

Lyapunov-type Functions and Invariant Sets for Riccati Matrix Differential Equations *

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Abstract: We present two different methods to obtain global existence results for solutions of nonsymmetric Riccati matrix differential equations. In the first approach we derive sufficient conditions ensuring that the spectral norm of the solutions remains uniformly bounded in an interval $(-\infty, t_0]$; in a second part we make use of the linearizability of the Riccati matrix differential equation. With the aid of an appropriate Lyapunov-type function we obtain sufficient conditions guaranteeing that no finite escape time of solutions can occur. These results are then applied to open loop Nash strategies as well as to H_∞ -type and related Riccati differential equations. A complete solution of a problem from [ThVo] is obtained and two examples show how the methods work.

1. Introduction and Problem Formulation

Riccati differential equations appear in a natural way in variational problems, control theory, differential games, factorizations of rational matrix functions, etc. (see [LaRo], [BO95]).

The most developed case is that of symmetric equations with constant coefficients. An important question is the existence of solutions on a fixed interval. The theory of these differential equations is closely connected with

algebraic Riccati equations.

Here we study nonsymmetric Riccati differential equations with possibly variable coefficients. These equations are of mathematical interest and appear also in several fields of applied sciences. Examples can be found in [Kui94], [Kui95] and [ThVo].

In this paper we concentrate on questions of global existence. Unlike in the case of periodic or constant coefficients, in general an analytical representation of solutions is not possible thus one has to look for qualitative methods to study their behavior. Our main tools are Lyapunov-type arguments by which we construct some invariant sets for nonsymmetric nonautonomous Riccati differential equations. With this general method we also treat some particular cases like Riccati equations appearing in open loop Nash games, and optimal control problems with linear state constraints.

We consider *nonsymmetric nonautonomous matrix differential equations* of the form

$$\dot{W} = B_{21}(t) + B_{22}(t)W - WB_{11}(t) - WB_{12}(t)W, \quad (1)$$

where

$$B_{11}(t) \in \mathbf{R}^{r_1 \times r_1}, \quad B_{12}(t) \in \mathbf{R}^{r_1 \times r_2}, \quad B_{21}(t) \in \mathbf{R}^{r_2 \times r_1},$$

$$B_{22}(t) \in \mathbf{R}^{r_2 \times r_2}, \quad W(t) \in \mathbf{R}^{r_2 \times r_1},$$

and with piecewise continuous (or Riemann-integrable) coefficient matrices $B_{ij}(t)$ in $(-\infty, T]$. Our main goal is to obtain conditions on the coefficients and initial data which guarantee the existence of solutions on $(-\infty, t_0]$ for $t_0 \leq T$. In particular, by using Lyapunov-type arguments, we construct bounded negative invariant sets for equation (1).

2. Preliminaries

Let us recall first the well known relation between matrix Riccati differential equations (1) and linear systems of differential equations.

If Y is a solution of the initial value problem

$$\dot{Y}(t) = B(t)Y(t), \quad Y(t_0) = \begin{pmatrix} I_{r_1} \\ W_0 \end{pmatrix} \quad (2)$$

with

$$B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix},$$

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then $W(t) = Y_2(t)Y_1^{-1}(t)$ is a solution of equation (1) as long as $Y_1^{-1}(t)$ exists. Notice that one of our main goals in this paper is to find sufficient conditions ensuring that $Y_1(t)$ is nonsingular on a given interval.

If the coefficient matrices are constant then there exists a similar relation between solutions of the algebraic Riccati equation

$$0 = B_{21} + B_{22}W - WB_{11} - WB_{12}W, \quad (3)$$

and the r_1 -dimensional B -invariant graph subspaces ; i.e. B -invariant subspaces of the form $\text{Im} \begin{pmatrix} I_{r_1} \\ W \end{pmatrix}$ of

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

If $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is a basis of such an invariant graph subspace then $\det Y_1 \neq 0$ and $W = Y_2Y_1^{-1}$ is a solution of (3). Provided the eigenvalues of B satisfy the dichotomy condition

$$\Re \lambda_1 \leq \Re \lambda_2 \leq \dots \leq \Re \lambda_{r_1} < \Re \lambda_{r_1+1} \leq \dots \leq \Re \lambda_{r_1+r_2}$$

and if Y is a basis of the invariant subspace of B corresponding to the eigenvalues $\lambda_1, \dots, \lambda_{r_1}$ then the dichotomic solution $W = Y_2Y_1^{-1}$ of (3) or (1) exists whenever Y_1 is nonsingular. As it has been shown in [FrJa] are all solutions of the differential equation (1) with initial data from some open and dense set in $\mathbf{R}^{r_2 \times r_1}$ attracted (as t tends to $-\infty$) either by this dichotomic solution or have finite escape time.

3. Negative invariant bounded domains for nonsymmetric nonautonomous matrix Riccati differential equations

In this section we derive uniform estimates on the operator norm $\|W(t)\|$ of solutions of (1) in an interval $(-\infty, t_0)$. By the operator norm of a rectangular matrix W in the sequel we mean

$$\begin{aligned} \|W\| &= \max\{\|Wx\|_{\mathbf{R}^{r_2}} : \|x\|_{\mathbf{R}^{r_1}} = 1\} \\ &= \max\{\lambda : \lambda = \sqrt{\sigma}, \sigma \in \text{spec}(W^T W)\} \end{aligned}$$

Theorem 3.1 (Negative invariance). *Let $B_{11}, B_{12}, B_{21}, B_{22}$ be Riemann-integrable on $(-\infty, T]$. If for some $\gamma > 0$ and every $t \leq t_0 \leq T$ there holds*

$$\begin{aligned} (\eta_1^T, \eta_2^T) \begin{pmatrix} -B_{11}(t) - B_{11}^T(t) & B_{21}^T(t) - B_{12}(t) \\ B_{21}(t) - B_{12}^T(t) & B_{22}(t) + B_{22}^T(t) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ \geq \gamma(|\eta_1|^2 + |\eta_2|^2) \end{aligned} \quad (4)$$

for all $\eta_1 \in \mathbf{R}^{r_1}, \eta_2 \in \mathbf{R}^{r_2}$, then the ball $\|W\| < 1$ is negative invariant for solutions W of (1), i.e. if

$W(t, W_0)$ denotes a solution of (1) with $W(t_0, W_0) = W_0$ and if $\|W_0\| < 1$ then also $\|W(t, W_0)\| < 1$ holds for all $t \leq t_0$.

Proof. (i) Since it is possible to approximate each of the coefficients of (1) uniformly by piecewise constant matrices and since the solutions of (1) depend continuously on the coefficients (see [CoLe], Chapter 2.4) it is sufficient to prove Theorem 3.1 for constant coefficients.

(ii) In the case of constant coefficients it is known that $W = W(\cdot, W_0)$ is a holomorphic matrix function as long as it is bounded.

Let us consider the symmetric, positive semi-definite $(r_1 \times r_1)$ -Matrix $W^T(t)W(t)$. As mentioned before the operator norm of $W(t)$ is equal to the square root of the maximal eigenvalue of $W^T(t)W(t)$, which is denoted by $\lambda(t)$.

Since $W(t)$ is holomorphic in t as long as $\|W\| < 1$ it follows from [Ba], section 3.5, that for these values of t , with the exception of at most a discrete subset, $\lambda(t)$ is differentiable; moreover for these t with $t \in \mathbf{R}$ there exists a basis of eigenvectors of $W^T(t)W(t)$ depending holomorphically on t (see [Ba], section 3.5.5, Corollary 2).

Assume now $\lambda(t) < 1$ and $x(t) \in \mathbf{R}^{r_1}$ is a corresponding holomorphic unit eigenvector:

$$W^T(t)W(t)x(t) = \lambda(t)x(t), \quad x^T(t)x(t) = 1.$$

Using

$$\lambda(t) = x^T(t)W^T(t)W(t)x(t)$$

we obtain by differentiation

$$\begin{aligned} \dot{\lambda} &= \frac{d}{dt}(x^T(t)W^T(t)W(t)x(t)) = \\ &= 2x^T(t)W^T(t) \frac{d}{dt}(W(t)x(t)) = \\ &= 2x^T(t)W^T(t)[\dot{W}(t)x(t) + W(t)\dot{x}(t)] = \\ &= 2x^T(t)W^T(t)\dot{W}(t)x(t) + 2\lambda(t)x^T(t)\dot{x}(t). \end{aligned}$$

From $x^T(t)x(t) \equiv 1$ we infer (by differentiation) that $x^T(t)\dot{x}(t) = 0$ a.e., hence

$$\dot{\lambda}(t) = 2x^T(t)W^T(t)\dot{W}(t)x(t).$$

Inserting (1) (and suppressing time parameters) yields

$$\begin{aligned} \frac{1}{2}\dot{\lambda} &= x^T W^T (B_{21} + B_{22}W - WB_{11} - WB_{12}W)x = \\ &= x^T W^T B_{21}x + x^T W^T B_{22}Wx - \lambda x^T B_{11}x - \lambda x^T B_{12}Wx. \end{aligned}$$

Since this equality depends only on the symmetric parts of the matrices involved, we may rewrite it in symmetric form

$$\begin{aligned} \dot{\lambda} &= x^T W^T (B_{21} - \lambda B_{12}^T)x + x^T W^T (B_{22} + B_{22}^T)Wx - \\ &\quad - \lambda x^T (B_{11} + B_{11}^T)x + x^T (B_{21}^T - \lambda B_{12})Wx. \end{aligned}$$

Let us put now $\eta_1 = x \in \mathbf{R}^{r_1}$, $\eta_2 = Wx \in \mathbf{R}_2^r$, then we obtain

$$\dot{\lambda} = (\eta_1^T, \eta_2^T) \begin{pmatrix} -\lambda(B_{11} + B_{11}^T) & B_{21}^T - \lambda B_{12} \\ B_{21} - \lambda B_{12}^T & B_{22} + B_{22}^T \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

If $\lambda(t)$ is sufficiently close to 1 then by virtue of condition (4) we obtain $\dot{\lambda}(t) > 0$ (for $x \neq 0$). Assume, there is a $t_1 < t_0$ such that $\lambda(t_1) \geq 1$. Since $\|W_0\| < 1$, there exists some $t^* \in [t_1, t_0]$ where $\lambda(t^*) = 1$ and $\dot{\lambda}(t^*) < 0$ which leads to a contradiction. This proves $\lambda(t) < 1$. From the proof it can also be seen that the set $\|W\| > 1$ is positive invariant.

The result of Theorem 3.1 is stronger than that derived recently by Kuiper [Kui94] using a similar method, since we removed superfluous assumptions on the coefficients. It should be pointed out that this method also can be used in the infinite dimensional case, i.e. in the case of operator Riccati differential equations (see [Kui94]). Another slight generalization of Theorem 3.1 is obtained by taking a modified norm; analogously to Theorem 3.1 it can be proved:

Corollary 3.2 *If for some $a, \gamma > 0$, a positive definite matrix $C \in \mathbf{R}^{r_2 \times r_2}$ and every $t \leq t_0$ there holds*

$$\begin{aligned} (\eta_1^T, \eta_2^T) \begin{pmatrix} -a(B_{11}(t) + B_{11}^T(t))B_{21}^T(t)C - aB_{12}(t) \\ CB_{21}(t) - aB_{12}^T(t) & CB_{22}(t) + B_{22}^T(t)C \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ \geq \gamma(|\eta_1|^2 + |\eta_2|^2) \end{aligned} \quad (5)$$

for all $\eta_1 \in \mathbf{R}^{r_1}, \eta_2 \in \mathbf{R}^{r_2}$, then the set $\|W\|_C < \sqrt{a}$ is negative invariant for solutions W of (1), where

$$\|W\|_C^2 = \max\{\text{spect}(W^T C W)\}.$$

The latter corollary allows to include several parameters in order to obtain reasonable results. These parameters should be adapted somehow to the problem under consideration. It turns out to be an advantage of this approach that it applies also to non linearizable differential equations, this will be discussed in a subsequent article.

4. Non-blow-up conditions for nonsymmetric nonautonomous matrix Riccati differential equations

A second approach is to obtain quadratic Lyapunov-type functions. Here we make use of the linearizability of Riccati differential equations as described in section 2. From the remarks made there it is clear that a *blow up* of the solution $W = W(\cdot, W_0)$ of the Riccati differential equation occurs at moments where $\det Y_1(t)$ vanishes. To formulate non-blow-up conditions we introduce the

quadratic Lyapunov-type function

$$V(t) := x^T (Y_1^T(t) C Y_1(t) + Y_1^T(t) D Y_2(t) + Y_2^T(t) D^T Y_1(t)) x. \quad (6)$$

Here $0 \neq x \in \mathbf{R}^{r_1}$, $C \in \mathbf{R}^{r_1 \times r_1}$, $C^T = C$, $D \in \mathbf{R}^{r_1 \times r_2}$ are again some parameters. Evidently

$$V_0 := V(t_0) = x^T (C + D W_0 + W_0^T D^T) x. \quad (7)$$

Calculating the derivative of $V(t)$ in (6), along a solution of (2), we obtain (suppressing t)

$$\begin{aligned} \dot{V} &= 2x^T [Y_1^T C (B_{11} Y_1 + B_{12} Y_2) + \\ &+ (Y_1^T B_{11}^T + Y_2^T B_{12}^T) D Y_2 + Y_1^T D (B_{21} Y_1 + B_{22} Y_2)] x \end{aligned}$$

and therefore

$$\begin{aligned} \dot{V} &= 2x^T Y_1^T (C B_{11} + D B_{21}) Y_1 x + \\ &+ 2x^T Y_1^T (C B_{12} + B_{11}^T D + D B_{22}) Y_2 x + \\ &+ 2x^T Y_2^T B_{12}^T D Y_2 x. \end{aligned}$$

This we may rewrite as

$$\begin{aligned} \frac{1}{2} \dot{V} &= (x^T Y_1^T, x^T Y_2^T) \times \\ &\times \begin{pmatrix} C B_{11} + D B_{21} & C B_{12} + B_{11}^T D + D B_{22} \\ 0 & B_{12}^T D \end{pmatrix} \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \\ &= (x^T Y_1^T, x^T Y_2^T) L \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \end{aligned}$$

or in symmetrized representation

$$\dot{V} = (x^T Y_1^T, x^T Y_2^T) (L + L^T) \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \quad (8)$$

where

$$L = \begin{pmatrix} C B_{11} + D B_{21} & C B_{12} + B_{11}^T D + D B_{22} \\ 0 & B_{12}^T D \end{pmatrix}. \quad (9)$$

Now we are ready to prove

Theorem 4.1 *Let $B_{11}, B_{12}, B_{21}, B_{22}$ be piecewise continuous on $(-\infty, T]$. If for some matrices $C \in \mathbf{R}^{r_1 \times r_1}$, with $C^T = C$, $D \in \mathbf{R}^{r_1 \times r_2}$ holds*

$$L(t) + L^T(t) \leq 0 \quad (10)$$

for all $t \leq t_0 \leq T$ and if

$$\begin{aligned} C + D W_0 + W_0^T D^T > 0, \text{ or} \\ C + D W_0 + W_0^T D^T \geq 0, \text{ (} D \text{ of full rank)} \end{aligned} \quad (11)$$

for some $W_0 \in \mathbf{R}^{r_2 \times r_1}$, then the solution $W(t, W_0)$ of (1) with $W(t_0, W_0) = W_0$ exists for all $t \leq t_0$.

Proof. From section 2 and with the notation used there we infer that $W(t, W_0)$ exists as long as $\det Y_1(t) \neq 0$, (here $Y_1(t), Y_2(t)$ define a solution of the initial value

problem (2)). Together with (2), (6), (8) and (10) we obtain that $\dot{V}(t)$ is piecewise continuous and $\dot{V}(t) \leq 0$ in $(-\infty, t_0]$.

Hence, $V(t)$ is monotonically decreasing in $(-\infty, t_0]$.

Assume now that $C + DW_0 + W_0^T D^T > 0$ holds; from (2), (6), (8) and (11) we then infer

$$V(t_0) = x^T(C + DW_0 + W_0^T D^T)x > 0,$$

which yields together with the monotonicity of $V(t)$ that $V(t) > 0$ for all $t \leq t_0$. Hence, $Y_1(t)$ must be regular in $(-\infty, t_0]$ which implies the existence of $W(t)$ in $(-\infty, t_0]$. If in (11) only $C + DW_0 + W_0^T D^T \geq 0$ holds, then for $\epsilon > 0$ we solve the initial value problem (2) with

$$Y_1^\epsilon(t_0) = I, Y_2^\epsilon(t_0) = W_0 + \epsilon D^T.$$

From (7) we infer for $\epsilon > 0$

$$V_0^\epsilon = x^T(C + DW_0 + W_0^T D^T + 2\epsilon D D^T)x > 0$$

which as before yields the regularity of $Y_1^\epsilon(t)$ in $(-\infty, t_0]$ for all $\epsilon > 0$.

From the continuous dependence of the existence interval of a solution of a differential equation from its initial value (see [CoLe], 1, Theorem 7.1) we infer that for an arbitrary closed subinterval $[\tilde{t}, t_0] \subset (-\infty, t_0]$ the matrix $Y_1^{\epsilon=0}(t)$ is regular on $(\tilde{t}, t_0]$. This again implies the existence of $W(t)$ in $(-\infty, t_0]$. In [Fe] a similar result was obtained for nonsymmetric but quadratic matrices W_0 .

5. Applications to open loop Nash games and related problems

In the preceding existence theorems it is not a priori clear how to choose the parameters in order to obtain reasonable good results. Here we will propose some possibilities of the choice and show by examples how these approaches work. First we start with the coupled Nash system .

5.1 Application of global existence results to Open Loop Nash differential games

In the theory of dynamical games there appear systems of coupled matrix Riccati differential equations; here we deal with open loop Nash games . In this case the system can be written as one system with rectangular matrices of the form (see for instance [AbFJ]):

$$\begin{aligned} \begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} &= - \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \\ &+ \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (S_1 S_2) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \quad A, K_i, Q_i, S_i \in \mathbf{R}^{n \times n}, \\ Q_i &= Q_i^T, S_i = S_i^T, \quad i = 1, 2. \end{aligned} \tag{12}$$

where all coefficients are constant. Therefore we also have

the associated algebraic (Nash-type) Riccati equation:

$$\begin{aligned} 0 &= - \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \\ &+ \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (S_1 S_2) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}. \end{aligned} \tag{13}$$

Here we are mainly interested in the dynamic behavior around a particular *stationary* solution - namely the dichotomic solution. If one wants to apply Theorem 3.1 or Corollary 3.2, the method will strongly depend on the existence of an attracting solution of (13) which should be contained in the considered ball. It clearly would be quite unnatural to take the ball ‘‘around’’ 0, although 0 is not a solution at all.

Therefore we propose to investigate instead of (12) left invariant domains for a transformed equation.

If (13) admits a dichotomic solution

$$K_d = \begin{pmatrix} K_{d,1} \\ K_{d,2} \end{pmatrix}$$

and if $K(t) = \begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix}$ is a solution of (12) then

$$\hat{K}(t) = \begin{pmatrix} \hat{K}_1(t) \\ \hat{K}_2(t) \end{pmatrix} = \begin{pmatrix} K_1(t) - K_{d,1} \\ K_2(t) - K_{d,2} \end{pmatrix}$$

is a solution of

$$\begin{aligned} \frac{d}{dt} \hat{K} &= \hat{B}_{22} \hat{K} - \\ &- \hat{K} (A - S_1 K_{d,1} - S_2 K_{d,2}) + \hat{K} (S_1, S_2) \hat{K}. \end{aligned} \tag{14}$$

where $\hat{B}_{22} = - \begin{pmatrix} A^T - K_{d,1} S_1 & -K_{d,1} S_2 \\ -K_{d,2} S_1 & A^T - K_{d,2} S_2 \end{pmatrix}$. and where $A_{cl} = A - S_1 K_{d,1} - S_2 K_{d,2}$ denotes the closed loop matrix of the system.

Applying for instance Corollary 3.2 with $C = I$ yields:

If for some $a > 0$

$$\begin{pmatrix} -a(A_{cl} + A_{cl}^T) & aS_1 & aS_2 \\ aS_1 & \hat{B}_{22}^T + \hat{B}_{22} & \\ aS_2 & & \end{pmatrix} > 0, \tag{15}$$

holds then $\| \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \| < \sqrt{a}$ is negative invariant.

From the results in [FrJa], page 278, it follows that (15) can only hold if $K_{d,1}, K_{d,2}$ define the *dichotomic solution*.

For all other constant solutions there cannot exist such a negative invariant set of initial data except if the solution

is already contained in the ball $\| \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \| < \sqrt{a}$, since

the dichotomic solution is the only attractive solution (as $t \rightarrow -\infty$).

As it can be seen by examples, the quality of the results depend on a suitable choice of the parameters in the criterion; in particular the method seems to work satisfactory if A is stable.

Let us now apply the *Lyapunov-function approach* to OL-Nash differential games.

of the eigenvalues of the matrix appearing in (5) is $\{.067860928, 2.633925415, 5.206934869, 9.184240949\}$. Hence Corollary 3.2 implies that the solution $W(t, W_0)$ exists for $t \leq t_0$.

In a similar way this method could also be used to obtain global existence results for Riccati differential equations as they are used by Thompson and Volz [ThVo] where the “only” difference to (SRDE) is that Q is *not* symmetric anymore. Then the solutions are nonsymmetric, but Corollary 3.2 may be applied if there is a dichotomic solution.

A more general Riccati differential equation is introduced by Kuiper [Kui94] in order to study a diffusion process. Even more general Riccati differential equations can be treated by the previous method where the coefficient matrices are arbitrary square matrices without symmetry properties. If the dichotomic solution exists an application of Corollary 3.2 would also lead to global existence results.

(ii) In the second part of this section we turn towards an application of Theorem 4.1 to nonsymmetric differential equations (17) with variable coefficients, where also Q and S are allowed to be nonsymmetric. We obtain, for the matrix appearing in the criterion of Theorem 4.1

$$(L + L^T)(t) = \begin{pmatrix} CA + A^T C - DQ - Q^T D^T & -CS + A^T D - D A^T \\ -S^T C + D^T A - A D^T & -S^T D - D^T S \end{pmatrix}. \quad (22)$$

If we choose here for instance, applying Theorem 4.1,

$$C = 0, D = I \quad (23)$$

then for an initial matrix P_0 with $P_0 + P_0^T \geq 0$ condition (11) is fulfilled and condition (10) here has the form

$$L + L^T(t) = \begin{pmatrix} -(Q + Q^T) & 0 \\ 0 & -(S + S^T) \end{pmatrix}(t) \leq 0,$$

which is fulfilled if and only if $Q + Q^T \geq 0$ and $S + S^T \geq 0$. This completely solves a problem of global existence as stated in [ThVo].

In the case of symmetric positive semidefinite matrices Q , S , and P_0 this yields a classical existence result which can also be derived by other methods.

Instead of the choice in (23) for instance we could choose, as we already did before,

$$C = 0, D = P_0 \quad (24)$$

with the initial matrix P_0 of full rank. From Theorem 4.1 and (22) we obtain

$$L + L^T = \begin{pmatrix} -P_0 Q - Q P_0^T & A^T P_0 - P_0 A^T \\ P_0 A - A P_0^T & -S P_0 - P_0^T S \end{pmatrix}$$

and if this matrix turns out to be negative semidefinite then the solution $P(t, P_0)$ of (17), where $P(t_0, P_0) = P_0$, does not have finite escape time in $(-\infty, t_0]$. This shows that beyond the classical results there exist solutions of (17) without finite escape time in $(-\infty, t_0]$ where S, Q, P_0 need not be positive semidefinite; moreover P_0 and hence also $P(t, P_0)$ need not be symmetric. This shows for instance the applicability of Theorem 4.1 in H_∞ -type problems.

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