Existence and Comparison Theorems for Algebraic Riccati Equations and Riccati Differential and Difference Equations

G. FREILING
FB 11 - Mathematik - G. Mercator Universität Duisburg, D-47048 Duisburg, Germany, email: freiling@math.uni-duisburg.de

G. JANK
Lehrstuhl II für Mathematik - RWTH Aachen, D-52056 Aachen, Germany, email: jank@math2.rwth-aachen.de

ABSTRACT: We present comparison and global existence theorems for solutions of generalized matrix Riccati differential and difference equations. Moreover we obtain existence and comparison results for the maximal solutions of the corresponding generalized algebraic Riccati equations. For the symplectic matrix Riccati differential equation we derive sufficient conditions ensuring the global existence of the solutions of the corresponding initial value problems.

KEY WORDS: Continuous-time Riccati equations, discrete-time Riccati equations, generalized Riccati equations, algebraic Riccati equations, Riccati inequalities, comparison theorems, global existence of solutions.

1. INTRODUCTION

Let $A, Q, S \in \mathbf{C}^{n,n}$ be $n \times n$ matrices with complex entries, $Q = Q^*, S = S^*$ and let $H = \begin{pmatrix} A & -S \\ -Q & -A^* \end{pmatrix}$ be the corresponding hamiltonian matrix. In the sequel $\Pi : \mathbf{C}^{n,n} \to \mathbf{C}^{n,n}, W \to \Pi(W)$ denotes a linear function with $\Pi(W^*) = (\Pi(W))^*$ and $\Pi(W_1) \leq \Pi(W_2)$ for $W_1 \leq W_2$; here $W_1 \leq W_2$ means that $W_2 - W_1$ is positive semidefinite $(W_2 - W_1 \geq 0)$. It is easy to construct functions Π having the aforementioned properties, they are fulfilled for example if $\Pi(W)$ is the sum of terms of the form $C^T W C$ or an infinite series in terms of this form.

One of our main goals is the study of the generalized Riccati differential equation

$$\dot{W} = -A^*W - WA - Q + WSW - \Pi(W) =: \mathrm{Ric}\,(W, H) - \Pi(W) \qquad (GRDE)$$

and of the corresponding generalized algebraic Riccati equation

$$0 = \mathrm{Ric}\,(W, H) - \Pi(W). \qquad (GARE)$$

It is worth while to point out that in the most of sections 2 and 3 of this paper we do not assume that Q and S are positive semidefinite.

Moreover we show that there are similar methods to study the associated discrete time equations. Generalized Riccati equations of this form appear when studying optimal control of linear systems with Markovian jumps (see [AFJ1], [M], [Wo] and [dSF2] for a detailed description and further references). Such equations also occur in robust control problems (see [KB]) and for the discrete time case (see [KA]).

In contrast to the *unperturbed* Riccati differential equation

$$\dot{P} = \mathrm{Ric}(P, H) \tag{RDE}$$

and the corresponding algebraic Riccati equation

$$0 = \mathrm{Ric}(P, H) \tag{ARE}$$

it is in general not possible to transform $(GRDE)$ (or$(GARE)$) to an equivalent $2n$-dimensional linear system. On the other hand it is known from the results of Wimmer [W1], Gohberg/Lancaster/Rodman [GLR], Faibusovich [F] and Ran/Vreugdenhil [RV] that there is an interesting interplay between the solutions of (ARE) and of the corresponding Riccati inequality and that for $S \geq 0$ the strong solution of (ARE) depends monotonically on JH, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and I_n is the unit matrix. It will be shown in this paper that similar results can be obtained for $(GRDE)$, $(GARE)$ and for the corresponding discrete-time equations. A first step in this direction has been made in [dSF1].

In section 2 of this paper we recall a monotonicity property and a comparison theorem for generalized Riccati equations which have been obtained recently and are generalizations of well known results for (RDE) (see [CO], pp. 51-53).

Using these results and the interconnections between $(GARE)$ and the corresponding generalized Riccati inequalities, we give in section 3 a *dynamic* proof for the existence of solutions of $(GARE)$ and $(GRDE)$. In addition we use the comparison results from section 2 for the proof of the global existence of the solutions of $(GRDE)$.

Analogous results are obtained in section 4 for the discrete-time case.

In section 5 we use a different approach in order to derive sufficient conditions for the global existence of solutions of (RDE).

2. PRELIMINARY RESULTS AND NOTATIONS

In the sequel we denote by $W(\cdot, X_0)$ the solution of $(GRDE)$ with $W(t_0, X_0) = X_0$, here t_0 is arbitrary but fixed.

LEMMA 2.1. *If $t_0 \in I$ and $W(\cdot, X_0)$ exists on the interval I then*

$$\dot{W}(t_0, X_0) = \mathrm{Ric}(X_0, H) - \Pi(X_0) \leq 0 \qquad (or\ \geq 0)$$

implies $\dot{W}(t, X_0) \leq 0$ (or ≥ 0, respectively) for $t \in (-\infty, t_0] \cap I$.

The proof of Lemma 2.1 is analogous to the proof of [AFJ1], Theorem 1, where we considered a special case of $(GRDE)$ with real coefficients; notice that X_0 is not assumed to be hermitian.

The next theorem is a generalization of a comparison result of Coppel (see [CO], pp. 51/52); it has been proved (for real coefficients) in [FJA], Theorem 2.1, and it holds true for complex coefficients as well. This theorem will be used for the investigation of (RDE) and $(GRDE)$; it could also be used to derive a generalized version of the monotonicity result proved by Fragoso and de Souza [FdS], Theorem 2.1 for periodic generalized Riccati differential equations.

THEOREM 2.2. *For* $i \in \{1,2\}$ *let* $A_i, Q_i, S_i : I \to \mathbf{C}^{n,n}$ *be piecewise continuous (or integrable) with* $Q_i(t) = Q_i^*(t)$ *and* $S_i(t) = S_i^*(t)$ *for* $t \in I$; *moreover let* K_i, $i \in \{1,2\}$ *be a solution of*

$$\dot{K}_i = -A_i^*(t)K_i - K_iA_i(t) - Q_i(t) + K_iS_i(t)K_i \tag{2.1}$$

on the interval I. *If for some* $t_f \in I$ $\quad K_1(t_f) \leq K_2(t_f)$ *(or* $K_1(t_f) < K_2(t_f)$*) and if*

$$\begin{pmatrix} Q_1 & A_1^* \\ A_1 & -S_1 \end{pmatrix}(t) \leq \begin{pmatrix} Q_2 & A_2^* \\ A_2 & -S_2 \end{pmatrix}(t) \quad \textit{for } t \in I,$$

then $K_1(t) \leq K_2(t) \qquad$ *(or* $K_1(t) < K_2(t)$*) for* $t \in I \cap (-\infty, t_f]$.

In the sequel we shall use the following abbreviations for the sets of the hermitian solutions of the (generalized) algebraic Riccati equations and inequalities:
For $\square \in \{\leq, =, \geq\}$ let

$$R_{\square}^{H} := \{P \in \mathbf{C}^{n,n} | P = P^* \text{ and } \operatorname{Ric}(P,H) \,\square\, 0\}$$

and

$$GR_{\square}^{H} := \{W \in \mathbf{C}^{n,n} | W = W^* \text{ and } \operatorname{Ric}(W,H) - \Pi(W) \;\square\; 0\,\}.$$

3. EXISTENCE THEOREMS FOR THE CONTINUOUS-TIME CASE

The following theorem shows the interplay between the existence of hermitian solutions of $(GARE)$ and $(GRDE)$ and the solvability of the corresponding generalized Riccati inequalities.

THEOREM 3.1. $GR_{=}^{H} \neq \emptyset$ *if and only if there exist* $W_1 \in GR_{\leq}^{H}$ *and* $W_2 \in GR_{\geq}^{H}$ *with* $W_1 \leq W_2$; *moreover in this case* $W_1 \leq W_0 \leq W_2$ *yields that* $W(t, W_0)$ *exists for* $t \in (-\infty, t_0]$ *with* $W_1 \leq W(t,W_1) \leq W(t,W_0) \leq W(t,W_2) \leq W_2$ *for* $t \in (-\infty, t_0]$.

Proof. (i) If $W_0 \in GR_{=}^{H}$ exists, we can choose $W_1 = W_2 = W_0$; then $W(t,W_0) \equiv W_0$.
(ii) Let $W_1 \in GR_{\leq}^{H}$ and $W_2 \in GR_{\geq}^{H}$ with $W_1 \leq W_2$ exist.
Then, according to Lemma 2.1, $\dot{W}(t,W_1) \leq 0$ and $\dot{W}(t,W_2) \geq 0$ for $t \leq t_0$ while $W(t,W_1)$ and $W(t,W_2)$ exist. On the other hand we infer from Theorem 2.2 that $W_1 \leq W(t,W_1) \leq W(t,W_0) \leq W(t,W_2) \leq W_2$ for $t \leq t_0$, hence the limits $\lim\limits_{t\to-\infty} W(t,W_1) = W_1^{\infty} \leq W_2^{\infty} = \lim\limits_{t\to-\infty} W(t,W_2)$ exist with $W_1^{\infty}, W_2^{\infty} \in GR_{=}^{H}$.
The existence of $W(t,W_0)$ for $t \leq t_0$ and $W_1 \leq W(t,W_1) \leq W(t,W_0) \leq W(t,W_2) \leq W_2$

is an immediate consequence of the preceding estimates and of Theorem 2.2. ■

REMARK 3.2. (i) Notice that up to this point we did not make any assumptions on the definiteness of S or Q. In this case the main problem in the application of Theorem 3.1 arises when we have to check the existence of the matrices $W_1 \in GR^H_{\leq}$ and $W_2 \in GR^H_{\geq}$. Fortunately in most applications at least one of the matrices Q and S is positive semidefinite and/or the existence of a solution of at least one Riccati inequality is guaranteed. If for example $Q \geq 0$ then obviously $W_1 := 0 \in GR^H_{\leq}$. If in addition $S = BB^*$ and if (A, B) is stabilizable then it can easily be shown that in this case $GR^H_{=}$ contains a positive semidefinite (and hence a maximal) element (see the proof of Theorem 3.3); this last result corresponds to [GLR], Theorem 2.2.
(ii) The assumptions of Theorem 3.1 on the existence of W_1, W_2 could be replaced by the weaker assumption that there exist solutions $W(\cdot, W_1) \leq W(\cdot, W_2)$ on $(-\infty, t_0]$. All known global existence results are based on hypotheses implying that this necessary and sufficient condition is satisfied. Usually one is looking for sufficient conditions for the existence of $W(\cdot, W_1)$ and $W(\cdot, W_2)$ which can be checked more easily.
(iii) According to Theorem 3.1 there exist a solution of $(GARE)$ and initial values W_0 such that $W(t, W_0)$ is defined for $t \leq t_0$ if and only if there exist adequate solutions of the corresponding algebraic Riccati inequalities. For $\Pi(W) \equiv 0$ there exist algebraic tests for the solvability of $\mathrm{Ric}(W, H) \leq 0$ (see [S] for details and further references).

The following theorems are generalizations of results obtained in [GLR], [RV] and [dSF1].

THEOREM 3.3. *Let* $S = BB^*, (A, B)$ *stabilizable and* $W_1 \in GR^H_{\leq}$. *If* Π *satisfies the condition (see [Wo])*

$$\inf_K \| \int_0^\infty e^{t(A-BK)^*} \Pi(I_n) e^{t(A-BK)} dt \| < 1, \tag{3.1}$$

then there exists $W_2 \in GR^H_{=}$ *with* $W_2 \geq W_1$. *Moreover* $GR^H_{=}$ *contains a unique maximal element* W_+ *and* $\sigma(A - SW_+) \subset \mathbf{C}_{\leq}$.
Here $\| M \|$ *is the spectral radius of* M *and* $\mathbf{C}_{\leq}$ *denotes the closed left half-plane.*

Proof. We choose matrices $Q_+, Q_-, W_{1+}, W_{1-} \geq 0$ with $Q = Q_+ - Q_-$ and $W_1 = W_{1+} - W_{1-}$. Further we denote by $X(\cdot, W_{1+})$ the solution of

$$\dot{X} = \mathrm{Ric}(X, H) - Q_- - \Pi(X), \; X(t_0) = W_{1+}.$$

Then it follows from the proof of [Wo], Theorem 2.1 (where real coefficients are considered an where (3.1) is used) that $X(t, W_{1+})$ exists and is bounded for $t \leq t_0$. From Lemma 2.1 and Theorem 2.2 we infer that

$$\dot{W}(t, W_1) \leq 0 \quad \text{and} \quad W_1 \leq W(t, W_1) \leq X(t, W_{1+}) \quad \text{for} \quad t \leq t_0.$$

Hence $W_2 := \lim_{t \to -\infty} W(t, W_1)$ exists and belongs to $GR^H_{=}$.
Since $GR^H_{=} \neq \emptyset$ it follows as in the proof of [dSF1], Theorem 2.1, that $GR^H_{=}$ contains a unique maximal element W_+ with $\sigma(A - SW_+) \subset \mathbf{C}_{\leq}$. ■

Theorem 2.2. implies that the solutions of (RDE) and $(GRDE)$ are monotonically dependent on JH. We show that the same monotonicity is shown by the maximal solution of $(GARE)$ - if it exists. For (ARE) corresponding results have been obtained in [W1], [GLR] and [RV].

THEOREM 3.4. *Let* $JH = \begin{pmatrix} Q & A^* \\ A & -S \end{pmatrix} \geq J\tilde{H} := \begin{pmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -\tilde{S} \end{pmatrix}, S = BB^*, (A, B)$ *stabilizable and let* Π *satisfy condition (3.1). Then there exists for any* $\tilde{W}_0 \in GR^{\tilde{H}}_=$ *a matrix* $W_0 \in GR^H_=$ *with* $\tilde{W}_0 \leq W_0$*; this inequality holds in particular if* $\tilde{W}_0$ *and* W_0 *are the maximal elements of* $GR^{\tilde{H}}_=$ *and* $GR^H_=$ *respectively.*

Proof. From $\mathrm{Ric}(\tilde{W}_0, \tilde{H}) = \Pi(\tilde{W}_0)$,

$$\begin{aligned} \tilde{W}(t_0, \tilde{W}_0) &= \begin{pmatrix} I_n \\ \tilde{W}_0 \end{pmatrix}^* (-JH) \begin{pmatrix} I_n \\ \tilde{W}_0 \end{pmatrix} - \Pi(\tilde{W}_0) \\ &= \begin{pmatrix} I_n \\ \tilde{W}_0 \end{pmatrix}^* (J\tilde{H} - JH) \begin{pmatrix} I_n \\ \tilde{W}_0 \end{pmatrix} \leq 0 \end{aligned}$$

and Theorem 3.3 it follows that there exists $W_0 \in GR^H_=$ with $\tilde{W}_0 \leq W_0$. The existence of the maximal elements of $G^{\tilde{H}}_=$ and $G^H_=$ is ensured by Theorem 3.3 since $GR^{\tilde{H}}_= \neq \emptyset$. ■

REMARK 3.5. Theorem 3.4 has been proved
a) for $\Pi(W) \equiv 0, A = \tilde{A}$ and $S = \tilde{S}$ in [GLR], Theorem 2.3,
b) for $\Pi(W) \equiv 0$ in [RV], Theorem 2.2 (see also [W1]) and
c) for $A = \tilde{A}$ and under the assumption $GR^{\tilde{H}}_= \neq \emptyset$ in [dSF1], Theorem 2.1.
Notice that $Q \geq 0$ implies that $GR^H_\leq \neq \emptyset$, hence in this case $GR^H_=$ contains a maximal (positive semidefinite) element W_+ if (A, B) is stabilizable and if condition (3.1.) holds - this situation was studied in [Wo]. In this case, according to [Wo], Theorem 2.1, (iv), W_+ is positive definite if in addition $(\sqrt{Q}, A)$ is observable. The last condition is sufficient but not necessary.
The complete structure of $GR^H_=$ is still unknown.

The next theorem shows how the preceding results can be used for the proof of a global existence theorem for generalized Riccati differential equations with variable coefficients. For this purpose we consider matrices $A_1, A_2, Q_1, Q_2, S_1, S_2 \in \mathbf{C}^{n,n}$ and matrix-valued functions $A, Q, S : (-\infty, t_0] \to \mathbf{C}^{n,n}$ with hermitian $Q_1, Q_2, S_1, S_2, Q(t)$ and $S(t), t \in (-\infty, t_0]$. By $H_i = \begin{pmatrix} A_1 & -S_i \\ -Q_i & -A_i^* \end{pmatrix}, i = 1, 2$ and $H(t) = \begin{pmatrix} A & -S \\ -Q & -A^* \end{pmatrix}(t)$ we denote the corresponding hamiltonian matrices.

THEOREM 3.6. *Let* $JH_1(t) \leq JH(t) \leq JH_2(t), t \in (-\infty, t_0]$ *and assume that there exists* $W_1 \in GR^{H_1}_\leq$ *and* $W_2 \in GR^{H_2}_\geq$ *with* $W_1 \leq W_2$*. Then the solution* $W(\cdot, W_0)$ *of*

$$\dot{W} = -A^*(t)W - WA(t) - Q(t) + WS(t)W - \Pi(W), \quad W(t_0) = W_0 \qquad (3.2)$$

exists for $t \leq t_0$ *and any* W_0 *with*

$$W^{H_1}(t, W_0) \leq W(t, W_0) \leq W^{H_2}(t, W_0), \quad t \leq t_0;$$

here $W^{H_i}(\cdot, W_0)$ *is the solution of*

$$\dot{W} = \mathrm{Ric}\,(W, H_i) - \Pi(W), \quad W(t_0) = W_0.$$

The proof of Theorem 3.6 is similar to second part of the proof of Theorem 3.1.
In general the solutions of (3.2) are not necessarily monotonic if $\dot{W}(t_0)$ is positive semidefinite; but this property of autonomous Riccati differential equations is maintained if for example

$$\begin{pmatrix} \dot{Q} & \dot{A} \\ \dot{A}^* & -\dot{S} \end{pmatrix}(t) \geq 0 \quad \text{for} \quad t \leq t_0. \tag{3.3}$$

LEMMA 3.7. *Let the solution of (3.2) satisfy* $\dot{W}(t_0, W_0) = \mathrm{Ric}\,(W_0, H(t_0)) - \Pi(W_0) \geq 0$ *and assumption (3.3); then* $\dot{W}(t, W_0) \geq 0$ *for* $t \leq t_0$ *while* $W(t, W_0)$ *exists.*

Proof. $\dot{W}(= \dot{W}(\cdot, W_0))$ satisfies $\dot{W}(t_0) \geq 0$ and

$$\begin{aligned} \ddot{W} &= -(A - SW)^* \dot{W} - \dot{W}(A - SW) - \Pi(\dot{W}) \\ &\quad -(I_n \; W) \begin{pmatrix} \dot{Q} & \dot{A} \\ \dot{A}^* & -\dot{S} \end{pmatrix} \begin{pmatrix} I_n \\ W \end{pmatrix} \\ &\leq -(A - SW)^* \dot{W} - \dot{W}(A - SW) - \Pi(\dot{W}) \\ &\leq -(A - SW)^* \dot{W} - \dot{W}(A - SW). \end{aligned}$$

Notice that $\dot{W}$ is the solution of a generalized Riccati equation of the type $(GRDE)$ with $\dot{W}(t_0) \geq 0$ therefore we infer from [Wo], Theorem 2.1 that $\dot{W}(t) \geq 0$ for $t \leq t_0$; this yields the last of the preceding inequalities. Hence the assertion of the Lemma results from $\dot{W}(t_0) \geq 0$, [Wo], Theorem 2.1 (ii) and well known properties of Lyapunov differential equations ([KK], Hilfssatz 10.2). ■

COROLLARY 3.8. *Assume in addition to the assumptions of Lemma 3.7 that* $W_0 \geq 0$, $Q(t) \geq 0$ *and* $S(t) \geq 0$ *for* $t \leq t_0$. *Then* $W(t, W_0)$ *exists for* $t \leq t_0$ *with*

$$0 \leq W(t, W_0) \leq W_0; \tag{3.4}$$

moreover $\lim_{t \to -\infty} W(t, W_0) = W_\infty \geq 0$ exists.

Proof. The first inequality in (3.4) results from [Wo], Theorem 2.1, and the second inequality and $\dot{W}(t, W_0) \geq 0$ follow from Lemma 3.7. Hence $W(t, W_0)$ exists for $t \leq t_0$ and is convergent as $t \to -\infty$. ■

4. EXISTENCE THEOREMS FOR THE DISCRETE-TIME CASE

The subsequent comparison theorem will be used for the study of generalized discrete time Riccati equations and has been obtained essentially in [W2], Theorem 2.2 and [WP], Theorem 3.1 (see also [dS], Lemma 3.1, [BG], Lemma 10.1 and [FJA], Theorem 3.1).
For $\nu \in \mathbf{N}$, let $A, \tilde{A}, A(\nu), \tilde{A}(\nu) \in \mathbf{C}^{n \times n}$; $B, \tilde{B}, B(\nu), \tilde{B}(\nu) \in \mathbf{C}^{n \times m}$ and let $Q, \tilde{Q}, Q(\nu), \tilde{Q}(\nu) \in \mathbf{C}^{n \times n}$ be positive semidefinite.
For $K \in \mathbf{C}^{n \times n}$, define Φ by

$$\Phi(A, B, Q, K) = A^* K A - A^* K B (I + B^* K B)^{-1} B^* K A + Q.$$

THEOREM 4.1 *(i) Let* K *and* $\tilde{K} \in \mathbf{C}^{n \times n}$ *be symmetric with* $K \geq \tilde{K}$ *and* $I_n + \tilde{B}^* \tilde{K} \tilde{B} > 0$. *If*

$$\begin{pmatrix} Q & A^* \\ A & -BB^* \end{pmatrix} \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -\tilde{B}\tilde{B}^* \end{pmatrix}$$

then

$$\Phi(A,B,Q,K) \geq \Phi(\tilde{A},\tilde{B},\tilde{Q},\tilde{K}).$$

(ii) Let $K_0 \geq \tilde{K}_0 \geq 0$ *and assume that the sequences* $(K(\nu))$ *and* $(\tilde{K}(\nu))$ *with*

$$K(0) = K_0, \qquad K(\nu+1) = \Phi(A(\nu), B(\nu), Q(\nu), K(\nu))$$

and

$$\tilde{K}(0) = \tilde{K}_0, \qquad \tilde{K}(\nu+1) = \Phi(\tilde{A}(\nu), \tilde{B}(\nu), \tilde{Q}(\nu), \tilde{K}(\nu))$$

for $\nu \geq 0$, *are well defined. If in addition*

$$\begin{pmatrix} Q & A^* \\ A & -BB^* \end{pmatrix}(\nu) \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -\tilde{B}\tilde{B}^* \end{pmatrix}(\nu) \quad \text{for } \nu \geq 0$$

then $K(\nu) \geq \tilde{K}(\nu) \geq 0$ *for* $\nu \geq 0$.

Theorem 4.1 is very useful for the study of the generalized discrete-time Riccati difference equation

$$K(\nu+1) = \Phi(A,B,Q,K(\nu)) + \Pi(K(\nu)), K(0) = K_0 \qquad (DGRDE)$$

and the corresponding generalized discrete-time algebraic Riccati equation

$$K = \Phi(A,B,Q,K) + \Pi(K); \qquad (DGARE)$$

notice that equations of this type appear in discrete-time markovian jump linear quadratic control problems (see [AFJ2]).

THEOREM 4.2. *Let* $K(0) := K_0$ *such that* $(I + B^*K_0B) > 0$ *and* $K(0) \leq K(1)$ $(= \Phi(A,B,Q,K(0)) + \Pi(K(0)))$. *Then there exists a solution* $\tilde{K} \geq K_0$ *of* $(DGARE)$ *if and only if there exists* $\tilde{K}_0 \geq K_0$ *with*

$$\tilde{K}(1) = \Phi(A,B,Q,\tilde{K}_0) + \Pi(\tilde{K}_0) \leq \tilde{K}_0. \qquad (DGARE)$$

Proof. Let $(K(\nu))$ and $(\tilde{K}(\nu))$ be the sequences satisfying $(DGRDE)$ with $K(0) = K_0$ and $\tilde{K}(0) = \tilde{K}_0$, respectively. From $K(0) \leq K(1)$ and $\tilde{K}_0 \geq \tilde{K}(1)$ it follows by induction from Theorem 4.1 that these sequences are well defined with

$$K_0 \leq K(\nu) \leq K(\nu+1) \leq \tilde{K}(\nu+1) \leq \tilde{K}(\nu) \leq \tilde{K}_0 \quad \text{for} \quad \nu \in \mathbf{N}.$$

Consequently $\lim_{\nu\to\infty} K(\nu) = K_\infty$ and $\lim_{\nu\to\infty} \tilde{K}(\nu) = \tilde{K}_\infty$ exist and are solutions of $(DGARE)$ with $K_\infty \leq \tilde{K}_\infty$. This proves the nontrivial part of the assertion of Theorem 4.2. ■

REMARK 4.3. (i) The assumptions $K(1) \geq K_0$ and $\tilde{K}(1) \leq \tilde{K}_0$ of Theorem 4.2 correspond to the Riccati inequalities fulfilled by $W_1 \in GR^H_\leq$ and $W_2 \in GR^H_\leq$ in Theorem 3.1. Notice that in the discrete-time case we need the additional assumption $I + B^*K_0B > 0$ which ensures the existence of the inverses of $I + B^*K(\nu)B$. As in the corresponding continuous-time case the assumptions of Theorem 4.2 could be replaced by a necessary

and sufficient condition (see Remark 3.2 (ii)).
(ii) If (A, B) is stabilizable, $\Pi(K) \equiv 0$ and if there exists $K(0)$ with $K(1) \geq K(0) = K_0$ and $I + B^*K(0)B > 0$, then it is easy to see that there exists $\tilde{K}(0) \geq K(0)$ with $\tilde{K}(1) \leq \tilde{K}(0)$:

As in the proof of Theorem 3.3 let $Q = Q_+ - Q_-$ and $K_0 = K_{0+} - K_{0-}$ with Q_+, Q_-, K_{0+}, $K_{0-} \geq 0$.
It follows from Theorem 4.1 by induction that the sequence $(\hat{K}(\nu))$ with $\hat{K}(0) = K_{0+}$ and

$$\hat{K}(\nu+1) = \Phi(A, B, Q_+, \hat{K}(\nu)) + \Pi(\hat{K}(\nu)) \quad \text{for} \quad \nu \in \mathbf{N}$$

is nondecreasing and (see [CM], Theorem 2.2) convergent. Hence $\lim\limits_{\nu \to \infty} \tilde{K}(\nu) = K_\infty \geq K_0$ exists and we can set $\tilde{K}(0) = K_\infty$.
Furthermore it is known (see [RV], Theorem 3.1) that under these assumptions the maximal solution P^+ of $P = \Phi(A, B, Q, P)$ exists.
(iii) As in the continuous-time case (see [LRR],[RV], [W2], [dSF1]) P^+ depends monotonically on $JH = \begin{pmatrix} Q & A^* \\ A & -BB^* \end{pmatrix}$ and is the strong solution of the unperturbed discrete-time algebraic Riccati equation. In order to ensure the existence of the maximal solution K^+ of $(DGARE)$ one needs in addition to the preceding assumptions the discrete-time versions of condition (3.1) and of the proof of [dSF1], Theorem 2.1 (see [RV], section 3, for the case $\Pi(K) \equiv 0$). We resign to discuss this problem in detail.
(iv) Let $K_0 \leq \hat{K}_0 \leq \tilde{K}_0$ and $I_n + B^*K_0B > 0$. If $K_0 = K(0) \leq K(1)$ and $\tilde{K}(1) \leq \tilde{K}(0) = \tilde{K}_0$, then the sequence $(\hat{K}(\nu))$ defined by

$$\hat{K}(0) = \hat{K}_0, \quad \hat{K}(\nu+1) = \Phi(A, B, Q, \hat{K}(\nu)) + \Pi(\hat{K}(\nu)) \;\; \text{for} \;\; \nu \in \mathbf{N},$$

is bounded.
If $K_\infty = \tilde{K}_\infty$ then $\lim\limits_{\nu \to \infty} \hat{K}(\nu) = K_\infty$ exists.
This existence result is a consequence of Theorem 4.1.

The next theorem shows that the maximal solution of $(DGARE)$ depends - if it exists - monotonically on JH.

THEOREM 4.4. *Let* $\tilde{K} \in \mathbf{C}^{n,n}$ *be hermitian with* $I + B^*\tilde{K}B > 0$ *be a solution of*

$$\tilde{K} = \Phi(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{K}) + \Pi(\tilde{K}).$$

If

$$\begin{pmatrix} Q & A^* \\ A & -BB^* \end{pmatrix} \geq \begin{pmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -\tilde{B}\tilde{B}^* \end{pmatrix}$$

then the sequence $(K(\nu))$ *defined by* $K(0) = \tilde{K}$ *and* $K(\nu+1) = \Phi(A, B, Q, K(\nu)) + \Pi(K(\nu))$, $\nu \in \mathbf{N}$, *is nondecreasing. If* $(K(\nu))$ *is bounded then* $K := \lim\limits_{\nu \to \infty} K(\nu) \geq \tilde{K}$ *solves* $K = \Phi(A, B, Q, K) + \Pi(K)$.

Notice that the sequence $(K(\nu))$ defined in Theorem 4.4 is bounded under the assumptions of Remark 4.3 (ii).

Further results on the existence of positive semidefinite solutions of (unperturbed) algebraic Riccati equations can be found in [GH] and [W3].

5. EXISTENCE THEOREMS FOR THE SYMPLECTIC CASE WITH GENERAL INITIAL VALUES

Apart from the preceding treatment, where we have been using comparison results, in this section we propose a quite different approach for the symplectic or unperturbed (RDE). Here we mainly use properties of hamiltonian systems and the fact that there exists a transformation to a higher dimensional linear system. Although this approach is not applicable to $(GRDE)$ it is possible to extend it partially to nonsymmetric Riccati-differential equations appearing in Nash games. This will be studied in a forthcoming paper.
Let

$$H = \begin{pmatrix} A & -S \\ -Q & -A^* \end{pmatrix} \in \mathbf{C}^{2n,2n}$$

be a hamiltonian matrix as before and let

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{C}^{2n,2n}, \quad \text{where } \; a, b, c, d, \in \mathbf{C}^{n,n},$$

be a symplectic matrix such that

$$V^{-1}HV = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1^* \end{pmatrix}.$$

For simplicity we assume in the sequel that J_1 is diagonal and has all its eigenvalues in the open left half plane, i. e. $\sigma(J_1) \subset \mathbf{C}^-$. This describes the generic case and leads to the dichotomic solution of the associated algebraic Riccati equation (ARE) (see [FJ]); we note that problems with multiple eigenvalues can be treated similarly.
Let $W_0 \in \mathbf{C}^{n,n}$ with

$$\det Z_1 \neq 0, \quad \text{where } \; \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := V^{-1} \begin{pmatrix} I_n \\ W_0 \end{pmatrix}; \tag{5.1}$$

notice that (see [FJ]) the set $D^- = \{W_0 \in \mathbf{C}^{n,n} | W_0 \; \textit{has property} \; (5.1)\}$ is open and dense in $\mathbf{C}^{n,n}$. Then we set $Z_0 = Z_2 Z_1^{-1}$ and introduce the Möbius-type transformation (see [K])

$$W_0 = (c + dZ_0)(a + bZ_0)^{-1}.$$

Furthermore let Y denote a solution of $\dot{Y} = HY, \quad Y(t) \in \mathbf{C}^{2n,n}$, with initial value

$$Y(0) = V \begin{pmatrix} I_n \\ Z_0 \end{pmatrix} = \begin{pmatrix} a + bZ_0 \\ c + dZ_0 \end{pmatrix} = \begin{pmatrix} I_n \\ W_0 \end{pmatrix} Z_1^{-1};$$

Y has the representation

$$Y(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{J_1 t} & 0 \\ 0 & e^{-J_1^* t} \end{pmatrix} \begin{pmatrix} I_n \\ Z_0 \end{pmatrix} = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix},$$

where $Q(t), P(t) \in \mathbf{C}^{n,n}$.
¿From (5.1) it follows that $(a + bZ_0)^{-1} (= Z_1)$ exists; using the fact that Z_0 is uniquely determined by W_0 we infer that $(-W_0 b + d)^{-1}$ exists as well with

$$Z_0 = (-W_0 b + d)^{-1}(W_0 a - c).$$

To Y there corresponds a solution W of (RDE) defined by (see [FJ] for further details)

$$\begin{aligned} W(t) := P(t)Q(t)^{-1} &= (ce^{J_1 t} + de^{-J_1^* t}Z_0)(ae^{J_1 t} + be^{-J_1^* t}Z_0)^{-1} \\ &= (c + de^{-J_1^* t}Z_0 e^{-J_1 t})(a + be^{-J_1^* t}Z_0 e^{-J_1 t})^{-1} \\ &= (c + dZ(t))(a + bZ(t))^{-1} \end{aligned} \tag{5.2}$$

as long as $(a + bZ(t))^{-1}$ exists. Here we used $Z(t) := e^{-J_1^* t}Z_0 e^{-J_1 t}$; hence $Z(0) = Z_0$ and therefore $W(0) = W_0$ and $Z(t) = (-W(t)b + d)^{-1}(W(t)a - c)$, while $(-W(t)b + d)^{-1}$ exists.
V is symplectic, consequently we have (see [K]) $a^*d - c^*b = I_n = ad^* - bc^*$, and the matrices a^*c, b^*d, ab^* and cd^* are hermitian. Using these properties we get moreover

$$V^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}. \tag{5.3}$$

¿From $H = V \begin{pmatrix} J_1 & 0 \\ 0 & -J_1^* \end{pmatrix} V^{-1}$ it follows that

$$\begin{aligned} S &= aJ_1 b^* + bJ_1^* a^* \\ A &= aJ_1 d^* + bJ_1^* c^* \\ Q &= -cJ_1 d^* - dJ_1^* c^*. \end{aligned} \tag{5.4}$$

After these preparations and under assumption (5.1) we get the following results:

THEOREM 5.1. *For* $\rho \in \mathbf{R}$ *and* $W_0 \in D^-$ *we set*

$$R(\rho, W_0) = W_0(S + \rho bb^*)W_0^* - W_0(A + \rho bd^*) - (A^* + \rho db^*)W_0^* - Q + \rho dd^*$$

and we denote by $W = W(., W_0)$ *the solution of* (RDE) *with* $W(0, W_0) = W_0$ *(as represented in (5.2)).*
a) If $R(0, W_0) \geq 0$ *(or* ≤ 0*) then* $R(0, W(t)) \geq 0$ *(or* ≤ 0*, respectively) while* $W(t)$ *exists.*
b) If sign$(\rho)R(\rho, W_0) \geq 0$ *(or* ≤ 0*) then sign*$(\rho)R(\rho, W(t)) \geq 0$ *(or* ≤ 0*) for* $t \leq 0$ *(or* $t \geq 0$*, respectively,) while* $W(t)$ *exists.*
c) If sign(ρ) $R(\rho, W_0) \leq 0$ *then sign*(ρ) $R(0, W(t)) \leq 0$ *for* $t \leq 0$ *while* $W(t)$ *exists; notice that here we do not claim that sign*(ρ) $R(\rho, W(t)) \leq 0$.

Proof. We give a detailed proof for $\rho \geq 0$ and $R(\rho, W_0) \geq 0$; for all remaining cases the proof is similar. Defining $\phi(\rho) := -Z_0 J_1 - J_1^* Z_0^* + \rho I_n$ we get together with (5.4) by an elementary calculation

$$\begin{aligned} (d - W_0 b)\phi(\rho)(d^* - b^* W_0^*) &= W_0(S + \rho bb^*)W_0^* + W_0(-A - \rho bd^*) \\ &\quad + (-A^* - \rho db^*)W_0^* - Q + \rho dd^* = R(\rho, W_0). \end{aligned} \tag{5.5}$$

¿From this identity and $R(\rho, W_0) \geq 0$ we infer that $\phi(\rho) \geq 0$. Multiplying this from the left by $e^{-J_1^* t}$ and from the right by $e^{-J_1 t}$ gives

$$-Z(t)J_1 - J_1^* Z^*(t) \geq -\rho e^{-(J_1 + J_1^*)t} = -\rho e^{-2\,\mathrm{diag}(\mathrm{Re}\lambda_1, \cdots, \mathrm{Re}\lambda_n)t} I_n.$$

By our assumption Re $\lambda_1 < 0, \ldots,$ Re $\lambda_n < 0$ we obtain

$$-Z(t)J_1 - J_1^* Z^*(t) \geq -\rho I_n \quad \text{for} \quad t \leq 0. \tag{5.6}$$

¿From the proof of (5.5) it follows that (5.5) also holds if therein W_0 and $\phi(\rho)$ are replaced by $W(t)$ and $-Z(t)J_1 - J_1^* Z(t)^* + \rho I_n$, respectively. Therefore we obtain

$$W(t)(S + \rho bb^*)W^*(t)^- W(t)(A + \rho bd^*) - (A^* + \rho db^*)W^*(t) - Q + \rho dd^* \geq 0.$$

This holds for $t \leq 0$ as long as $W(t)$ exists. ■

REMARK 5.2. (i) Theorem 5.1 means that a generalized Riccati inequality remains valid for $t \leq 0$ or $t \geq 0$, respectively, if it holds for a terminal or initial value at $t = 0$. For $\rho = 0$ and hermitian matrices W_0 this is well known and follows from properties of Ljapunov equations (see Lemma 2.1 or [KK], Hilfssatz 10.4).
(ii) Theorem 5.1 is based on the assumption $W_0 \in D^-$ (i. e. $\det Z_1 \neq 0$), which has been used to ensure that W_0 is in the basin of attraction of the dichotomic solution as $t \to -\infty$ (see Remark 5.4).
If we assume instead of $\det Z_1 \neq 0$ that W_0 belongs to

$$D^+ = \{W_0 \in \mathbf{C}^{n,n} | \det Z_2 \neq 0 \; in \; (5.1)\},$$

then we can prove as before a modified version of Theorem 5.1. ¿From $\tilde{Z}_0 = Z_1 Z_2^{-1}$ it follows that $a\tilde{Z}_0 + b$ is regular with $W_0 = (c\tilde{Z}_0 + d)(a\tilde{Z}_0 + b)^{-1}$; moreover $W_0 a - c$ is regular and $\tilde{Z}_0 = (W_0 a - c)^{-1}(d - W_0 b)$.
Using $\tilde{Z}(t) := e^{J_1 t}\tilde{Z}_0 e^{J_1^* t}$ and $\tilde{\phi} := -\tilde{Z}_0 J_1 - J_1^* \tilde{Z}_0 + \rho I_n$ (instead of $Z(t)$ and $\phi(\rho)$), we obtain as before and with the abbreviation

$$\tilde{R}(\rho, W_0) = W_0(S + \rho aa^*)W_0^* - W_0(A + \rho ac^*) - (A^* + \rho ca^*)W_0^* - Q + \rho cc^* :$$

a) If $\tilde{R}(0, W_0) \geq 0$ *(or* ≤ 0*) then* $\tilde{R}(0, W(t)) \geq 0$ *(or* ≤ 0*, respectively,) while* $W(t)$ *exists.*
b) If sign$(\rho)\tilde{R}(\rho, W_0) \geq 0$ *(or* ≤ 0*) then sign*$(\rho)\tilde{R}(\rho, W(t)) \geq 0$ *(or* ≤ 0*) for* $t \geq 0$ *(or* $t \leq 0$*, respectively,) while* $W(t)$ *exists.*
c) If sign(ρ) $\tilde{R}(\rho, W_0) \leq 0$ *then sign*(ρ) $\tilde{R}(0, W(t)) \leq 0$ *for* $t \geq 0$ *while* $W(t)$ *exists; notice that here we do not claim that sign*(ρ) $R(\rho, W(t)) \geq 0$.
(iii) From the proof of Theorem 5.1 it follows that (5.6) and consequently also the assertions of Theorem 5.1 b), c) and Remark 5.2 b), c) could be sharpened; we omit details.

Theorem 5.1 and Remark 5.2 (ii) can be used to obtain *global* existence theorems.

THEOREM 5.3. *Let* $W_0 \in D^-$ *such that* $R(\rho, W_0) \leq 0$ *holds for some* $\rho \leq 0$. *If in addition* $S + \rho bb^* > 0$, *then the solution* $W = W(., W_0)$ *of* (RDE) *with* $W(0) = W_0$ *remains bounded (with respect to some norm on* $\mathbf{C}^{n,n}$*) on each bounded subinterval of* $(-\infty, 0]$, *i. e.* $W(\cdot, W_0)$ *has no poles on the negative real axis.*

Proof. Notice that from representation (5.2) it follows that the singularities of W are poles. Given any pole of W at $t_0 \in (-\infty, 0)$ there is a local series representation

$$W(t) = W_{-p}\frac{1}{(t - t_0)^p} + \cdots, \qquad W_{-p} \in \mathbf{C}^{n,n} \setminus \{0\}.$$

¿From $R(\rho, W(t)) \leq 0$ we infer (by taking the limit as $t \to t_0$)

$$W_{-p}(S + \rho bb^*)W_{-p}^* \leq 0.$$

Since $S + \rho bb^* > 0$ we infer that $W_{-p} = 0$, hence poles do not exist in $(-\infty, 0]$. ■

REMARK 5.4. (i) From the representation (5.2) we infer that $\lim_{t\to-\infty} W(t) = W^-$ exists if (5.1) holds and a^{-1} exists; we obtain in this case $W^- = ca^{-1}$, which obviously is a solution of (ARE). Notice that the *dichotomic solution* W^- of (ARE) exists (under our standing assumption $\sigma(J_1) \subset \mathbf{C}^-$) if and only if a^{-1} exists; for $S \geq 0$ it is well known that a^{-1} exists if and only if $(A, \sqrt{S})$ is stabilizable.
(ii) Let a^{-1} exist. It is known (see [FJ]) that

$$\mathbf{A\hat{C}}(W^-) = \{W_0 \in \mathbf{C}^{n,n} | \lim_{\mathbf{R}\ni t\to-\infty} W(t, W_0) = W^- \text{ with respect to the chordal metric}\},$$

the *(generalized) basin of attraction* of W^-, is open and dense in $\mathbf{C}^{n,n}$. Moreover it is known that

$$\mathbf{A\hat{C}}(W^-) = D^- = \{W_0 \in \mathbf{C}^{n,n} | W_0 \text{ has property (5.1)}\}$$

and that $W(\cdot, W_0)$ has at most a finite number of poles on $(-\infty, 0]$ if $W_0 \in \mathbf{A\hat{C}}(W^-)$. In general the form of

$$\mathbf{AC}(W^-) = \{W_0 \in \mathbf{C}^{n,n} | \lim_{\mathbf{R}\ni t\to-\infty} W(t, W_0) = W^- \text{ with respect to the euclidean metric,}$$
$$\text{where } W(\cdot, W_0) \text{ has no poles in } (-\infty, 0]\},$$

which is the *(restricted) basin of attraction* of W^-, cannot be described completely. Under the assumptions of Theorem 5.3 it follows that $W_0 \in \mathbf{A\hat{C}}(W^-)$ implies $W_0 \in \mathbf{AC}(W^-)$.
(iii) If $S + \rho bb^* > 0$ and $-Q + \rho dd^* \leq 0$ for some $\rho \leq 0$ then obviously $\|W_0\|$ can be chosen small enough such that the assumptions of Theorem 5.3 are fulfilled. In this case for example $W_0 = 0$ is an appropriate initial value for a solution $W(t)$ without singularities on $(-\infty, 0]$ if $0 \in D^-$.

Analogously to Theorem 5.3 we get the following existence results:

5.5 COROLLARY. *The solution W of (RDE) with $W(0) = W_0$ has no pole on the interval I if $\rho \in \mathbf{R}, W_0 \in \mathbf{C}^{n,n}$ and $I \subset \mathbf{R}$ satisfy at least one of the following 12 conditions:*

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S + \rho bb^* < 0,\ \rho \geq 0,\ I = (-\infty, 0] \tag{5.7}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S + \rho bb^* < 0,\ \rho \leq 0,\ I = [0, \infty) \tag{5.8}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \geq 0,\ S < 0,\ \rho \leq 0,\ I = (-\infty, 0] \tag{5.9}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S + \rho bb^* > 0,\ \rho \leq 0,\ I = (-\infty, 0] \tag{5.10}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S + \rho bb^* > 0,\ \rho \geq 0,\ I = [0, \infty) \tag{5.11}$$

or

$$W_0 \in D^-,\ R(\rho, W_0) \leq 0,\ S > 0,\ \rho \geq 0,\ I = (-\infty, 0] \tag{5.12}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0,\ S + \rho aa^* < 0,\ \rho \geq 0,\ I = [0, \infty) \tag{5.13}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0,\ S + \rho a a^* < 0,\ \rho \leq 0,\ I = (-\infty, 0] \tag{5.14}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \geq 0,\ S < 0,\ \rho \leq 0,\ I = [0, \infty) \tag{5.15}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S + \rho a a^* > 0,\ \rho \leq 0,\ I = [0, \infty) \tag{5.16}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S + \rho a a^* > 0,\ \rho \geq 0,\ I = (-\infty, 0] \tag{5.17}$$

or

$$W_0 \in D^+,\ \tilde{R}(\rho, W_0) \leq 0,\ S > 0,\ \rho \geq 0,\ I = [0, \infty). \tag{5.18}$$

Corollary 5.5 yields twelve different sufficient conditions for the global existence of $W(t)$, for $t \in I$; here condition (5.10) corresponds to Theorem 5.3.

5.6 EXAMPLE. Using a computer-algebra system like MAPLE it is very easy to produce nontrivial examples verifying one of the hypotheses of Corollary 5.5. Here we reproduce one of several examples which have been provided by our student Seung-Rae Lee using MAPLE:
Let

$$H = \begin{pmatrix} 1 & 0 & \frac{1}{15} & 0 \\ 0 & 2 & 0 & -1 \\ -4 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} A & -S \\ -Q & -A^T \end{pmatrix},$$

then it follows with

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & \frac{1}{330}\sqrt{165} \\ 1 & 0 & \frac{-1}{1\sqrt{6}} & 0 \\ 0 & -\sqrt{165} - 15 & 0 & \frac{1}{2} - \frac{1}{22}\sqrt{165} \\ 2 + \sqrt{6} & 0 & \frac{1}{6+2\sqrt{6}} & 0 \end{pmatrix}$$

that

$$V^{-1} H V = \text{diag}\ (-\sqrt{6}, -\frac{1}{15}\sqrt{165}, \sqrt{6}, \frac{1}{15}\sqrt{165}).$$

Moreover for example $W_0 = \begin{pmatrix} -31 & 0 \\ 0 & 3 \end{pmatrix} \in D^+ \cap D^-$ and $S + \rho a a^* = \begin{pmatrix} -\frac{1}{15} & 0 \\ 0 & 1 \end{pmatrix} + \rho I_2 > 0$ for $\rho > 1/15$, and

$$\tilde{R}(\frac{1}{2}, W_0) = \ \text{diag}\ (\frac{6133}{30} - 16\sqrt{165}, -\frac{3}{2} - \sqrt{6}) < 0.$$

Obviously here (5.17) is fulfilled, and consequently $W(t, W_0)$ exists for $t \leq 0$ with $\lim_{t \to -\infty} W(t, W_0) = W^- = \begin{pmatrix} -15 - \sqrt{165} & 0 \\ 0 & 2 + \sqrt{6} \end{pmatrix}$.
Notice that here S is indefinite (like for example in H^∞ control problems) moreover, the initial matrix W_0 and the stabilizing solution W^- of the algebraic Riccati equation are indefinite as well.

In the same way it follows from $S + \rho bb^* = \begin{pmatrix} -\frac{1}{15} + \frac{\rho}{660} & 0 \\ 0 & 1 + \frac{\rho(2+\sqrt{6})^2}{16(3+\sqrt{6})^2} \end{pmatrix} > 0$ for $\rho > 44$ that (5.11) is fulfilled for $\rho = 45$ and $W_0 := \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \in D^+$; hence in this case $W(t, W_0)$ exists for $t \geq 0$ with $\lim_{t\to\infty} W(t, W_0) = W^+ = db^{-1} = \begin{pmatrix} \sqrt{165} - 15 & 0 \\ 0 & 2 - \sqrt{6} \end{pmatrix}$.
Since $-Q + \rho dd^* < 0$ for $\rho < \frac{24}{2\sqrt{6}-5} = 237.5755..$, it follows from (5.11) and $S + \rho bb^* > 0$ for $\rho > 44$ that there exists a neighborhood U of 0 such that $W(t, W_1)$ exists for $t \geq 0$ and $W_1 \in U \cap D^+$ with $\lim_{t\to\infty} W(t, W_1) = W^+$.
Examples of this and of similar type show how Corollary 5.5 can be used to ensure the global existence of solutions of initial value problems for Riccati differential equations.

Another example, which shows how Corollary 5.5 can be used to check the existence of $W(t, W_f)$ in $(-\infty, t_f]$ for different terminal values W_f is given in [FLJA]; this example indicates how one can use Corollary 5.5 to approximate the restricted basin of attraction of W_s.

References

[AFJ1] H. Abou-Kandil, G. Freiling and G. Jank. Solution and asymptotic behavior of coupled Riccati equations in jump-linear systems. *IEEE Transactions on Automatic Control* **AC-39**,(1994) No. 8, 1631-1635.

[AFJ2] H. Abou-Kandil, G. Freiling and G. Jank. On the solution of discrete-time Markovian jump linear quadratic control problems, *Automatica* **31** (1995) 765 - 768.

[BG] R. R. Bitmead, M. R. Gevers. Riccati Difference and Differential Equations: Convergence, Monotonicity and Stability. In: S. Bittanti et al., Eds., *The Riccati Equation,* (1991) pp. 263 - 291, Springer, New York.

[CM] P. E. Caines, D.Q. Maine. On the discrete time matrix Riccati equation. *Internat. J. Control* **12** (1970), 785-794.

[CO] W. A. Coppel. *Disconjugacy.* Springer Lecture Notes in Math., Springer, Berlin, 1971.

[dS] C. E. De Souza.(1989). On Stabilizing Properties of Solutions of the Riccati Difference Equation. *IEEE Transactions on Automatic Control*, **AC-34**, No.12, 1313-1316.

[dSF1] C. E. de Souza, M.D. Fragoso. On the existence of maximal solution for generalized algebraic Riccati equations arising in stochastic control. *Systems & Control Letters* **14** (1990), 233-239.

[dSF2] C. E. de Souza, M.D. Fragoso. H^∞-Control for linear systems with Markovian jumping parameters. *Control Theory and Advanced Technology*, **9** (1993), No 2, 457 - 466.

[F] L. E. Faibusovich. Matrix Riccati inequality: Existence of solutions. *Systems & Control Letters* **9** (1987), 59-64.

[FdS] M. C. Fragoso, C.E. de Souza. Maximal solution of a certain class of periodic Riccati differential equations. *Linear Algebra Appl.* **169** (1992), 61-73.

[FJ] G. Freiling, G. Jank. Non-Symmetric Matrix Riccati Equations. *Zeitsch rift für Analysis und ihre Anwendungen* **14** (1995) 2, 259 - 284 .

[FJA] G. Freiling, G. Jank and H. Abou-Kandil. Generalized Riccati differential and difference equations. *Lin. Alg. Appl. 237/238*, (1996), to appear.

[FLJA] G. Freiling, S.-R. Lee, G. Jank and H. Abou-Kandil. On the Dependence of the Solutions of Algebraic and Differential Game Riccati Equations on the Parameter μ. *Europ. J. Contr.* , to appear.

[GH] A. H. W. Geerts and M. L. J. Hautus. The output-stabilizable subspace and linear optimal control, *Proceedings MTNS 89*, (M. A. Kaashoek et al., eds.), pp. 113-120, Birkhäuser, Boston 1990.

[GLR] I. Gohberg, P. Lancaster and L. Rodman. On hermitian solutions of the symmetric algebraic Riccati equation. *SIAM J. Contr. Opt.* **24** (1986), 1323-133.

[K] H. Klingen. Introductory lectures on Siegel modular forms. *Cambridge University Press*, Cambridge, 1988.

[KK] H. W. Knobloch, H. Kwakernaak. *Lineare Kontrolltheorie.* Springer-Verlag, Berlin, Heidelberg, 1985.

[KA] O.I. Kosmidou, H. Abou-Kandil, Robust control for systems with structured uncertainties: the discrete time case. *Proceedings of the European Control Conference, Groningen* (1993) 745 - 749.

[KB] O.I. Kosmidou, P. Bertrand, Robust controller design for systems with large parameter variations. *Int. J. Control* **45** (1987) 927 - 938.

[LRR] P. Lancaster, A.C.M. Ran and L. Rodman. An existence and monotonicity theorem for the discrete algebraic matrix Riccati equation. *Linear and Multilinear Algebra*, **20** (1987), 353-361.

[M] M. Mariton, *Jump Linear Systems in Automatic Control.* Marcel Dekker Inc., New York, 1990.

[RV] A. C. M. Ran, R. Vreugdenhil. Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems. *Linear Algebra Appl.* **99** (1988), 63-83.

[S] C. W. Scherer. The General Nonstrict Algebraic Riccati Inequality. Preprint, submitted.

[W1] H. K. Wimmer. Monotonicity of maximal solutions of algebraic Riccati equations. *Systems & Contr. Letters*, **5** (1985), 317-319.

[W2] H. K. Wimmer. Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations. *J. Math. Systems, Estim. & Contr.* , **2** (1992), 219-235.

[W3] H. K. Wimmer, Existence of positive-definite and semidefinite solutions of discrete-time algebraic Riccati equations, *Int. J. Control*, **59** (1994), 463-471.

[WP] H. K. Wimmer and M. Pavon. A comparison theorem for matrix Riccati difference equations. *Systems & Contr. Letters*, **19** (1992), 233-239.

[Wo] W. M. Wonham. On a matrix Riccati equation of stochastic control. *SIAM J. Contr.*, **6** (1968), No. 4, 681-697.