilde{A} := A - S_{11}K_1(t) - \frac{1}{2}S_{22}K_2(t).$$

Proof. $K_1(t) \geq 0$ follows from Lemma 2.1 and $K_1(t) \leq P_0(t)$ is a consequence of assumption (2.5),

$$\dot{K}_1 = -A^T K_1 - K_1 A - (Q_1 + Q) + K_1(S_{11} - S_{22})K_1 + R_2(K_1, K_2, Q)$$

and Theorem 7.1. Hence $K_1(t)$ cannot blow up on $[t_0, t_f]$ at least as long as (2.5) is fulfilled.
Since K_2 is on this interval a solution of a standard Riccati differential equation with variable coefficients (depending on $K_1(t)$), it follows from Lemma 2.1 and [17], Theorem 2.1, (iii), that $K_2(t)$ cannot blow up and is bounded from above by the solution of the linear differential equation (2.6). ∎

Remark 2.6: Similarly to Remark 2.4 it follows that the implicit condition (2.6) is automatically fulfilled in several special cases; moreover this condition can be interpreted as a geometric condition on the curves $t \mapsto K_i(t),\ t \leq t_f,\ \ 1 \leq i \leq 2$.
Since there are well known conditions ensuring the global existence of the solution P_0 of (2.4) for $t \leq t_f$ and since the solution L, appearing in Theorem 2.5, exists for $t \leq t_f$ (as long as $K_1(t)$ exists) we have obtained sufficient conditions for the global existence of $K_1(t)$ and $K_2(t)$ for $t \leq t_f$.

III. EXISTENCE OF SOLUTIONS FOR A MODIFIED EQUATION

In this section we are applying again the method of section II in order to obtain global existence results for solutions of the initial value problem (1.4) but here we use a modified assumption concerning S_{12}, S_{21}.
Throughout this section we assume for the coefficient matrices in (1.4):

$$S_{11}, S_{22}, K_{1f}, K_{2f} \geq 0 \text{ and } S_{12}, S_{21} \leq 0. \tag{3.1}$$

Lemma 3.1: Let $S_{11} + S_{12} \geq 0$ and $S_{22} + S_{21} \geq 0$, then the solutions of (1.4) satisfy $0 \leq K_1(t) + K_2(t)$ for $t \leq t_f$ while $K_1(t)$ and $K_2(t)$ exist.

Proof. From (3.1) we infer that $K_1(t_f) + K_2(t_f) \geq 0$. Moreover it follows from the assumptions of the lemma that $K_1 + K_2$ satisfies the following Ljapunov inequality:

$$(\dot{K}_1 + \dot{K}_2) = -A^T(K_1 + K_2) - (K_1 + K_2)A$$

$$-(Q_1+Q_2)+(K_1+K_2)(S_{11}+S_{22})(K_1+K_2)-K_1S_{21}K_1-K_2S_{12}K_2-(K_1S_{22}K_1+K_2S_{11}K_2)$$

$$\leq -\tilde{A}^T(K_1 + K_2) - (K_1 + K_2)\tilde{A},$$

where $\tilde{A} = A - \frac{1}{2}(S_{11} + S_{22})(K_1 + K_2)$.
Consequently the assertion of the lemma follows from standard arguments on Ljapunov equations, see for instance [6], Hilfsatz 10.3. ∎

The previous lemma yields a lower bound for $K_1 + K_2$, an upper bound for the solutions will now be obtained from the following modified version of theorem 2.5.

Theorem 3.2: Let $Q \in \mathbf{R}^{n,n}$ be symmetric and assume that the solution P_0 of the H_∞-type Riccati equation

$$\dot{P}_0 = -A^T P_0 - P_0 A - (Q_1 + Q) + P_0(S_{11} - S_{22})P_0, \quad P_0(t_f) = K_{1f} \tag{3.2}$$

exists on $[t_0, t_f]$. Then the solutions K_1 and K_2 of (1.4) exist on $[t_0, t_f]$ while

$$R_2(K_1(t), K_2(t), Q) := [Q + (K_1 + K_2)S_{22}(K_1 + K_2) - K_2 S_{22} K_2](t) \geq 0. \tag{3.3}$$

Moreover for these $t \in [t_0, t_f]$

$$-K_2(t) \leq K_1(t) \leq P_0(t) \text{ and } -K_1(t) \leq K_2(t) \leq L(t), \tag{3.4}$$

where L is the solution of the linear differential equation (2.6).

Proof. According to Lemma 3.1 we have $0 \leq K_1(t) + K_2(t)$ for $t \leq t_f$ while $K_1(t)$ and $K_2(t)$ exist, this yields the left inequalities in (3.4). Since $S_{12}, S_{21} \leq 0$ it follows (from the comparison theorem 7.1) that the solutions of the initial value problem (1.4) are bounded from above by the solutions of the initial value problem obtained from (1.4) by replacing therein S_{12} and S_{21} by 0. Hence the remaining assertions of the theorem follow from Theorem 2.5. ■

IV. EXISTENCE OF SOLUTIONS FOR MIXED H_2/H_∞-TYPE PROBLEMS

Recently in [7] a mixed H_2/H_∞ control problem was formulated as a Nash game leading to the following terminal value problem for coupled Riccati equations:

$$\begin{aligned} \dot{P}_1 &= -A^T P_1 - P_1 A + Q_1 + \frac{1}{\gamma^2} P_1 S_1 P_1 + P_1 S_2 P_2 + P_2 S_2 P_1 + P_2 S_2 P_2, \\ \dot{P}_2 &= -A^T P_2 - P_2 A - Q_1 + P_2 S_2 P_2 + \frac{1}{\gamma^2}[P_1 S_1 P_2 + P_2 S_1 P_1], \end{aligned} \tag{4.1}$$

with $P_1(t_f) = P_2(t_f) = 0$, $Q_1 \geq 0$ and $S_i = B_i B_i^T$, $(i = 1, 2)$.
It is shown in [7] that $P_1(t) \leq 0 \leq P_2(t)$ and $P_1(t) + P_2(t) \geq 0$ for $t \in [0, t_f]$ as long as $P_2(t)$ and $P_1(t)$ exist. Hence it is sufficient for the application of the results obtained in [7] to have a condition ensuring the existence of $P_2(t)$ on $[0, t_f]$.
Let $Q = Q^T \in \mathbf{R}^{n,n}$ be a given matrix; then we compare P_2 with the solutions of the following three initial value problems:

$$\dot{L} = -A^T L - LA - (Q + Q_1), \quad L(t_f) = 0, \tag{4.2}$$

$$\dot{P} = -A^T P - PA - (Q + Q_1) + P(S_2 - \frac{1}{\gamma^2} S_1)P, \quad P(t_f) = 0, \tag{4.3}$$

$$\dot{M} = -A^T M - MA - (Q + Q_1) + M(S_2 - \frac{2}{\gamma^2} S_1)M, \quad M(t_f) = 0. \tag{4.4}$$

Theorem 4.1: For $t < t_f$ the following holds:
(i) While

$$Q + P_2(t) S_2 P_2(t) + \frac{1}{\gamma^2}[P_1(t) S_1 P_2(t) + P_2(t) S_1 P_1(t)] \geq 0 \tag{4.5}$$

$$0 \leq P_2(t) \leq L(t).$$

(ii) While $P(t)$ exists and while

$$Q + \frac{1}{\gamma^2}[(P_1(t) + P_2(t)) S_1 (P_1(t) + P_2(t)) - P_1(t) S_1 P_1(t)] \geq 0 \tag{4.6}$$

$$0 \leq P_2(t) \leq P(t).$$

(iii) While $M(t)$ exists and while

$$Q + \frac{1}{\gamma^2}[(P_1(t) + P_2(t)) S_1 (P_1(t) + P_2(t)) + P_2(t) S_1 P_2(t) - P_1(t) S_1 P_1(t)] \geq 0 \tag{4.7}$$

$$0 \leq P_2(t) \leq M(t).$$

Proof. As in the proof of Theorem 2.5 the upper estimates in (i), (ii) and (iii) are direct consequences of Theorem 7.1; $P_2(t) \geq 0$ follows from [7], Theorem 2.1. ■

Remark 4.2: (i) Notice that $P(t)$ or $M(t)$ exists for $t \leq t_f$ if the algebraic Riccati equation corresponding to (4.3) or (4.4), respectively, has a positive semidefinite solution; this could be checked easily.
(ii) The conditions (4.5), (4.6) and (4.7) are implicit conditions since they are containing the (unknown) solutions P_1 and P_2 but, as we have already indicated in Remark 2.4, there are situations where these conditions are automatically fulfilled for $Q \geq 0$: for example in the case $n = 1$ (4.7) is always fulfilled and (4.5) is fulfilled at least for $S_2 \geq \frac{2}{\gamma^2} S_1$.

V. CONVERGENCE OF SOLUTIONS

In sections II and III we have derived sufficient conditions for the global existence of the solutions of (1.4) for $t \leq t_f$. Numerical examples indicate that these solutions are convergent as $t \to -\infty$ to a solution of the corresponding algebraic Riccati equation.
For the general case as studied in sections II and III there are no convergence results known; here we obtain conditions for convergence in the diagonal case.

Theorem 5.1: Assume that $S_{12} = S_{21} = 0$, that A is diagonal and that all matrices $Q_1,\ Q_2,\ S_{11},\ S_{22},\ K_{1f},\ K_{2f}$ are diagonal and positive semidefinite with $Q_1,\ Q_2 > 0$. Then the solutions K_i of (1.4) exist on $(-\infty, t_f]$ with

$$0 \leq K_i(t) \leq P_i(t), \quad t \leq t_f,\ i = 1, 2, \tag{5.1}$$

where P_1, P_2 are the solutions of the open loop Nash Riccati equations

$$\dot{P}_1 = -A^T P_1 - P_1 A - Q_1 + 2P_1 S_{11} P_1 + 2P_1 S_{22} P_2, \; P_1(t_f) = K_1(t_f),$$

$$\dot{P}_2 = -A^T P_2 - P_2 A - Q_2 + 2P_2 S_{22} P_2 + 2P_2 S_{11} P_1, \;\; P_2(t_f) = K_2(t_f).$$

Moreover, if K_1 and K_2 are bounded on $(-\infty, t_f]$, then the limits

$$\lim_{t \to -\infty} K_i(t) =: K_i^\infty \geq 0, \quad i = 1, 2,$$

exist and define solutions of the algebraic Riccati equation corresponding to (1.4).

Proof. Since all matrices considered are diagonal it is sufficient to give the proof for the one dimensional case; hence we assume in the sequel that $n = 1$.
The estimate (5.1) and the existence of K_1 and K_2 follow from Theorem 7.1 and Lemma 7.2 since for $n = 1$ (1.4) can be written as

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + 2K_1 S_{11} K_1 + 2K_1 S_{22} K_2 - K_1 S_{11} K_1;$$

$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + 2K_2 S_{22} K_2 + 2K_2 S_{11} K_1 - K_2 S_{22} K_2.$$

Notice that K_1 and K_2 are monotonic and hence convergent (if they are bounded) as $t \to -\infty$ if $\pm\dot{K}_1(t) > 0$ and if $\pm\dot{K}_2(t) > 0$ (i. e. $\dot{K}_1$ and $\dot{K}_2$ are both positive or both negative definite) for $t \leq t_f$. Moreover $\dot{K}_1(t)$ and $\dot{K}_2(t)$ vanish simultaneously if and only if $K_{1f},\ K_{2f}$ are solutions of the algebraic Riccati equation.
Next we assume that $\dot{K}_1(t_f) \leq 0$ and $\dot{K}_2(t_f) \leq 0$ and that there exists $t_0 \leq t_f$ such that t_0 is the maximal $t \leq t_f$ where one of the functions $\dot{K}_1$ or $\dot{K}_2$ is changing its sign; without loss of generality let $\dot{K}_1(t_0) = 0 > \dot{K}_2(t_0)$ (all other cases can be treated similarly). Notice that in this situation it follows that $S_{22} > 0$ since otherwise the first equation in (1.4) would be a standard Riccati differential equation and in this case $\pm\dot{K}_1(t_f) \geq 0$ would imply $\pm\dot{K}_1(t) \geq 0$ for $t \leq t_f$; moreover $K_1(t_0) > 0$ since otherwise $\dot{K}_1(t_0) = -Q_1 < 0$.
In a neighborhood of t_0 we have $\dot{K}_2(t) < 0$ and Taylor approximation yields

$$\dot{K}_1(t) = [-A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2 + K_2 S_{22} K_1](t)$$

$$= -A^T K_1(t) - K_1(t) A - Q_1 + K_1(t) S_{11} K_1(t) + K_1(t_0) S_{22} K_2(t_0) + K_2(t_0) S_{22} K_1(t_0)$$

$$+[K_1(t) S_{22} K_2(t) + K_2(t) S_{22} K_1(t) - K_1(t_0) S_{22} K_2(t_0) - K_2(t_0) S_{22} K_1(t_0)]$$

$$= 2K_1(t_0) S_{22} \dot{K}_2(t_0)(t - t_0) + O((t - t_0)^2).$$

Notice that from this identity and from $S_{22} > 0$, $\dot{K}_2(t_0) < 0$ and $K_1(t_0) > 0$ it follows that $2K_1(t_0) S_{22} \dot{K}_2(t_0)(t - t_0) > 0$ for $t < t_0$; hence $\dot{K}_1(t) > 0$ for $t \in [t_0 - \epsilon, t_0)$ for some $\epsilon > 0$.
Moreover, in the same way it follows that $\dot{K}_1(t)$ and $\dot{K}_2(t)$ cannot change their sign on $(-\infty, t_0]$ if $\pm\dot{K}_1(t_0) \geq 0$ and $\pm\dot{K}_2(t_0) \leq 0$.
Consequently at most one of the functions $\dot{K}_1$ and $\dot{K}_2$ can change its sign on $(-\infty, t_f]$ (and this can happen at most once). This implies that there is $t_1 \leq t_f$ such that K_1 and K_2 are both monotonic on $(-\infty, t_1]$ and, if they are bounded, also convergent as $t \to -\infty$.■

Remark 5.2: (i) If all matrices A, Q_1, S_1, S_2, appearing in (4.1), are diagonal with $Q_1 > 0$ then the solutions $P_1,\ P_2$ of (4.1) are monotonic for $t \leq t_f$ (this follows as with the proof of Theorem 5.1). If the solutions are bounded they are also convergent with

$$\lim_{t \to -\infty} P_1(t) =: P_1^\infty \leq 0 \leq P_2^\infty := \lim_{t \to -\infty} P_2(t).$$

Using the fact that $P_1(t) + P_2(t) \geq 0$ (see [7]) this can be proved like Theorem 5.1; we omitt details.

Notice that in this case and also in the case considered in Theorem 5.1 there are no non-trivial periodic solutions. Moreover we cannot apply here Lemma 7.2 in order to get an estimate of the form (5.1).

(ii) The preceding investigations indicate that the theory of coupled matrix Riccati equations of the form (1.4) or (4.1) is (in particular for $n > 1$) much more involved than the corresponding theory of uncoupled Riccati equations since they cannot be transformed to an equivalent linear system.

Of course it is possible to use for the investigation of (1.4) and (4.1) standard methods for nonlinear (polynomial) dynamical systems, but in order to get a clearer picture of the corresponding dynamics one has to take into account the special structure of these systems; a first step into this direction has been made in [16].

(iii) The set Γ of all solutions of the algebraic Riccati equation corresponding to (1.4) can be interpreted geometrically as the intersection of the sets

$$\{(K_1, K_2) \in \mathbf{C}^{n,2n} | 0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2 + K_2 S_{22} K_1 - K_2 S_{12} K_2\}$$

and

$$\{(K_1, K_2) \in \mathbf{C}^{n,2n} | 0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_{22} K_2 + K_2 S_{11} K_1 + K_1 S_{11} K_2 - K_1 S_{21} K_1\}.$$

It would be very helpful to have a nice geometric description of Γ.

For $n = 1$ it is easy to determine geometrically all real solutions of the algebraic Riccati equations. If for example $n = 1$ and $Q_1,\ Q_2\ S_{11},\ S_{22} > 0$ then the real solutions of the algebraic Riccati equation corresponding to (1.4) are the points on the intersection of two hyperbolas; if we do not restrict the sign of the coefficients then the real solutions are lying on the intersection of two curves of degree ≤ 2.

VI. SOLUTION OF THE ALGEBRAIC EQUATIONS

In contrast to standard Riccati equations where the solutions of the corresponding algebraic equations can be obtained from the generalized eigenvectors, the constant solutions of (1.4) or (4.1) cannot be determined explicitly. Moreover, in general there is no sufficient condition ensuring the existence of solutions for the corresponding algebraic equations. Numerical examples indicate that the solutions of (1.4) with positive semidefinite terminal values $K_{1f},\ K_{2f}$ are usually convergent to solutions $K_{1\infty},\ K_{2\infty}$ of the corresponding algebraic Riccati equations. In order to obtain $K_{1\infty},\ K_{2\infty}$ we propose the following algorithm. Here we assume that $(A, \sqrt{S_{11}})$ and $(A, \sqrt{S_{22}})$ are stabilizable; this guarantees the existence of the strong solutions of the Riccati equations appearing in the algorithm.

Algorithm

- Start with initial matrices K_1^0 and K_2^0.
- Compute the strong solutions K_1^{c+1}, K_2^{c+1} of the decoupled Riccati equations

$$0 = K_1^{c+1}(A - S_{22}K_2^c) + (A - S_{22}K_2^c)^T K_1^{c+1} + Q_1 - K_1^{c+1} S_{11} K_1^{c+1} + K_2^c S_{12} K_2^c$$
$$0 = K_2^{c+1}(A - S_{11}K_1^c) + (A - S_{11}K_1^c)^T K_2^{c+1} + Q_2 - K_2^{c+1} S_{22} K_2^{c+1} + K_1^c S_{21} K_1^c$$

  while the norm of $\dot{K}_1^{c+1}$ or $\dot{K}_2^{c+1}$ (defined by the right hand side of (1.4) for $K_1 = K_1^{c+1}$ and $K_2 = K_2^{c+1}$) is bigger than ϵ.

- End while.

A natural choice for matrices K_1^0 and K_2^0 is to take the strong solutions of the standard decoupled algebraic Riccati equations

$$-A^T K_1^0 - K_1^0 A - Q_1 + K_1^0 S_{11} K_1^0 = 0$$
$$-A^T K_2^0 - K_2^0 A - Q_2 + K_2^0 S_{22} K_2^0 = 0.$$

An alternative algorithm has been proposed in [5], Appendix 12.2, under the assumption that either $(A, B_1, \sqrt{Q_1})$ or $(A, B_2, \sqrt{Q_2})$ is stabilizable and detectable. For both algorithms there exists up to now no proof for the convergence.

Example 1 First we give an example for algebraic Riccati equations corresponding to (1.4). Consider the matrices

$$A = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0.4422 \\ 3.0447 \\ -5.52 \\ 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0.1761 \\ -7.5922 \\ 4.99 \\ 0 \end{pmatrix}.$$

The numerical values for A, B_1 and B_2 are taken from [14]. The weighting matrices in the cost functionals are given by:

$$Q_1 = \mathrm{diag}(3.5; 2; 4; 5), \quad Q_2 = \mathrm{diag}(1.5; 6; 3; 1),$$
$$R_{11} = 1, \quad R_{12} = 0.25, \quad R_{21} = 0.6 \text{ and } \quad R_{22} = 2.$$

For $\epsilon = 10^{-6}$ our algorithm converges to the following solution of the algebraic equations:

$$K_{1\infty} = \begin{pmatrix} 7.6586 & 0.64379 & 0.63982 & -3.0831 \\ 0.64379 & 0.28775 & 0.28555 & -0.094471 \\ 0.63982 & 0.28555 & 0.56201 & 0.22702 \\ -3.0831 & -0.094471 & 0.22702 & 6.6987 \end{pmatrix},$$

$$K_{2\infty} = \begin{pmatrix} 3.4579 & 0.15681 & 0.20465 & -1.8480 \\ 0.15681 & 0.62348 & 0.28895 & -0.071133 \\ 0.20465 & 0.28895 & 0.40138 & 0.072898 \\ -1.8480 & -0.071133 & 0.072898 & 3.7850 \end{pmatrix}.$$

In this example it turned out that these limits were independent of the positive semidefinite initial values K_1^0 and K_2^0; nevertheless we were not able to prove that in our example these solutions are unique positive semidefinite solutions of the algebraic Riccati equations.
It could be easily checked that $R_1(K_{1\infty}, K_{2\infty}, 0.5\ I_4) \geq 0$, this implies that condition (2.3) is verified near the limits.

Example 2 For mixed H_2/H_∞-type problems the preceding algorithm has to be modified slightly since $Q_1 = -Q_2 \leq 0$; here in each step of the algorithm K_1^{c+1} is the solution of

$$0 = K_1^{c+1}(A - S_{22}K_2^c) + (A - S_{22}K_2^c)^T K_1^{c+1} + Q_1 - K_1^{c+1}S_{11}K_1^{c+1} + K_2^c S_{12} K_2^c$$

corresponding to the eigenvalues in the open left half-plane (which is assumed to exist). We consider the following numerical example:

$$A = \begin{pmatrix} -0.016896 & -0.066099 & -28.891 & 0 & 0 \\ -0.24501 & -0.015623 & -0.14625 & 0 & 0 \\ 0.53061 & -0.034715 & -0.50025 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100.0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.19474 & 0 \\ 0.08988 & 0 \\ 0.21103 & 0 \\ 0 & -0.08 \\ 0.8 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0.24342 \\ 0.11235 \\ 0.26379 \\ 0 \\ 1 \end{pmatrix}.$$

$$Q_1 = \mathrm{diag}(0; 0; 0; -0.36; -100), \quad Q_2 = -Q_1,$$

The solutions $P_1 = K_{1\infty}$ and $P_2 = K_{2\infty}$ of the algebraic equations corresponding to (4.1) with $\gamma = 1$ are

$$P_1 = \begin{pmatrix} -8.8238e-03 & 0.55187 & 0.25212 & 0.11480 & -2.2276e-04 \\ 0.55187 & -35.849 & -16.407 & -6.4599 & 0.014496 \\ 0.25212 & -16.407 & -7.5152 & -2.9214 & 6.6376e-03 \\ 0.11480 & -6.4599 & -2.9214 & -2.1828 & 2.5872e-03 \\ -2.2276e-04 & 0.014496 & 6.6376e-03 & 2.5872e-03 & -0.49956 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0.012488 & -0.79204 & -0.36211 & -0.15616 & -7.1170e-05 \\ -0.79204 & 51.627 & 23.634 & 9.1436 & 4.5677e-03 \\ -0.36211 & 23.634 & 10.825 & 4.1498 & 2.0906e-03 \\ -0.15616 & 9.1436 & 4.1498 & 2.6718 & 8.7170e-04 \\ -7.1170e-05 & 4.5677e-03 & 2.0906e-03 & 8.7170e-04 & 0.50034 \end{pmatrix}.$$

Here (near the limits) condition (4.5) is verified for $Q := I_5$.

VII. APPENDIX

For convenience of the reader we present here two known results.

Theorem 7.1: (see [6], Satz 10.2) Let P_i $1 \leq i \leq 2$, be solutions of the matrix Riccati equations

$$\dot{P} = PM_i(t)P - R_i(t) - PA(t) - A^T(t)P, \ \ t \in I,$$

with continuous coefficients $M_i,\ R_i,\ A$, and let $M_1(t) \geq M_2(t),\ \ R_1(t) \leq R_2(t)$ for $t \in I$. If $P_1(t_0) \leq P_2(t_0)$ for some $t_o \in I$ then $P_1(t) \leq P_2(t)$ for $t \in (-\infty, t_0] \cap I$.

This means that the solutions of the Riccati equations are monotonically depending on the coefficients and the terminal value.

Lemma 7.2: For $N \geq 2$ and $1 \leq i \leq N$ let $K_{if} \geq 0$ and let $q_i, s_i : (-\infty, t_f] \longrightarrow [0, \infty)$ and $a : (-\infty, t_f] \longrightarrow \mathbf{R}$ be continuous functions. Then the solutions $K_1, ...K_N$ of the coupled system of open loop Nash Riccati equations

$$\dot{K}_i = -2a(t) - q_i(t) - \sum_{j=1}^{N} K_i s_j(t) K_j, \quad K_i(t_f) = K_i f,$$

exist on $(-\infty, t_f]$.

For the proof of Lemma 7.2 see [4], Lemma 5.5. A similar result is given in [3], section VI.

VIII. CONCLUSIONS

In this paper we obtained global existence results for solutions of coupled Riccati type differential equations occuring in closed loop Nash games and in mixed H_2/H_∞ control problems. Since these equations are much more involved than standard Riccati equations there are no direct conditions ensuring existence, uniqueness and convergence of solutions. In most of the applications solutions of the associated algebraic equations are needed, the discussion and the numerical examples of section VI show that future research should concentrate on these topics, in particular on constructing convergent numerical algorithms.

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