DISCRETE TIME RICCATI EQUATIONS IN OPEN LOOP NASH AND STACKELBERG GAMES

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Abstract :We study the asymptotic behavior of difference equations appearing in the necessary optimality conditions of noncooperative open loop Nash and Stackelberg games. Moreover we study also the properties of the solutions of the corresponding algebraic Riccati-type equations.

1. Introduction and Problem Formulation.
Generalized Riccati equations play a prominent role in dynamic nonzero-sum games. In this paper we consider nonsymmetric coupled Riccati equations occurring in open loop discrete-time linear-quadratic Nash and Stackelberg games. For the continuous time case, the corresponding equations were examined in [2], [3] and [4]. The purpose of this paper is to study the discrete-time equations as well as their asymptotic behavior. For the sake of clarity, we restrict the presentation here to two player games of the form

$$x(k+1) = Ax(k) + B_1 u_1(k) + B_2 u_2(k), \quad x(0) = x_0, \tag{1}$$

with $x(k) \in \mathbf{R}^n$, $u_i(k) \in \mathbf{R}^{r_i}$, $1 \le i \le 2,\ 0 \le k \le N-1$.

The cost functionals of the players are defined by

$$J_1 = \frac{1}{2} x^T(N) K_{1N} x(N) + \tag{2a}$$

$$\frac{1}{2}\sum_{k=0}^{N-1} \left[x^T(k)Q_1x(k) + u_1^T(k)R_{11}u_1(k) + u_2^T(k)R_{12}u_2(k)\right]$$

$$J_2 = \frac{1}{2} x^T(N) K_{2N} x(N) + \tag{2b}$$

$$\frac{1}{2}\sum_{k=0}^{N-1} \left[x^T(k)Q_2x(k) + u_1^T(k)R_{21}u_1(k) + u_2^T(k)R_{22}u_2(k)\right],$$

where all matrices are symmetric with

$$K_{iN}, Q_i \ge 0 \qquad \text{and} \qquad R_{ii} > 0 \qquad \text{for} \qquad i = 1, 2.$$

Moreover we assume here that the information structure of both players is of open loop type.

1.1 Nash Strategy. It has been shown by Pindyck [7] (see also [5]) that the necessary conditions for an open loop Nash strategy for the game defined by (1), (2) are given by

$$u_1(k) = -R_{11}^{-1} B_1^T \psi_1(k+1),$$
$$u_2(k) = -R_{22}^{-1} B_2^T \psi_2(k+1), \tag{3}$$

where the costate vectors $\psi_i(k)$ must satisfy

$$\psi_i(k) = Q_i x(k) + A^T \psi_i(k+1), \tag{4}$$

$$\psi_i(N) = K_{iN} x(N), \;\; 1 \le i \le 2, \;\; 0 \le k \le N-1.$$

Due to the linearity of the above equations *we suppose here* that

$$\psi_i(k) = K_i(k)x(k), \;\; 1 \le i \le 2, \;\; 0 \le k \le N, \tag{5}$$

which implies that (1) can be written as

$$0 = Ax(k) - M_{k+1}x(k+1) \tag{6}$$

where $M_{k+1} = [I + S_1K_1(k+1) + S_2K_2(k+1)]$, or, if M_{k+1} is invertible,

$$x(k+1) = M_{k+1}^{-1} A x(k), \tag{7}$$

$0 \le k \le N-1$ with $x(0) = x_0$ and $S_i = B_i R_{ii}^{-1} B_i^T,\ 1 \le i \le 2$. Here and in the sequel $I \in \mathbf{R}^{n\times n}$ denotes the identity matrix.
¿From (4),(5) and (6) we infer that K_1, K_2 must be solutions of the discrete -time open loop Nash Riccati difference equations

$$K_1(k) = Q_1 + A^T K_1(k+1) M_{k+1}^{-1} A, \quad K_1(N) = K_{1N},$$
$$K_2(k) = Q_2 + A^T K_2(k+1) M_{k+1}^{-1} A, \quad K_2(N) = K_{2N}, \tag{8}$$

provided the inverses in (8) exist.
The asymptotic behavior of such difference equations is discussed in section 2 of this paper where necessary conditions are established for the existence of constant solutions of (8), i.e. solutions of the coupled algebraic Riccati equations

$$K_1 = Q_1 + A^T K_1 \left[I + S_1K_1 + S_2K_2\right]^{-1} A,$$

$$K_2 = Q_2 + A^T K_2 [I + S_1 K_1 + S_2 K_2]^{-1} A. \tag{9}$$

In the sequel we write the (pairs of) solutions of (9) in the form $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$.

1.2 Stackelberg Strategy. If one of the players (the leader) has the ability to enforce his strategy on the second player (the follower) then one has to introduce a hierarchical equilibrium solution concept as proposed by H. von Stackelberg in 1934 (see [5] for details).
When player 2 acts as a leader and player 1 as follower the necessary conditions for an open loop Stackelberg strategy (3) are given by (see [5], Theorem 7.1 and Corollary 7.1)

$$\begin{gathered} \psi_1(k) = Q_1 x(k) + A^T \psi_1(k+1), \\ \psi_2(k) = Q_2 x(k) + A^T \psi_2(k+1) + Q_1 \gamma(k), \\ \gamma(k+1) = S_{21}\psi_1(k+1) - S_1 \psi_2(k+1) + A\gamma(k), \\ \psi_1(N) = K_{1N} x(N), \;\; \psi_2(N) = K_{2N} x(N) + K_{1N}\gamma(N), \\ \gamma(0) = 0, \end{gathered} \tag{10}$$

with $S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T$ and where S_1 is defined as in (7).
If $\psi_i(k) = K_i(k)x(k), 1 \le i \le 2$, and $\gamma(k) = P(k)x(k)$, then again (1) can be rewritten in the form (7), where K_1, K_2 are obtained from the coupled system of difference equations

$$\begin{gathered} K_1(k) = Q_1 + A^T K_1(k+1) M_{k+1}^{-1} A \\ K_2(k) = Q_2 + A^T K_2(k+1) M_{k+1}^{-1} A + Q_1 P(k) \\ P(k+1) M_{k+1}^{-1} A = \\ [S_{21} K_1(k+1) - S_1 K_2(k+1)] M_{k+1}^{-1} A + A P(k), \\ K_1(N) = K_{1N}, K_2(N) = K_{2N} + K_{1N} P(N), P(0) = 0, \end{gathered} \tag{11}$$

provided the inverses in (11) exist.
Notice that - in contrast to (8) - the equations (11) cannot be integrated backwards since they are not decoupled at the terminal condition.

2. Nash Games. In the sequel we *assume that A is regular*; this implies that the difference equations in (4) and (6) can be rewritten as

$$\begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} (m+1) = M_{Na} \begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} (m) \tag{12}$$

where $m := N - k$, $\tilde{x}(m) = x(N+1-m)$, $\tilde{\psi}_i(m) = \psi_i(N+1-m)$, $1 \le i \le 2$, and

$$M_{Na} = \begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1 A^{-1} & A^T + Q_1 A^{-1} S_1 & Q_1 A^{-1} S_2 \\ Q_2 A^{-1} & Q_2 A^{-1} S_1 & A^T + Q_2 A^{-1} S_2 \end{pmatrix}.$$

Obviously the sequence $\begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} (m)$, $m \ge 1$, is uniquely defined if its initial value is known.
Since it turns out that the asymptotic behavior of this sequence as $m \to \infty$ is related to the behavior of the solutions of the algebraic Riccati equation (9), we present next some results concerning the interconnection of (9) and the corresponding linear difference equation (12).

Theorem 1 *(i) If* $S(K_1, K_2) :=$ span $\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} \subset \mathbf{C}^{3n \times n}$ *is an invariant subspace of* M_{Na} *with* *det* $(I + S_1 K_1 + S_2 K_2) \neq 0$ *then* $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ *is a solution of (9).*
(ii) If $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \in \mathbf{C}^{2n \times n}$ *is a solution of the algebraic Riccati equation (9), then* $S(K_1, K_2) \subset \mathbf{C}^{3n \times n}$ *is an invariant subspace of* M_{Na}*. Moreover* $F_{c\ell}^{-1} = A^{-1}(I + S_1 K_1 + S_2 K_2)$*, which is the inverse of the corresponding* closed loop matrix $F_{c\ell}$*, is the matrix of the restriction of* M_{Na} *to* $S(K_1, K_2)$ *with respect to the basis defined by the columns of* $\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$.

Proof (i) If $S(K_1, K_2)$ is M_{Na}-invariant there exists $R \in \mathbf{C}^{n \times n}$ with

$$\begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1 A^{-1} & A^T + Q_1 A^{-1} S_1 & Q_1 A^{-1} S_2 \\ Q_2 A^{-1} & Q_2 A^{-1} S_1 & A^T + Q_2 A^{-1} S_2 \end{pmatrix} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$$

$$= M_{Na} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} R \tag{13}$$

The first *row* of (13) yields $R = A^{-1}(I + S_1 K_1 + S_2 K_2)$, hence we obtain, using the second and third *row* of (13), that

$$\begin{aligned} & Q_1 A^{-1}(I + S_1 K_1 + S_2 K_2) + A^T K_1 \\ & \quad = K_1 A^{-1}(I + S_1 K_1 + S_2 K_2), \\ & Q_2 A^{-1}(I + S_1 K_1 + S_2 K_2) + A^T K_2 \\ & \quad = K_2 A^{-1}(I + S_1 K_1 + S_2 K_2). \end{aligned} \tag{14}$$

This means that $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is a solution of (9) since we assumed $\det(I + S_1 K_1 + S_2 K_2) \neq 0$.
(ii) If vice versa $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is a solution of (9) then (14) is verified and (13) holds with $R := A^{-1}(I + S_1 K_1 + S_2 K_2)$.
Remark 1. Theorem 1 shows that the solutions of the algebraic Riccati equation (9) can be determined from the generalized eigenvectors of M_{Na}; more precisely we have: Let span$(v_{\nu_1}, \ldots, v_{\nu_n})$ be an M_{Na}-invariant subspace such

that $\det X \neq 0$ for $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} := (v_{\nu_1}, \ldots, v_{\nu_n})$, then $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} := \begin{pmatrix} Y_1 X^{-1} \\ Y_2 X^{-1} \end{pmatrix}$ is a solution of (9) if $\det(I+S_1K_1+S_2K_2) \neq 0$.

If M_{Na} has at least one eigenvalue of geometric multiplicity $\mu > 1$ then (9) may have an uncountable number of (real or complex) solutions; if all eigenvalues of M_{Na} have geometric multiplicity 1 then there exist at most $\binom{3n}{n}$ solutions of (9).

Notice that a solution $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ of (9) corresponding to $S(K_1, K_2) = \text{span}\{v_{i_1}, \ldots, v_{i_n}\}$ is real if the generalized eigenvectors v_{i_k} corresponding to nonreal eigenvalues of M_{Na} are appearing in $\{v_{i_1}, \ldots, v_{i_n}\}$ in conjugate complex pairs.

(ii) From Theorem 1, (ii) we infer that the closed loop matrix $F_{c\ell} = (I + S_1K_1 + S_2K_2)^{-1}A$ is stable (i.e. here that all the eigenvalues of $F_{c\ell}$ have modulus less than 1) if the generalized eigenvectors spanning $S(K_1, K_2)$ are corresponding to eigenvalues of M_{Na} lying in the exterior of the closed unit circle; in this case $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is called a *stabilizing solution* of (9). Since M_{Na} has (counting multiplicity) $3n$ eigenvalues it is obvious that (9) may have several stabilizing solutions. Notice that on account of the substitution $m = N - k$ we have to use a solution of (9) corresponding to an invariant subspace of M_{Na} belonging to unstable eigenvalues if we want to have a corresponding stable closed loop matrix.

For the formulation of our next results we introduce the following notations:

Notation (i) λ is called an *unobservable mode (of rank r)* of the pair (Q_ν, A^{-1}) (with $\nu \in \{1, 2\}$) if there exist vectors $p_{i+j} \in \mathbf{C}^n \setminus \{0\}$, $0 \le j \le r-1$, and $p_{i-1} = 0$ such that

$$(A^{-1} - \lambda I)p_{i+j} = p_{i+j-1} \quad \text{and} \quad Q_\nu p_{i+j} = 0 \quad \text{for} \quad 0 \le j \le r-1.$$

(ii) Let $B_0 = (B_1, B_2)$. λ is called an *uncontrollable mode (of rank r)* of the pair (A, B_ν) (with $\nu \in \{0,\ 1,\ 2\}$) if there exist vectors $y_{i+j} \in \mathbf{C}^n \setminus \{0\}$, $0 \le j \le r-1$, and $y_{i-1} = 0$ such that

$$y_{i+j}^T(A - \lambda I) = y_{i+j-1}^T \quad \text{and} \quad B_\nu^T y_{i+j} = 0 \quad \text{for} \quad 0 \le j \le r-1.$$

Using these notations we get:

Lemma 1 *(i) λ is for $1 \le \nu \le 2$ an uncontrollable mode of (A, B_ν) of rank r corresponding to the chain y_{i+j}, $0 \le j \le r-1$, if and only if $\begin{pmatrix} 0 \\ y_{i+j} \\ y_{i+j} \end{pmatrix}$, $0 \le j \le r-1$, is a chain of generalized eigenvectors of M_{Na} corresponding to the eigenvalue λ. (Notice that this implies that λ is also an uncontrollable mode of rank r of (A, B_0) with the same chain).*

(ii) λ is an uncontrollable mode of rank r of (A, B_1) (respectively (A, B_2)) corresponding to the chain y_{i+j}, $0 \le j \le r-1$, if and only if $\begin{pmatrix} 0 \\ y_{i+j} \\ 0 \end{pmatrix}$, $0 \le j \le r-1$, (respectively $\begin{pmatrix} 0 \\ 0 \\ y_{i+j} \end{pmatrix}$, $0 \le j \le r-1$,) is a chain of generalized eigenvectors of M_{Na} corresponding to the eigenvalue λ; subspaces of C^{3n} spanned by chains of such generalized eigenvectors of M_{Na} and unions of such subspaces are called (A, B_1) (respectively (A, B_2)) uncontrollable subspaces of M_{Na}.

Proof. Let $y_{i-1} := 0$. From $\det A \neq 0, R_{\nu\nu} > 0$ and $S_\nu = B_\nu R_{\nu\nu}^{-1} B_\nu^T$ it follows that $A^{-1}S_\nu y = 0$ if and only if $B_\nu^T y = 0$. Therefore it follows from

$$M_{Na} \begin{pmatrix} 0 \\ y_{i+j} \\ y_{i+j} \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1}S_1 y_{i+j} + A^{-1}S_2 y_{i+j} \\ Q_1 A^{-1}S_1 y_{i+j} + Q_1 A^{-1} S_2 y_{i+j} + A^T y_{i+j} \\ Q_2 A^{-1}S_1 y_{i+j} + Q_2 A^{-1} S_2 y_{i+j} + A^T y_{i+j} \end{pmatrix}$$

that $y_{i+j}^T(A - \lambda I) = y_{i+j-1}^T$ and $B_\nu^T y_{i+j} = 0$ for $0 \le j \le r-1, 1 \le \nu \le 2$ holds if and only if

$$M_{Na} \begin{pmatrix} 0 \\ y_{i+j} \\ y_{i+j} \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ y_{i+j} \\ y_{i+j} \end{pmatrix}, \quad 0 \le j \le r-1.$$

This proves (i); similarly one can prove (ii) and the assertion of the following Lemma. □

Lemma 2 *$\lambda \neq 0$ is an unobservable mode of rank r of the pairs (Q_1, A^{-1}) and (Q_2, A^{-1}) corresponding to the chain y_{i+j}, $0 \le j \le r-1$, if and only if $\begin{pmatrix} y_{i+j} \\ 0 \\ 0 \end{pmatrix}$, $0 \le j \le r-1$, is a chain of generalized eigenvectors of M_{Na} corresponding to the eigenvalue λ; subspaces of C^{3n} spanned by chains of such generalized eigenvectors of M_{Na} and unions of such subspaces are called (Q_1, A^{-1}) and (Q_2, A^{-1}) unobservable subspaces of M_{Na}.*

The next Theorem is an immediate consequence of Lemma 1 and Lemma 2.

Theorem 2 *Let $S = span \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$ with $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} = (v_{\nu_1}, \ldots, v_{\nu_n})$ be a n-dimensional M_{Na}-invariant subspace of $\mathbf{C}^{3n}$.*

(i) If S is containing a nontrivial (A, B_ν) uncontrollable

subspace for some $\nu \in \{0,\ 1,\ 2\}$ then det $X = 0$; i.e. S does not correspond to a (finite) solution of (9).
(ii) If S is containing a nontrivial (Q_1, A^{-1}) and (Q_2, A^{-1}) unobservable subspace then $\det Y_1 = \det Y_2 = 0$.

!t is well known that (A, B_ν) is controllable if and only if

$$\text{rank } (B_\nu, AB_\nu, \cdots, A^{n-1}B_\nu) = n,$$

which is equivalent to

$$L_\nu := \text{ Kernel of } \begin{pmatrix} B_\nu^T \\ B_\nu^T A \\ \vdots \\ B_\nu^T A^{n-1} \end{pmatrix} = \{0\}.$$

Notice that $L = L_0 = L_1 \cap L_2$. Now we prove another characterization of the controllability of (A, B_0).

Theorem 3 *(i) The subspace*

$$S_0 = \{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \in \mathbf{C}^{3n} | y \in L_0\} \tag{15}$$

is the maximal M_{Na}-invariant subspace having a basis-matrix of the form $\begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix}$.

(ii) (A, B_0) is controllable if and only if M_{Na} does not have a nontrivial invariant subspace of the form (15).

Proof. (i) implies (ii), therefore we give a proof for (i).

a) Let $y \in L$. Using the theorem of Cayley-Hamilton it follows from the definition of L that $B^T(A^T)^j Y = 0$ for $j \in \mathbf{N} \cup \{0\}$, therefore $A^T y \in L$.
For $y \in L$ we get

$$M_{Na} \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = \begin{pmatrix} A^{-1}S_1 y + A^{-1}S_2 y \\ Q_1 A^{-1} S_1 y + Q_1 A^{-1} S_2 y + A^T y \\ Q_2 A^{-1} S_1 y + Q_2 A^{-1} S_2 y + A^T y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ A^T y \\ A^T y \end{pmatrix} \in S_0$$

which shows that S_0 is M_{Na}-invariant.

b) Let $S = \text{ span } \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix}$ be M_{Na}-invariant. Then there exists $P \in \mathbf{C}^{k \times k}$ with

$$M_{Na} \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ YP \\ YP \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1}S_1 Y & + & A^{-1}S_2 Y \\ Q_1 A^{-1} S_1 Y & + & Q_1 A^{-1} S_2 Y + A^T Y \\ Q_2 A^{-1} S_1 Y & + & Q_2 A^{-1} S_2 Y + A^T Y \end{pmatrix}. \tag{16}$$

Since det $A \neq 0$ we infer from the first *line* of (16) that $(S_1 + S_2)Y = 0$. Consequently it follows with $R_{ii} > 0, i = 1, 2$, that

$$Y^T B_i R_{ii}^{-1} B_i^T Y = 0, \ \ 1 \le i \le 2.$$

This implies

$$B_1^T Y = 0 \ \text{ and } \ B_2^T Y = 0 \tag{17}$$

and, on account of (16),

$$M_{Na} \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ A^T Y \\ A^T Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ YP \\ YP \end{pmatrix}. \tag{18}$$

¿From (17) and (18) we infer that

$$B_i^T A^T Y = B_i^T Y P = 0_{r_{i,k}} P = 0_{r_{i,k}} \text{ for } i = 1, 2.$$

Repeating this step we obtain, using for $j \ge n$ the Cayley-Hamilton theorem, that

$$B_i^T (A^T)^j Y = 0_{r_{i,k}} \text{ for } i = 1, 2 \text{ and } j \in \mathbf{N} \cup \{0\}.$$

Hence span $Y \subset L$ and $S \subset S_0$. □

Remark 2. In analogy to the continuous-time case we can show that the Riccati difference equation (8) and the linear difference equation

$$\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} (m+1) = M_{Na} \begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} (m), \ 1 \le m \le N,$$

$$\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} (1) = \begin{pmatrix} I \\ K_{1N} \\ K_{2N} \end{pmatrix} \tag{19}$$

are equivalent in the following sense:

(i) Let $K_1(k), K_2(k), N \ge k \ge 0$, be solutions of (8) with $K_i(N) = K_{iN},\ 1 \le i \le 2$.
If (with $m := N - k$) for $1 \le m \le N$,

$$\tilde{X}(m+1) = A^{-1}\tilde{X}(m) + A^{-1}S_1\tilde{\Psi}_1(m) + A^{-1}S_2\tilde{\Psi}_2(m), \tag{20}$$

$$\tilde{\Psi}_i(m+1) := K_i(N-m)\tilde{X}(m+1), \tag{21}$$

$$\text{with } \tilde{X}(1) = I, \ \ \tilde{\Psi}_i(1) = K_{iN},\ 1 \le i \le 2,$$

and if the matrices $\tilde{X}(m), 1 \le m \le N+1$, are invertible, then $\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} (m), 1 \le m \le N+1$, is a solution of (19).

(ii) If vice versa $\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m), 1 \le m \le N+1$, is a solution of (19) with invertible matrices $\tilde{X}(m)$ and $I + S_1\tilde{\Psi}(m)^{-1} + S_2\tilde{\Psi}(m)^{-1}, 1 \le m \le N+1$, then $K_i(N-m) := \tilde{\Psi}_i(m+1)\tilde{X}(m+1)^{-1}$,
$1 \le i \le 2, 1 \le m \le N+1$, defines a solution of (8).

Proof. (i) Let $K_1(k), K_2(k)$ solve (8) and let $\tilde{X}(m)$ and $\tilde{\Psi}_i(m),\ 1 \le m \le N+1$, be defined by (20) and (21). Obviously (20) implies the first *row* of (19). Using this identity it follows from (8) and (21) that for $i = 1, 2$ and $1 \le m \le N$

$$\begin{aligned}
&\tilde{\Psi}_i(m+1) \\
&= K_i(N-m)\tilde{X}(m+1) \\
&= \{Q_i + A^T K_i(N-m+1)M_{N-m+1}^{-1}A\}\tilde{X}(m+1) \\
&= Q_i A^{-1} M_{N-m+1}\tilde{X}(m) \\
&\quad + A^T K_i(N-m+1)M_{N-m+1}^{-1}AA^{-1} \\
&\quad [\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)] \\
&= Q_i A^{-1}[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)] + A^T\tilde{\Psi}_i(m),
\end{aligned}$$

which is equivalent to the second and third *row* of (19).

(ii) (19) implies for $1 \le m \le N$

$$\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m+1) =$$

$$\begin{pmatrix} A^{-1}[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)] \\ A^T\tilde{\Psi}_1(m) + Q_1A^{-1}[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)] \\ A^T\tilde{\Psi}_2(m) + Q_2A^{-1}[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)] \end{pmatrix}.$$

Replacing $\tilde{X}(m+1)$ by !l the expression on the right hand side of this identity we get

$$\begin{pmatrix} \tilde{\Psi}_1(m+1)\tilde{X}(m+1)^{-1} \\ \tilde{\Psi}_2(m+1)\tilde{X}(m+1)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} Q_1 + A^T\tilde{\Psi}_1(m)[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)]^{-1}A \\ Q_2 + A^T\tilde{\Psi}_2(m)[\tilde{X}(m) + S_1\tilde{\Psi}_1(m) + S_2\tilde{\Psi}_2(m)]^{-1}A \end{pmatrix},$$

which shows that $K_i(N-m) = \tilde{\Psi}_i(m)\tilde{X}(m)^{-1},\ 1 \le i \le 2,\ 1 \le m \le N$, solve (8). □

Since the dynamics of the difference equation is very simple it can be used to investigate the asymptotic behavior of the sequences $K_i(k)$ for $k \to -\infty$.
Let $V = (v_1, \cdots, v_{3n}) \in \mathbf{C}^{3nx3n}$ be a matrix defined by a Jordan basis of generalized eigenvectors of M_{Na} such that

$$V^{-1}\, M_{Na}\, V = diag\,(J_1, \cdots, J_p) = \begin{pmatrix} \lambda_1 & * & 0 \\ \ddots & \ddots & * \\ 0 & \ddots & \lambda_{3n} \end{pmatrix}$$

with $* \in \{0, 1\}$ and (without loss of generality)

$$0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_{3n}|.$$

M_{Na} (and the system (8) of Riccati difference equations) is called *dichotomically separable* if $|\lambda_{2n}| < |\lambda_{2n+1}|$. If in this case in addition the n-dimensional matrix X defined by $\begin{pmatrix} X \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = (v_{2n+1}, \cdots, v_{3n})$ is invertible then it follows from Theorem 1 that $K_1^d = \Psi_1 X^{-1},\ K_2^d = \Psi_2 X^{-1}$ defines a real solution of the algebraic Riccati equation (9), this solution is called the *dichotomic solution* of (9).
We say that $\begin{pmatrix} K_{1N} \\ K_{2N} \end{pmatrix}$ belongs to $\mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$, the *generalized basin of attraction* of $\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$, if the first n rows of the matrix $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{3n} \end{pmatrix} = V^{-1}\begin{pmatrix} I \\ K_{1N} \\ K_{2N} \end{pmatrix}$ are linearly independent.
Notice that $\mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$ is an open and dense subset of $\mathbf{C}^{2nxn}$.

Theorem 4 *Assume that the dichotomic solution* $\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$ *of (9) exists and that*
$\begin{pmatrix} K_{1N} \\ K_{2N} \end{pmatrix} \in \mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$.
If the sequence $\begin{pmatrix} K_1(k) \\ K_2(k) \end{pmatrix}$ *corresponding to the solution of (8) with* $K_i(N) = K_{iN}$ *(for i = 1, 2) is defined for all* $k \le N$ *then*

$$\lim_{k \to -\infty} K_i(k) = K_i^d \ for\ i = 1, 2.$$

Proof. Let the sequence $\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m),\ m \ge 1$, be defined like in (19). Then it follows (under our assumptions) as with the proof of the classical power method that

$$\lim_{m \to \infty} span \begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m)$$

$$= span\ (v_{2n+1}, \cdots, v_{3n}) = span \begin{pmatrix} I_n \\ K_1^d \\ K_2^d \end{pmatrix}.$$

This proves the assertion of Theorem 4. □

Notice that here the order of convergence depends essentially on the difference $\delta = |\lambda_{2n+1}| - |\lambda_{2n}|$.

Theorem 4 indicates that for large values of N and for problems having a dichotomic solution with a stable closed loop matrix $(I + S_1K_1^d + S_2K_2^d)^{-1}A$ it makes sense to replace the controls $u_i(k)$ defined in (3) by the constant (or static) stabilizing controls

$$\tilde{u}_i(k) = -R_{ii}^{-1}B_i^TK_i^dx(k),\ 1 \le i \le 2,\ 0 \le k \le N.$$

3. Stackelberg Games. In this section we assume like in section 2 that A is regular. It is possible to formulate most of the results of section 2 in the context of Stackelberg games; detailed versions of these results will be presented elsewhere.

Notice that the difference equations in (1), (10) can be rewritten as

$$\begin{pmatrix} x \\ \gamma \\ \psi_2 \\ \psi_1 \end{pmatrix}(k) = M_{St}\begin{pmatrix} x \\ \gamma \\ \psi_2 \\ \psi_1 \end{pmatrix}(k+1), 0 \le k \le N-1, \tag{22}$$

where (with $\hat{S} = Q_1A^{-1}S_1 + Q_2A^{-1}S_2$)

$$M_{St} = \begin{pmatrix} A^{-1} & 0 & A^{-1}S_2 & A^{-1}S_1 \\ 0 & A^{-1} & A^{-1}S_1 & -A^{-1}S_{21} \\ Q_2A^{-1} & Q_1A^{-1} & A^T + \hat{S} & Q_2A^{-1}S_1 - Q_1A^{-1}S_{21} \\ Q_1A^{-1} & 0 & Q_1A^{-1}S_2 & A^T + Q_1A^{-1}S_1 \end{pmatrix}$$

is symplectic; i.e. with $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ we have $J^{-1}M_{St}^TJ = M_{St}^{-1}$.

The algebraic (Stackelberg-) Riccati equation corresponding to (11) is

$$\begin{array}{l} K_1 = Q_1 + A^TK_1[I + S_1K_1 + S_2K_2]^{-1}A, \\ K_2 = Q_2 + A^TK_2[I + S_1K_1 + S_2K_2]^{-1}A + Q_1P, \\ P[I + S_1K_1 + S_2K_2]^{-1}A = \\ \quad [S_{21}K_1 - S_1K_2][I + S_1K_1 + S_2K_2]^{-1}A + AP. \end{array} \tag{23}$$

The following theorem is proved like Theorem 1 and shows that the solutions of (23) can be determined from the n-dimensional invariant subspaces of M_{St}.

Theorem 5.
(i) If $\tilde{S}(K_1, K_2, P) :=$ span $(I, P^T, K_2^T, K_1^T)^T \subset \mathbf{C}^{4n\times n}$ is an invariant subspace of M_{St} with $\det(I+S_1K_1+S_2K_2) \neq 0$, then $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ is a solution of (23).
(ii) If $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ is a solution of (23) then $\tilde{S}(K_1, K_2, P)$ is an invariant subspace of M_{St} and $A^{-1}(I + S_1K_1 + S_2K_2)$ is the matrix of the restriction of M_{St} to $\tilde{S}(K_1, K_2, P)$ with respect to the basis defined by the columns of $(I, K_1^T, K_2^T, P^T)^T$.

If we define (Q_1, A^{-1}) unobservable and (A, B_1)- uncontrollable subspaces of M_{St} similarly to section 2, we obtain:

Theorem 6. *Let $S = span(X^T, Y_1^T, Y_2^T, Y_3^T)^T$ with $(X^T, Y_1^T, Y_2^T, Y_3^T) =: (v_{\nu_1}, \ldots, v_{\nu_n})$ be a n-dimensional M_{St}-invariant subspace of $\mathbf{C}^{4n}$.*
(i) If S contains a nontrivial (Q_1, A^{-1}) unobservable or (A, B_i), $1 \le i \le 2$, uncontrollable subspace of M_{St}, then $\det X = 0$, i.e. S does not correspond to a (finite) solution of the algebraic Riccati equation (23).
(ii) If S contains a nontrivial (Q_1, A) unobservable subspace then $\det Y_2 = 0$ and $\det Y_3 = 0$.

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