Sufficient conditions for the existence of open loop Nash and Stackelberg strategies *

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The main object of this note ist to derive sufficient conditions for the existence of the solutions of coupled Riccati-type differential equations appearing in linear quadratic differential games. For convenience we confine here to the two-player case. A detailed discussion of these differential games can be found in the book of Basar and Olsder [3]. We consider two-player linear quadratic differential games

$$\begin{array}{l} \dot{x} = Ax + B_1u_1 + B_2u_2; \quad x(0) = x_0 \\ x \in \mathbf{R}^n, \ u_i \in \mathbf{R}^{r_i}, \ \ i = 1, 2, \end{array} \tag{1.1}$$

with cost functionals

$$\begin{aligned} J_1(u_1, u_2) &= \frac{1}{2}x_f^T K_{1f} x_f + \frac{1}{2}\int_0^{t_f} (x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2) d\tau, \\ J_2(u_1, u_2) &= \frac{1}{2}x_f^T K_{2f} x_f + \frac{1}{2}\int_0^{t_f} (x^T Q_2 x + u_2^T R_{22} u_2 + u_1^T R_{21} u_1) d\tau, \end{aligned} \tag{1.2}$$

where all matrices K_{if}, Q_i, R_{ij} are symmetric with $R_{ii} > 0,\ 1 \le i,\ j \le 2$ and $x_f = x(t_f)$. Here we are interested in equilibrium solutions of the differential game of the form

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) \Phi(t, 0) x_0,$$

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$$u_2^*(t) = -R_{22}^{-1} B_2^T K_2(t)\Phi(t,0)x_0, \tag{1.3}$$

where $\Phi(t,0)$ is the transition matrix of the corresponding closed loop system, i.e.

$$\dot{\Phi}(t,0) = (A - S_{11}K_1(t) - S_{22}K_2(t))\Phi(t,0),\ \Phi(t,t) = I_n. \tag{1.4}$$

with $S_{ij} = B_i R_{ii}^{-1} R_{ij} R_{ii}^{-1} B_i^T,\ 1 \le i,j \le 2.$

The functions K_i appearing in (1.3) are solutions of coupled systems of Riccati-type matrix differential equations. The form and type of these differential equations depend essentially on the information available to each player and on the strategies used; in this paper it is assumed that both players are using open loop strategies. We further assume that $A,\ B_1,\ B_2,\ R_{ij}$ for $1 \le i,j \le 2,\ Q_1,\ Q_2$ are time independent.

a) **Open loop Nash strategies.** The pair (u_1^*, u_2^*) is called *Nash (equilibrium) strategy* if

$$J_1(u_1^*, u_2^*) \le J_1(u_1, u_2^*)$$

and (1.5)

$$J_2(u_1^*, u_2^*) \le J_2(u_1^*, u_2)$$

for all admissible strategies u_1, u_2.
It is known (see [3], section 6.5 and also [5]) that for the existence of an (open loop) Nash strategy of the form (1.3) it is necessary and sufficient that the functions K_i appearing in (1.3) are on $[0, t_f]$ solutions of the terminal value problem

$$\begin{aligned}\dot{K}_1 &= -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2;\quad K_1(t_f) = K_{1f}\\ \dot{K}_2 &= -A^T K_2 - K_2 A - Q_2 + K_2 S_{22} K_2 + K_2 S_{11} K_1;\quad K_2(t_f) = K_{2f}.\end{aligned} \tag{1.6}$$

It is obvious that $K_1(t)$ and $K_2(t)$ are definied in $[t_f - \delta; t_f]$ for sufficiently small $\delta > 0$ but up to now in the literature there are no nice (i.e. easily checkable) conditions ensuring the existence of K_1 and K_2 on $[0, t_f]$.

b) **Open loop Stackelberg strategies.** If player 2 is allowed to select his strategy first, he is called the leader, and player 1 who selects his strategy second is called the follower. A *Stackelberg strategy* is the optimal strategy for the leader under the assumption that the follower reacts by playing optimally (see Simaan and Cruz [8] for details). In this case the functions K_1, K_2 determining the Stackelberg strategies (1.3) are determined by (see [8], [4], [3]) the following boundary value problem

$$\begin{aligned}\dot{K}_1 &= -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2;\quad K_1(t_f) = K_{1f};\\ \dot{K}_2 &= -A^T K_2 - K_2 A - Q_2 + Q_1 P + K_2 S_{11} K_1 + K_2 S_{22} K_2;\quad K_2(t_f) = K_{2f} - K_{1f} P(t_f);\\ \dot{P} &= AP - PA + PS_{11} K_1 + PS_{22} K_2 - S_{21} K_1 + S_{11} K_2;\quad P(0) = 0.\end{aligned} \tag{1.8}$$

c) **The corresponding linear systems.** It is well known (see [2], [1]) that (1.6) and (1.8) are intimately related to the following linear systems:

If $Y = \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$ with $X, Y_1, Y_2 \in \mathbf{R}^{n\times n}$ is a matrix solution of the terminal value problem

$$Y' = H_{Na}Y =: \begin{pmatrix} A & -S_{11} & -S_{22} \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{pmatrix} Y, \;\; Y(t_f) = \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \tag{1.9}$$

and if $\det X(t) \neq 0$ for $0 \le t \le t_f$, then $K_1 := Y_1X^{-1},\; K_2 := Y_2X^{-1}$ defines the solution of (1.6); i.e. if K_1, K_2 do not blow up as long as $X(t)$ is regular. A similar relationship holds between the solutions of (1.8) and of the boundary value problem

$$\dot{Z} = H_{St}Z =: \begin{pmatrix} A & -S_{11} & -S_{22} & 0 \\ -Q_1 & -A^T & 0 & 0 \\ -Q_2 & 0 & -A^T & Q_1 \\ 0 & -S_{21} & S_{11} & A \end{pmatrix} \begin{pmatrix} X \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \tag{1.10}$$

$$X(t_f) = I,\; Y_1(t_f) = K_{1f},\; Y_2(t_f) = K_{2f} - K_{1f}Y_3(t_f),\; Y_3(0) = 0.$$

2 Existence of Nash strategies

In this section we assume that $Q_1 = 0$.

In order to derive sufficient conditions for the existence of the solutions K_i of (1.6) on $[0, t_f]$ we extend the matrix H_{Na} of the corresponding linear system to an $4n \times 4n$ hamiltonian matrix

$$H_{Na}^e := \begin{pmatrix} A & -S_{11} & -S_{22} & 0 \\ 0 & -A^T & 0 & 0 \\ -Q_2 & 0 & -A^T & 0 \\ 0 & -Q_0 & S_{11} & A \end{pmatrix} =: \begin{pmatrix} \tilde{A} & -\tilde{S} \\ -\tilde{Q} & -\tilde{A}^T \end{pmatrix} \tag{2.1}$$

with $Q_0 \ge 0$ and we use the properties of the standard Riccati differential equation

$$\dot{K} = -\tilde{A}^TK - K\tilde{A} - \tilde{Q} + K\tilde{S}K, \;\; K(t_f) = K_f. \tag{2.2}$$

From the theory of symmetric Riccati equations (see [9], Theorem 2.1 and [6]) we infer :

Lemma 2.1 *Let $\tilde{Q}, \tilde{S} \ge 0$, then the solution of (2.2) exists if either $K_f \ge 0$ or if $K_f \ge K_m$; where K_m is an arbitrary symmetric solution of the algebraic Riccati equation corresponding to (2.2).*

Using $n \times n$ matrices X_i^0, Y_i^0 we set

$$M_0 = \begin{pmatrix} I & 0 \\ X_1^0 & I \\ X_2^0 & Y_2^0 \\ X_3^0 & Y_3^0 \end{pmatrix} \in \mathbf{R}^{4n\times 2n}.$$

If Y_2^0 is an element of

$$L := \{Y \in \mathbf{R}^{n\times n} | S_{22}(A^T)^j Y = -S_{11}(A^T)^j \;\; \forall\, j \in \mathbf{N} \cup \{\mathbf{0}\}\} \tag{2.3}$$

then we infer from

$$(H_{Na}^e)^j M_0 = \begin{pmatrix} * & -S_{11}(A^T)^{j-1} - S_{22}(A^T)^{j-1}Y_2^0 \\ * & (-A^T)^j \\ * & (-A^T)^j Y_2^0 \\ * & * \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \\ * & * \\ * & * \end{pmatrix}$$

that

$$e^{H_{Na}^e t} M_0 =: \begin{pmatrix} U(t) & 0 \\ X_1(t) & Y_1(t) \\ X_2(t) & Y_2(t) \\ X_3(t) & Y_3(t) \end{pmatrix}. \tag{2.4}$$

Theorem 2.2 *Let $\tilde{Q}, \tilde{S} \geq 0$ and let $Y_2^0 \in L \neq \emptyset$.*
If

$$K_f = \begin{pmatrix} X_2^0 - Y_2^0 X_1^0 & Y_2^0 \\ X_3^0 - Y_3^0 X_1^0 & Y_3^0 \end{pmatrix} \geq 0 \tag{2.5}$$

(or $K_f \geq K_m$) then the solution of the open loop Nash Riccati equation in (1.6) with $K_1(t_f) = X_1^0,\; K_2(t_f) = X_2^0$ exists for $t \leq t_f$.

Proof. Under the assumptions of Theorem 2.2 it follows from Lemma 2.1 and (2.4) that $\det \begin{pmatrix} U(t) & 0 \\ X_1(t) & Y_1(t) \end{pmatrix} \neq 0$ for $t \leq t_f$. Consequently we have $\det U(t) \neq 0$ for $t \leq t_f$ and the solutions $K_i(t) = X_i(t)U(t)^{-1},\; 1 \leq i \leq 2$, of our terminal value problem cannot blow up. □

Remark 2.3 The matrices $Y_2^0 \in L,\; Y_3^0$ and X_3^0 appearing in (2.5) are parameters which we try to chose such that (for given X_1^0, X_2^0) $K_f \geq 0$ (or $K_f \geq K_m$).
If for example $X_1^0 = 0,\; X_2^0 > 0$ and $Y_2^0 \in L \neq \emptyset$, then we choose for example $X_3^0 = (Y_2^0)^T$ and $Y_3^0 = \alpha I$; in this case (2.5) is obviously fulfilled for $\alpha \geq \alpha_0$.

Example 2.4 Let $Q_1 = 0$ and

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix},\; S_{11} = \begin{pmatrix} \frac{16}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{19}{5} \end{pmatrix},\; S_{22} = \begin{pmatrix} \frac{26}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{29}{5} \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}.$$

Now we choose $S_0 = Q_0 = 0$ and

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{-46}{75} & \frac{2}{75} \\ 0 & \frac{1}{3} & \frac{2}{75} & \frac{-49}{75} \\ 1 & 0 & \frac{121}{75} & \frac{-2}{75} \\ 0 & \frac{1}{2} & \frac{-2}{75} & \frac{173}{150} \end{pmatrix} =: \begin{pmatrix} I & 0 \\ K_{1f} & I \\ K_{2f} & L_{1f} \\ P_f & L_{2f} \end{pmatrix}, \text{ then } K_f = \begin{pmatrix} \frac{167}{150} & \frac{-2}{75} & \frac{-46}{75} & \frac{2}{75} \\ \frac{-2}{75} & \frac{74}{75} & \frac{2}{75} & \frac{-49}{75} \\ \frac{-46}{75} & \frac{2}{75} & \frac{121}{75} & \frac{-2}{75} \\ \frac{2}{75} & \frac{-49}{75} & \frac{-2}{75} & \frac{173}{150} \end{pmatrix}$$

is positive definite and Theorem 2.2 implies that the solution of the terminal value problem (1.6) exists for $t \leq t_f$.

3 Existence of solutions of Stackelberg differential equations.

In this section we use the same idea as in section 2 in order to derive sufficient conditions for the existence of the solution of the coupled system of differential equations in (1.8) satisfying certain terminal conditions in $t = t_f$ or the boundary conditions given in (1.8).

Theorem 3.1 *Let $Q_1 = K_{1f} = 0,\ Q_2 \geq 0;\ K_{2f} \geq 0$ and $L \neq \emptyset$ (see (2.3)). Then the solution of the terminal value problem (1.8) exists for $t \leq t_f$; this implies a Stackelberg equilibrium is defined by (1.3).*

Proof. Under the assumptions of Theorem 3.1 the system in (1.8) is partially decoupled, since in this case K_1 and K_2 are independent of P and are solutions of a terminal value problem of open loop Nash type; according to remark 2.3 under our assumptions these solutions cannot blow up for $t \leq t_f$. Consequently P is solution of a linear differential equations (with variable coefficients depending on $K_1(t), K_2(t)$) and exists at least for $t \in (-\infty, t_f]$, □

If we do not want to use the restrictive assumption that $Q_1 = K_{1f} = 0$, we can apply the method of section 2 in order to prove the following existence theorem.

Theorem 3.2 *Let $\tilde{Q}$ and $\tilde{S}$ be defined as in (2.1) with $Q_0 := S_{21}$ and let $\tilde{Q}, \tilde{S} \geq 0$ and $Y_2^0 \in L \neq \emptyset$. If the matrices Y_2^0 and Y_3^0 can be chosen such that the solution of*

$$\dot{K} = -\tilde{A}^T K - K\tilde{A} - \tilde{Q} + K\tilde{S}K,\ K(t_f) = \begin{pmatrix} X_2^0 - Y_2^0 X_1^0 & Y_2^0 \\ X_3^0 - Y_3^0 X_1^0 & Y_3^0 \end{pmatrix}$$

exists for $t \leq t_f$ (for example, if $K(t_f) \geq 0$), then the solution of (1.8) with $K_1(t_f) = X_1^0,\ K_2(t_f) = X_2^0$ and $P(t_f) = X_3^0$ exists for $t \leq t_f$.

The proof of Theorem 3.2 is similar to the proof of Theorem 2.2 and is omitted.

Example 3.3 Let $Q_1 := 0,\quad A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix},\ S_{11} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix},\ S_{21} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$

$S_{22} = \begin{pmatrix} 4 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$, and $Q_2 = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix}$. If we have the terminal values $X_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X_2^0 = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ and $X_3^0 = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$, then we have

$Y_2^0 := \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \in L \neq \emptyset$ and the matrix $K_f = \begin{pmatrix} \frac{11}{2} & 0 & \frac{-1}{2} & 0 \\ 0 & \frac{9}{2} & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 & \frac{15}{2} & 0 \\ 0 & \frac{-1}{2} & 0 & \frac{7}{2} \end{pmatrix}$ is positive definite. Consequently Theorem 3.1 implies that the solution of the terminal value problem (1.8) exists for $t \leq t_f$.

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