# Time-varying discrete Riccati equation: Some monotonicity results

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**Abstract.** Using a Frechet derivative based approach some monotonicity and comparison results concerning the solutions of the time-varying discrete time Riccati equation are obtained. As a consequence of monotonicity sufficient conditions ensuring the existence of the stablizing solution are also derived.

**Keywords:** Discrete time systems, Riccati equation, monotonicity and comparison results.

## 1. Introduction and problem statement

One of the most remarkable properties of the (matrix) Riccati differential equation RDE

$$-\dot{X} = A^T(t)X + XA(t) - XS(t)X + Q(t), \;\; X(t_0) = X_0,$$

where $A(t)$, $S(t) = S^T(t)$, $Q(t) = Q^T(t)$ and $X_0 = X_0^T \in \mathbb{R}^{n\times n}$, is that concerning *monotonicity* with respect to both initial value $X_0$ and input data matrix

$$E(t) = \begin{bmatrix} Q & A^T \\ A & -S \end{bmatrix} (t).$$

Stokes [23] proved the striking fact that RDE is, for $n > 1$, the only (matrix) differential $\dot{X} = \Phi(t, X)$, $X(t_0) = X_0$ possessing the so-called *order-preserving* property, i.e. $X$ depends monotonically on $X_0$. Notice also that under different assumptions the monotonicity of $X$ with respect to $E$ was under the attention of many authors (see for instance [5], [6], [8], [9] and [21]).

Furthermore, various types of monotonicity results have been also obtained for differential and difference Riccati equations as well as for the algebraic Riccati equation (see [2], [3], [9], [17], [21], [22], [25], [26]).

It is worthwhile to point out that in the discrete-time case the proofs need a rather intricate technical machinery and, this is the reason that up to now, stronger assumptions in comparison with the continuous-time case have been considered.

One of the main purposes of the present paper was that of generalizing the aforementioned monotonicity and comparison results and, furthermore, to relax the assumptions imposed on the coefficients. To this end a Frechet derivative based approach has been subsequently proposed. Notice that in connection with Riccati equations, this idea has been used already by Delchamps in order to prove the analyticity of the stablizing solution with respect to the coefficients [7]. Furthermore, similar techniques are used in a forthcoming paper in order to prove and generalize convexity results obtained in [10], [14], [18], [24] and [27].

In the sequel the following notations will be used. By $\mathbb{Z}$, $\mathbb{R}^n$ $(\mathbb{C}^n)$ and $\mathbb{R}^{n\times m}$ we denote the ring of integers, the real (complex) $n$–dimensional Euclidian space and the set of $n \times m$ matrices with real entries. If $M \in \mathbb{R}^{n\times m}$ then $M^T$ stands for its transpose. By $I_n$ we denote the $n \times n$ unit matrix. The spectral radius of a linear bounded operator $T$ on a Hilbert $\mathcal{H}$ space will be denoted by $\rho(T)$ and $T^*$ will stands for the adjoint of $T$. Further, if $T$ is selfadjoint, i.e. $T = T^*$ then we shall term it *coercive* if there exists $\nu_0 > 0$ such that $\langle Tx, x\rangle \geq \nu_0 \|x\|^2$ $\forall x \in \mathcal{H}$. Here $\langle , \rangle$ and $\| \ \|$ stand for the inner product and the *associated* norm on $\mathcal{H}$, respectively. The coerciveness of $T$ will be denoted by $T >> 0$. Let $\ell^{2m}$ be the Hilbert space of norm-square doubly infinite $\mathbb{C}^m$-valued sequences $u = (u_k)_{k\in\mathbb{Z}}$, i.e. $u_k \in \mathbb{C}^m$ and $||u\|_2 := \left(\sum_{k=-\infty}^{\infty} \|u_k\|^2\right)^{\frac{1}{2}} < \infty$ is the $\ell^2$-norm, here $\| \ \|$ stands for the usual Euclidian norm. If $B = (B_k)_{k\in\mathbb{Z}}$ with $B_k \in \mathbb{R}^{n\times m}$, is any *bounded matrix sequence*, that is, $\|B_k\| \leq c_0$ $\forall k \in \mathbb{Z}$ for some $c_0 \geq 0$ then $B$ will be interpreted as a linear bounded *multiplication or block diagonal operator* from $\ell^{2,m}$ into $\ell^{2,n}$. This means that if $u \in \ell^{2,m}$ then we shall adopt for it the doubly infinite column representation $u = col(u_k)_{k=-\infty}^{\infty}$ and the action of $B$ on $u$, i.e. $y := Bu$ will be explicitly described by $y = col(y_i)_{i=-\infty}^{\infty} = mat(\delta_{ij} B_i)_{i,j=-\infty}^{\infty}\, col(u_j)_{j=-\infty}^{\infty} = diag(B_i)_{i=-\infty}^{\infty} col(u_i)_{i=-\infty}^{\infty}$. Here $\delta_{ij}$ stands for the Kronecker symbol. If $B = mat(\delta_{ij} B_i)_{i,j=-\infty}^{\infty} : \ell^{2,m} \to \ell^{2,n}$ is a block diagonal operator then we shall introduce the following four linear bounded operators from $\ell^{2,m}$ into $\ell^{2,n}$ : $\sigma B = mat(\delta_{ij}B_{i+1})_{i,j=-\infty}^{\infty}$, $\sigma^{-1}B = mat(\delta_{ij}B_{i-1})_{i,j=-\infty}^{\infty}$, $B\sigma = mat(\delta_{i,j-1}B_i)_{i,j=-\infty}^{\infty}$ and $B\sigma^{-1} = mat(\delta_{i,j+1}B_i)_{i,j=-\infty}^{\infty}$. Let $I_{\ell^{2,n}} = diag(A_i)_{i=-\infty}^{\infty}$, $A_i = I_n$, be the identity operator on $\ell^{2,n}$. Then $I_{\ell^{2,n}}\sigma$ and $I_{\ell^{2,n}}\sigma^{-1}$ describe the action of the *bilateral shift* $\sigma$ and its inverse $\sigma^{-1}$ on $\ell^{2,n}$, repectively. Thus if $x \in \ell^{2,n}$ then $(I_{\ell^{2,n}}\sigma)x = \sigma x$ and $(I_{\ell^{2,n}}\sigma^{-1})x = \sigma^{-1}x$ where $(\sigma^{\pm 1}x)_k := x_{k\pm 1}$. Let $A = \mathrm{diag}(A_i)_{i=-\infty}^{\infty}$ be a block diagonal operator on $\ell^{2,n}$. We shall say that $A$ defines an exponentially stable evolution (anticausal exponentially stable evolution) if $\rho(A\sigma^{-1}) < 1$ $(\rho(A\sigma) < 1)$. If $A$ defines an exponentially stable evolution (anticausal exponentially stable evolution) then clearly the operator $I_{\ell^{2,n}}\sigma - A$ $(I_{\ell^{2,n}}\sigma^{-1} - A)$ is boundedly invertible on $\ell^{2,n}$. For more details concerning shift operator algebra, exponential stability (both causal and anticausal) in terms of the spectral radius see [1], [12].

For any $k \in \mathbb{Z}$ let $\ell_{k,+}^{2,n}(\ell_{k,-}^{2,n})$ be the (closed) subspace of $\ell^{2,n}$ consisting of those sequences with the support in $[k,\infty)$ $((-\infty, k-1])$. Clearly $\ell^{2,n} = \ell_{k,-}^{2,n} \oplus \ell_{k,+}^{2,n}$ where $\oplus$ denotes the direct sum. Denote by $P_{k,+}^n$ $(P_{k,-}^n)$ the orthogonal projection of $\ell^{2,n}$ onto $\ell_{k,+}^{2,n}$ $(\ell_{k,-}^{2,n})$. If $T$ is a linear bounded operator from $\ell^{2,n}$ into $\ell^{2,m}$ then for any $k \in \mathbb{Z}$, $T_k := P_{k,+}^m T P_{k,+}^n$ is called the Toeplitz operator associated with $T$ at $k$. $T$ is called causal (anticausal) if $T_k = TP_{k,+}^n$ $(T_k = P_{k,+}^m T)$. Clearly if $A = diag(A_i)_{i=-\infty}^{\infty} : \ell^{2,n} \to \ell^{2,n}$ defines an exponentially stable evolution (anticausal exponentially stable evolution) then $A$ is causal (anticausal).
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ be the Banach space of the linear bounded operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. Let $T : M \mapsto T(M)$ be any function from $\mathcal{B}_1$ to $\mathcal{B}_2$ where $M$ ranges the domain $\mathcal{D} \subset \mathcal{B}_1$, and let $P \in \mathcal{D}$. We shall say that $T$ is Frechet differentiable (with respect to $M$) in $P$ if there exists $T_M(P) \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ such that $\lim_{\varepsilon\to 0} [T(P+\varepsilon N) - T(P)]/\varepsilon \to T_M(P)(N)$ uniformly with respect to all $N \in \mathcal{B}_1$. $T_M(P)(\cdot)$ is called the Frechet derivative of $T$ (with respect to $M$) in $P$. If $T$ is Frechet differentiable in each point of the domain $\mathcal{D} \subset \mathcal{B}$ then $T$ is called Frechet differentiable on $\mathcal{D}$. If $T_M(\cdot) : \mathcal{D} \subset \mathcal{B}_1 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is Frechet differentiable on $\mathcal{D}$ then $T$ is called twicely Frechet differentiable on $\mathcal{D}$ and $T_{MM}(P)$ denotes its second Frechet derivative (in $P$). For more details see [16].

The present paper is organized as follows. Section 1 has an introductory character. Section 2 is devoted to the evaluation of Frechet derivatives for the time varying discrete Riccati equation (TVDRE) and local monotonicity of any solution with respect to the input data. In Section 3 some global monotonicity and comparison results are given. Some limiting cases are also studied in the context of the perturbation of the nominal data. Section 4 deals with the global existence and monotonicity of the stabilizing solution with respect to the input data. Some conclusions are given in Section 5.

Let us be now more specific concerning the topics in order. First introduce

$$\mathbf{F}(D,X) := A^*\sigma XA - X - (L + A^*\sigma XB)(R + B^*\sigma XB)^{-1}(B^*\sigma XA + L^*) + Q \quad (1.1)$$

where

$$D := \begin{bmatrix} Q & A^* & L \\ A & 0 & B \\ L^* & B^* & R \end{bmatrix} = D^*, \quad (1.2)$$

is termed as the *input data matrix sequence.* Here $A : \ell^{2,n} \to \ell^{2,n}$, $B : \ell^{2,m} \to \ell^{2,n}$, $Q : \ell^{2,n} \to \ell^{2,n}$, $L : \ell^{2,m} \to \ell^{2,n}$ and $R : \ell^{2,m} \to \ell^{2,m}$ are *block diagonal operators.* Whenever $R^{-1}$ is well defined and bounded, (1.1) will be replaced by the following *reduced equivalent form*

$$\begin{aligned} \mathbf{G}(E, X) &:= \widetilde{A}^*\sigma X\widetilde{A} - X - \widetilde{A}^*\sigma XB(R + B^*\sigma XB)^{-1}B^*\sigma X\widetilde{A} + \widetilde{Q} \\ &= \widetilde{A}^*\sigma X(I + S\sigma X)^{-1}\widetilde{A} - X + \widetilde{Q} \end{aligned} \tag{1.3}$$

where $\widetilde{A} := A - BR^{-1}L^*$, $\widetilde{Q} := Q - LR^{-1}L^*$ and

$$E = \begin{bmatrix} \widetilde{Q} & \widetilde{A}^* \\ \widetilde{A} & -S \end{bmatrix} = E^* \tag{1.4}$$

Here $E$ is termed as the *reduced input data matrix sequence* and $S := BR^{-1}B^*$. Notice that if $R$ is boundely invertible then both right-hand sides of (1.1) and (1.3) coincide. In fact, it is easy checkable that the right-hand side of (1.3) is obtained from the right-hand side of (1.1) by making $L = 0$ and by replacing $A$ and $Q$ by $\widetilde{A}$ and $\widetilde{Q}$, respectively.

**Definition 1.1.** *Let $D$ be given and let $\mathbf{I}$ be any interval of $\mathbb{Z}$. Then any matrix sequence $X = X^*$ restricted to $\mathbf{I}$ for which the following two conditions*

*a) $(R + B^*\sigma XB)^{-1}$ is well defined on $\mathbf{I}$*

*b) $\mathbf{F}(D, X) = 0$ on $\mathbf{I}$,*

*all hold is called a (selfadjoint) solution on $\mathbf{I}$ to the TVDRE*

$$A^*\sigma XA - X - (L + A^*\sigma XB)(R + B^*\sigma XB)^{-1}(B^*\sigma XA + L^*) + Q = 0. \tag{1.5}$$

*If $\mathbf{I} = [l, s]$ then $X_s = (\sigma X)_{s-1}$ is termed as the terminal condition of the solution $X$.*
*If $\mathbf{I} = \mathbb{Z}$, $X$ is bounded (on $\mathbb{Z}$), a) and b) both hold and $(R+B^*\sigma XB)^{-1}$ is bounded, then $X$ is called a global solution to the TVDRE (1.5).*

*Any global solution $X$ for which $(A_{cl}(X) =)A_{cl} := A+BF$ defines an exponentially stable evolution is called a stabilizing solution to the TVDRE (1.5). Here*

$$(F(X) =)F := -(R + B^*\sigma XB)^{-1}(B^*\sigma XA + L^*), \tag{1.6}$$

*and it is termed, in this case, as the stabilizing feedback gain.* □

**Remark 1.2.** *If $X$ is any solution on any $\mathbf{I} = [l, s]$ to (1.5) then it is generated by the backward rule $X = \mathbf{H}(D, \sigma X)$ with the terminal condition $X_s = (\sigma X)_{s-1}$. Here $\mathbf{H}(D, \sigma X) := \mathbf{F}(D, X) + X$.* □

The subsequent sections will be devoted to the following two main topics.

1. Monotonicity of any solution $X$ to (1.5), in particular of the stabilizing solution if it exists, with respect to the *partial* input data matrix seqence

$$D_p = \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} = D_p^*. \tag{1.7}$$

2. Monotonicity of any solution $X$ to *the reduced form* of the TVDRE coming from (1.3), i.e.

$$\widetilde{A}^* \sigma X (I + S\sigma X)^{-1} \widetilde{A} - X + \widetilde{Q} = 0, \tag{1.8}$$

in particular monotonicity of the stablizing solution, if it exists, with respect to the reduced input data matrix $E$ (provided $R$ is boundedly invertible). Some existence conditions for the stabilizing solution are implicitly given.

## 2. Frechet derivatives and local monotonicity

We shall start with

**Proposition 2.1.** *For each pair $(D, X)$ with $X = X^*$ the following statements hold*

*1.*

$$\mathbf{F}_D(D, X)(\Delta D) = M^*(D, X) \Delta D M(D, X), \tag{2.1}$$

*where*

$$M(D, X) = \begin{bmatrix} I \\ \sigma X A_{cl} \\ F \end{bmatrix} \tag{2.2}$$

*and*

$$\Delta D = \begin{bmatrix} \Delta Q & (\Delta A)^* & \Delta L \\ \Delta A & 0 & \Delta B \\ (\Delta L)^* & (\Delta B)^* & \Delta R \end{bmatrix} = (\Delta D)^* \tag{2.3}$$

*with $F$ given in (1.6).*

*Furthermore, if $R$ ist boundedly invertible, then*

$$\mathbf{G}_E(E, X)(\Delta E) = N^*(E, X) \Delta E N(E, X) \tag{2.4}$$

*where*

$$N(E, X) = \begin{bmatrix} I \\ \sigma X A_{cl} \end{bmatrix} \tag{2.5}$$

*with the particular form for $A_{cl}$ given by $A_{cl} = (I + S\sigma X)^{-1} \widetilde{A}$, and*

$$\Delta E = \begin{bmatrix} \Delta \widetilde{Q} & (\Delta \widetilde{A})^* \\ \Delta \widetilde{A} & -\Delta S \end{bmatrix} = (\Delta E)^*. \tag{2.6}$$

2.
$$\mathbf{F}_X(D,X)(\Delta X) = A_{cl}^*\sigma\Delta X A_{cl} - \Delta X = \mathbf{G}_X(E,X)(\Delta X) \tag{2.7}$$
*where $\Delta X = (\Delta X)^*$ and the rightmost term in (2.7) makes sense provided $R$ is boundedly invertible.*

3.
$$\begin{aligned}\mathbf{F}_{X,X}(D,X)(\Delta X,\Delta Z) &= -A_{cl}^*[\sigma\Delta X P\sigma\Delta Z + \sigma\Delta Z P\sigma\Delta X]A_{cl} \\ &= \mathbf{G}_{X,X}(E,X)(\Delta X,\Delta Z)\end{aligned} \tag{2.8}$$
*with $\Delta X = (\Delta X)^*$, $\Delta Z = (\Delta Z)^*$ and*
$$P := B(R + B^*\sigma XB)^{-1}B^* = P^* = S(I+\sigma XS)^{-1} \tag{2.9}$$
*and where the rightmost terms in (2.8) and (2.9) make sense provided $R$ is boundedly invertible.*

**Proof.** See Appendix. □

**Remark 2.2.** *It must be noted that the essential difference between the Frechet derivatives shown in (2.1) and (2.4) consists in their signature namely, while (2.1) is for $\Delta A, \Delta B \neq 0$ of* **indefinite** *signature (see (2.3)), (1.7) could be of definite signature. This fact will be exploited further.* □

**Theorem 2.3.** *Let the input data matrix sequence $D$ (see (1.2)) together with any interval $\mathbf{I} = [l,s] \subset \mathbb{Z}$ be given. Assume that the TVDRE (1.5) has a solution $X$ on $\mathbf{I}$. Then the following statements are true*

1. *$X$ is locally monotonic with respect to both terminal condition $X_s$ and partial input data matrix sequence $D_p$ (see (1.7)).*
2. *If $R$ is boundedly invertible then $X$, seen as a solution to (1.8), is locally monotonic with respect to both terminal condition $X_s$ and reduced input data sequence $E$ (see (1.4)).*

**Proof.** 1. Clearly we have $\mathbf{F}(D,X) = 0$ on $\mathbf{I}$ with $\mathbf{F}$ given explicitly in (1.1). In addition the equation
$$A_{cl}^*\sigma\Delta X A_{cl} - \Delta X + W = 0 \tag{2.10}$$
has, for $(\Delta X)_s (= (\Delta X^*)_s)$ specified, and arbitrary free term $W = W^*$, a unique solution $\Delta X (= \Delta X^*)$ on $\mathbf{I}$. Hence, according to (2.7), this means that the Frechet derivative $\mathbf{F}_X(D,X)$ is invertible. In addition both $\mathbf{F}_X$ and $\mathbf{F}_{XX}$ (see (2.8), (2.9)) are continous in $(D,X)$.

Hence, according to the implicit function theorem, we can write

$$\begin{aligned}
&\mathbf{F}_D(D,X)(\Delta D) + \mathbf{F}_X(D,X)(X_{(X_s,D)}(X_s,D)(\Delta Z,\Delta D)) = 0,\\
&(X_{(X_s,D)}(X_s,D)(\Delta Z,\Delta D))_s = \Delta Z
\end{aligned} \tag{2.11}$$

where $\Delta Z = \Delta Z^T \in \mathbb{R}^{n\times n}$. Here $X_{(X_s,D)}(X_s,D)$ stands for the Frechet derivative of $X$ with respect to the pair $(X_s,D)$ taken in the product of the spaces to which $X_s$ and $D$ belong, and evaluated in the actual data $(X_s,D)$. Using now (2.1) in conjunction with (2.7) , (2.11) yields explicitly

$$\begin{aligned}
&A_{cl}^*\sigma X_{(X_s,D)}(X_s,D)(\Delta Z,\Delta D))A_{cl} - X_{(X_s,D)}(X_s,D)(\Delta Z,\Delta D)\\
&+M^*(D,X)\Delta D M(D,X) = 0, \quad (X_{(X_s,D)}(X_s,D)(\Delta Z,\Delta D))_s = \Delta Z.
\end{aligned} \tag{2.12}$$

As we already mentioned in Remark 2.2, $\Delta D$ is, for $\Delta A, \Delta B \neq 0$, of indefinite sign. Assuming $\Delta A = 0$ and $\Delta B = 0$, (2.12) receives the particular form (see also (A11) in the Appendix)

$$\begin{aligned}
&A_{cl}^*\sigma X_{(X_s,D_p)}(X_s,D_p)(\Delta Z,\Delta D_p)A_{cl} - X_{(X_s,D_p)}(X_s,D_p)(\Delta Z,\Delta D_p)\\
&+M_p^*(D,X)\Delta D_p M_p(D,X) = 0, \quad (X_{(X_s,D_p)}(X_s,D_p)(\Delta Z,\Delta D_p))_s = \Delta Z
\end{aligned} \tag{2.13}$$

where $\Delta D_p$ comes from (1.7) and has the explicit form

$$\Delta D_p = \begin{bmatrix} \Delta Q & \Delta L \\ (\Delta L)^* & \Delta R \end{bmatrix} \tag{2.14}$$

and $M_p^*(D,X) = [I, F^*]$.

Since the Lyapunov equation (2.13) has, for $\Delta Z \geq 0$ and $\Delta D_p \geq 0$, a unique positive semidefinite solution $X_{(X_s,D_p)}((\Delta Z,\Delta D_p)) \geq 0$ on $\mathbf{I}$, the conclusion follows.

2. In this case (2.11) is replaced by

$$\begin{aligned}
&\mathbf{G}_E(E,X)(\Delta E) + \mathbf{G}_X(E,X)(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)) = 0,\\
&(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s = \Delta Z
\end{aligned} \tag{2.15}$$

or explicitly (see (2.4) and (2.7))

$$\begin{aligned}
&A_{cl}^*\sigma X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)A_{cl} - X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)+\\
&N^*(E,X)\Delta E N(E,X) = 0, \ (X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s = \Delta Z.
\end{aligned} \tag{16}$$

Since (2.16) has, for $\Delta Z \geq 0$ and $\Delta E \geq 0$, a unique positive semidefinite solution $X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E) \geq 0$ on $\mathbf{I}$, the conclusion follows. □

**Corollary 2.3.** *Let $D$ as in (1.2) be given and assume that the TVDRE (1.5) has a stabilizing solution $X$. Then*

1. *$X$ is locally monotonic with respect to $D_p$.*

2. *If $R$ is boundedly invertible then $X$ (seen as a solution to (1.8)) is locally monotonic with respect to $E$.*

**Proof.** The proof runs similarly like in Theorem 2.3 by ignoring the terminal condition $X_s$ and, with the additional key remark that, since $A_{cl}$ defines an exponentially stable evolution, (2.10), (2.13) and (2.16) have unique *global* solutions. This conclusion leads automatically to the bounded invertibility of the Frechet derivatives $\mathbf{F}_X$ and $\mathbf{G}_X$. Therefore (2.11) and (2.15), with the additional terminal conditions ignored, hold globally.

□

**Remark 2.4.** *(i) Theorem 3.1 and its proof suggest a new Frechet derivative based, proof for well known monotonicity results (see for instance [6] and [21]) concerning the maximal (stabilizing) solution of algebraic Riccati equations both in continuous and discrete time.*

*(ii) In both statements of Theorem 2.2 and Corollary 2.3, the existence of the solution to the TVDRE was preassumed. The following lemma which is a slight generalization of the result given in [9] gives some sufficient conditions for the above mentioned existence.*

□

**Lemma 2.5.** *Assume $D_p \geq 0$ and $R > 0$. Then for any interval $\mathbf{I} = [l, s] \subset \mathbb{Z}$ and any terminal conditon $X_s \geq 0$ the TVDRE (1.5) has a solution $X \geq 0$ on $\mathbf{I}$.*

**Proof.** Rewrite (1.5) as

$$X = A_{cl}^* \sigma X A_{cl} + [I \ \ F^*] D_p \begin{bmatrix} I \\ F \end{bmatrix}, \quad X_s = (\sigma X)_{s-1} \geq 0$$

and, since $R_s + B_s^* X_s B_s > 0$ (see also (1.6)) the conclusion follows trivially by induction.

## 3. Global monotonicity and some comparison results

The main result of this section is stated in the subsequent theorem.

**Theorem 3.1.** *Let $\mathbf{I} = [l, s] \subset \mathbb{Z}$ be given. Let $\Xi_1 = \Xi_1^T$, $\Xi_2 = \Xi_2^T \in \mathbb{R}^{n \times n}$ with $\Xi_1 \leq \Xi_2$, and $D_{p_1}, D_{p_2}, E_1, E_2$ with $D_{p_1} \leq D_{p_2}$ and $E_1 \leq E_2$ be also given. Then the following statements hold.*

1. *If for all $\Xi$ and $D_p$ with $\Xi_1 \leq \Xi \leq \Xi_2$ and $D_{p_1} \leq D_p \leq D_{p_2}$, the TVDRE (1.5) has, for the terminal condition $X_s = \Xi$, a solution $X(\Xi, D_p)$ on $\mathbf{I}$ then*

$$X(\Xi_1, D_{p_1}) \leq X(\Xi_2, D_{p_2}). \tag{3.1}$$

2. *If $R$ is boundedly invertible and for all $\Xi$ and $E$ with $\Xi_1 \leq \Xi \leq \Xi_2$ and $E_1 \leq E \leq E_2$ the TVDRE (1.8) has, for the terminal condition $X_s = \Xi$, a solution $X(\Xi, E)$ on $I$ then*

$$X(\Xi_1, E_1) \leq X(\Xi_2, E_2). \tag{3.2}$$

**Proof.** 1. It is easy to see that for all terminal conditions and partial input data given by $\Xi_\lambda = (1-\lambda)\Xi_1 + \lambda\Xi_2$ and $D_{p_\lambda} = (1-\lambda)D_{p_1} + \lambda D_{p_2}$, respectively where $\lambda \in [0,1]$, the solution $X(\Xi_\lambda, D_{p_\lambda})$ exists on $\mathbf{I}$. Hence by applying the finite increments theorem one gets

$$X(\Xi_2, D_{p_2}) - X(\Xi_1, D_{p_1}) = X_{(\Xi,D_p)}(\tilde{\Xi}, \tilde{D}_p)((\Xi_2 - \Xi_1, D_{p_2} - D_{p_1}))$$

where $\tilde{\Xi}$ and $\tilde{D}_p$ are each on the segments $[\Xi_1, \Xi_2]$ and $[D_{p_1}, D_{p_2}]$, respectively. Since for $\Delta\Xi := \Xi_2 - \Xi_1$, $\Delta D_p := D_{p_2} - D_{p_1}$ we have $X_{(\Xi,D_p)}(\tilde{\Xi}, \tilde{D}_p)(\Delta\Xi, \Delta D_p) \geq 0$ as has been shown in the proof of 1. of Theorem 2.3, (3.1) follows.

2. Here the proof runs similarly to 1. with the Frechet derivative $X_{(\Xi,D_p)}(\tilde{\Xi}, \tilde{D}_p)$ replaced by $X_{(\Xi,E)}(\tilde{\Xi}, \tilde{E})$ and with the appropriate significance of the data. □

The next theorem generalize to the time-varying case, those results obtained in [9].

**Theorem 3.2.** *Let the partial input data matrix sequence be positive semidefinite, i.e.* $D_p \geq 0$ *and assume* $R >> 0$. *Assume also that the reduced input data matrix sequence* $E = (E_k)_{k\in\mathbb{Z}}$ *is a monotonically decreasing one, i.e.* $E_i \leq E_j$ $\forall i < j$. *If for some* $s \in \mathbb{Z}$ *there exists* $\Xi \geq 0$ *such that for the terminal condition* $X_s = \Xi$ *we have* $X_{s-1} \leq X_s$ *then the TVDRE (1.8) has a monotonically decreasing positive semidefinite solution* $X = (X_k)_{k\leq s}$, *i.e.* $0 \leq \ldots \leq X_k \leq \ldots \leq X_s$. *Here* $X_{s-1} := \tilde{A}_{s-1}^T X_s (I + S_{s-1}X_s)^{-1}\tilde{A}_{s-1} + \tilde{Q}_{s-1}$. *If in addition* $\lim_{k\to-\infty} E_k = \overline{E} \in \mathbb{R}^{2n\times 2n}$ *then the TVDRE converges, for* $k \to -\infty$, *to the discrete algebraic Riccati equation associated with* $\overline{E}$, *i.e. to* $\overline{A}^T\overline{X}(I+\overline{S}\,\overline{X})^{-1}\overline{A} - \overline{X} + \overline{Q} = 0$, *fulfilled by* $\overline{X} = \lim_{k\to-\infty} X_k$.

**Proof.** According to Lemma 2.5 the solution $X = (X_k)_{l\leq k\leq s}$ exists for *arbitrary* $l < s$ and it is positive semidefinite, i.e. $X_k \geq 0$ for $l \leq k \leq s$. Let us show by induction that $(X_k)_{k\leq s}$ is monotonically decreasing.

As $X_{s-1} \leq X_s$ we have to show that $X_{k-1} \leq X_k \Rightarrow X_{k-2} \leq X_{k-1}$. Assume temporarily that $X_{k-1} > 0$. Then we can write

$$-X_k^{-1} \geq -X_{k-1}^{-1} \tag{3.2}$$

By combining the monotonicity of $E$ with (3.2) one gets

$$\begin{bmatrix} \tilde{Q}_{k-1} & \tilde{A}_{k-1}^T \\ \tilde{A}_{k-1} & -S_{k-1} - X_k^{-1} \end{bmatrix} \geq \begin{bmatrix} \tilde{Q}_{k-2} & \tilde{A}_{k-2}^T \\ \tilde{A}_{k-2} & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix} \tag{3.3}$$

As $S_k = B_k R_k^{-1} B_k^T \geq 0$ $\forall k$, (3.3) yields

$$\begin{bmatrix} I & W_{k-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{Q}_{k-1} + \tilde{A}_{k-1}^T (S_{k-1} + X_k^{-1})^{-1} \tilde{A}_{k-1} & 0 \\ 0 & -S_{k-1} - X_k^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ W_{k-1}^T & I \end{bmatrix}$$

$$\geq \begin{bmatrix} I & W_{k-2} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{Q}_{k-2} + \tilde{A}_{k-2}^T (S_{k-2} + X_{k-1}^{-1})^{-1} \tilde{A}_{k-2} & 0 \\ 0 & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ W_{k-2}^T & I \end{bmatrix} \tag{3.4}$$

where $W_i := -\tilde{A}_i^T (S_i + X_{i+1}^{-1})^{-1}$ for $i = k-2, k-1$. Let

$$\begin{bmatrix} I & \widehat{W}_{k-1} \\ 0 & I \end{bmatrix} := \begin{bmatrix} I & -W_{k-2} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & W_{k-1} \\ 0 & I \end{bmatrix}. \tag{3.5}$$

Since $X_{i-1} = \tilde{A}_{i-1}^T (X_i^{-1} + S_{i-1})^{-1} \tilde{A}_{i-1} + \tilde{Q}_i$ for $i = k-1, k$, as directly follows from (1.8), (3.4) yields with (3.5)

$$\begin{bmatrix} I & \widehat{W}_{k-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{k-1} & 0 \\ 0 & -S_{k-1} - X_k^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \widehat{W}_{k-1}^T & I \end{bmatrix}$$

$$= \begin{bmatrix} X_{k-1} - \widehat{W}_{k-1}(S_{k-1} + X_k^{-1}) \widehat{W}_{k-1}^T & -\widehat{W}_{k-1}(S_{k-1} + X_k) \\ -(S_{k-1} + X_k^{-1}) \widehat{W}_{k-1}^T & -S_{k-1} - X_k^{-1} \end{bmatrix} \geq \begin{bmatrix} X_{k-2} & 0 \\ 0 & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix}.$$

Therefore $X_{k-1} - \widehat{W}_{k-1}(S_{k-1} + X_k^{-1}) \widehat{W}_{k-1}^T \geq X_{k-2}$. Since $S_{k-1} + X_k^{-1} > 0$ we get eventually $X_{k-1} \geq X_{k-2}$. If now the strict positiveness of $X_{k-1}$ is not fulfilled, i.e. $X_{k-1} \geq 0$, let $X_{k-1}^\varepsilon := X_{k-1} + \varepsilon I$ and $X_k^\varepsilon := X_k + \varepsilon I$ for $\varepsilon > 0$. Let $\tilde{X}_{k-1}^\varepsilon$ and $\tilde{X}_{k-2}^\varepsilon$ be obtained recurrently from $X_k^\varepsilon$ and $X_{k-1}^\varepsilon$ via (1.8), respectively. Since clearly (3.2) updated with $X_k^\varepsilon$ and $X_{k-1}^\varepsilon$ holds as well, we shall get as above $\tilde{X}_{k-1}^\varepsilon \geq \tilde{X}_{k-2}^\varepsilon$. Since both limits of $\tilde{X}_{k-1}^\varepsilon$ and $\tilde{X}_{k-2}^\varepsilon$ exist for $\varepsilon \downarrow 0$ one gets finally $X_{k-1} \geq X_{k-2}$. As $X_k \geq 0$, $\overline{X} = \lim_{k \to -\infty} X_k \geq 0$ is well defind. Taking into acount that $\lim_{i \to -\infty} E_i = \overline{E}$, the last part of the theorem follows trivially. □

A direct consequence of Theorem 3.2, which is essentially based on 1. of Theorem 3.1, is

**Theorem 3.3.** *Let*

$$D_i = \begin{bmatrix} Q_i & A_i^* & L_i \\ A_i & 0 & B_i \\ L_i^* & B_i^* & R_i \end{bmatrix}, \qquad i = 1, 2,$$

*be two input data matrix sequences with $R_i >> 0$ and let TVDRE 1 and TVDRE 2 be the Riccati equation of the reduced form (1.8) ($R_i >> 0$) associated with the reduced input data matrix sequences $E_1$ and $E_2$ respectively, where*

$$E_i = \begin{bmatrix} \tilde{Q}_i & \tilde{A}_i^* \\ \tilde{A}_i & -S_i \end{bmatrix}, \qquad i = 1, 2,$$

($S_i = B_i R_i^{-1} B_i^*$). *Let*

$$D_{p_i} = \begin{bmatrix} Q_i & L_i \\ L_i^* & R_i \end{bmatrix}, \qquad i = 1, 2,$$

*be the associated partial input data matrix sequences. Assume that* $0 \leq D_{p_1} \leq D_{p_2}$ *and that both sequences* $E_1 = (E_{1,k})_{k\in\mathbb{Z}}$ *and* $E_2 = (E_{2,k})_{k\in\mathbb{Z}}$ *are monotonically decreasing and in addition* $\lim_{k\to-\infty} E_{i,k} = \overline{E}_i \in \mathbb{R}^{2n\times 2n}$*,* $i = 1, 2$*. Assume that for some* $s \in \mathbb{Z}$ *there exists* $\Xi \geq 0$ *such that if* $X_{2,s} = \Xi$ *then* $X_{2,s-1} \leq X_{2,s} = \Xi$ *where* $X_{2,s-1} = \tilde{A}^T_{2,s-1}X_{2,s}(I + S_{2,s-1}X_{2,s})^{-1}\tilde{A}_{s-1} + \tilde{Q}_{2,s-1}$*. If* $\Xi$ *is taken as common terminal condition at* $s \in \mathbb{Z}$ *for both TVDRE 1 and TVDRE 2 then both TVDRE 1 and TVDRE 2 have the solutions* $X_1 = (X_{1,k})_{k\leq s}$*,* $X_2 = (X_{2,k})_{k\leq s}$*, respectively with the following properties*

$$0 \leq X_{1,k} \leq X_{2,k} \forall k \leq s \tag{3.6}$$

$$X_{1,k} \downarrow \text{ and } X_{2,k} \downarrow \text{ for } k \downarrow -\infty \ (k \leq s). \tag{3.7}$$

*Furthermore, TVDRE 1 and TVDRE 2 converge for* $k \to -\infty$ *to the discrete algebraic Riccati equations* $\overline{A}_1^T \overline{X}_1(I + \overline{S}_1\overline{X}_1)^{-1}\overline{A}_1 - \overline{X}_1 + \overline{Q}_1 = 0$ *and* $\overline{A}_2^T \overline{X}_2(I + \overline{S}_2\overline{X}_2)^{-1}\overline{A}_2 - \overline{X}_2 + \overline{Q}_2 = 0$*, each of them fulfilled by* $\overline{X}_1 = \lim_{k\to-\infty} X_{1,k}$*,* $\overline{X}_2 = \lim_{k\to-\infty} X_{2,k}$*, respectively.*

**Proof.** Let

$$D = \begin{bmatrix} Q & A^* & L \\ A & 0 & B \\ L^* & B^* & R \end{bmatrix},$$

be any input data matrix sequence such that

$$0 \leq D_{p_1} \leq D_p \leq D_{p_2} \tag{3.8}$$

and let $\Xi$ be the terminal condition at $s$. Then, according to Lemma 2.5, the TVDRE associated with $D$ has a positive semidefinite solution $X = (X_k)_{l\leq k\leq s}$ for arbitrary $l < s$. As $D_p$ in (3.7) was taken arbitrarily it follows from 1. of Theorem 3.1 that (3.6) holds. Hence in particular

$$X_{1,s-1} \leq X_{2,s-1} \leq \Xi$$

that is all the conditions stated in Theorem 3.2 hold for both TVDREs, furthermore (3.7) as well as the remainder of the theorem are true. □

**Remark 3.4.** *If we put in the context of the above theorem* $A_1 = A_2$*,* $B_1 = B_2$*,* $L_1 = L_2$*,* $R_1 = R_2$ *and* $\Delta\tilde{Q} = \tilde{Q}_2 - \tilde{Q}_1$ *then the TVDREs (of type (1.8)) can be written as* $X = \tilde{A}^*\sigma X(I+S\sigma X)^{-1}\tilde{A}+\tilde{Q}_1$ *and* $X = A^*\sigma X(I+S\sigma X)^{-1}\tilde{A}+\tilde{Q}_1+\Delta\tilde{Q}$*, respectively, that is, the second equation can be seen as a perturbed form of the first one. In these circumstances Theorem 3.3 makes explicit the asymptotic behaviour of the associated solutions. More specific results are obtained in the time invariant case in [9] and for a modified Riccati difference equation in [19].* □

## 4. Global monotonicity of the stablizing solution

Let

$$D = \begin{bmatrix} Q & A^* & L \\ A & 0 & B \\ L^* & B^* & R \end{bmatrix} = D^* \tag{4.1}$$

be a (bounded) input data matrix sequences. If $A$ *defines an expontenially stable evolution* then

$$\sqcap_D := [B^*(I_{\ell^{2,n}}\sigma^{-1} - A^*)^{-1} \quad I_{\ell^{2,m}}] \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} (I_{\ell^{2,n}}\sigma - A)^{-1}B \\ I_{\ell^{2,m}} \end{bmatrix} = \sqcap_D^* \tag{4.2}$$

is a linear bounded selfadjoint operator form $\ell^{2,m}$ to $\ell^{2,m}$ and is called the *Popov operator* associated with $D$ (see [11], [12], [13], [15]). For each $k \in \mathbb{Z}$ let the Toeplitz operator $\sqcap_{D,k} : \ell^{2,m}_{k,+} \to \ell^{2,m}_{k,+}$ associated with $\sqcap_D$ at $k$ be also introduced, that is,

$$\sqcap_{D,k} = P^m_{k,+} \sqcap_D P^m_{k,+} = [\mathcal{L}_k^* \;\; I_{\ell^{2,m}_{k,+}}] \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} \mathcal{L}_k \\ I_{\ell^{2,m}_{k,+}} \end{bmatrix} \tag{4.3}$$

with $\mathcal{L}_k := P^n_{k,+}(I_{\ell^{2,n}}\sigma - A)^{-1}BP^m_{k,+} = (I_{\ell^{2,n}}\sigma - A)^{-1}BP^m_{k,+}$, and where the last written equality is a trivial consequence of the causality of the $\ell^2$-linear bounded operator $(I_{\ell^{2,n}}\sigma - A)^{-1}B$ (see [15]). If $A$ in $D$ does not define an exponentially stable evolution but *the pair* $(A, B)$ *is stabilizable*, i.e. there exists a bounded $\widehat{F}$ such that $A + B\widehat{F}$ defines an exponentially stable evolution, then let the $\widehat{F}$-*equivalent* $\widehat{D}$ of $D$ be introduced by

$$\widehat{D} := T^*_{\widehat{F}} D T_{\widehat{F}} = \begin{bmatrix} \widehat{Q} & \widehat{A}^* & \widehat{L} \\ \widehat{A} & 0 & \widehat{B} \\ \widehat{L}^* & \widehat{B}^* & \widehat{R} \end{bmatrix} \tag{4.4}$$

where

$$T_{\widehat{F}} := \begin{bmatrix} I_{\ell^{2,n}} & 0 & 0 \\ 0 & I_{\ell^{2,n}} & 0 \\ \widehat{F} & 0 & I_{\ell^{2,n}} \end{bmatrix}. \tag{4.5}$$

Performing the product $T^*_{\widehat{F}} D T_{\widehat{F}}$ we obtain the explicit connection between the entries of $D$ and $\widehat{D}$ which is

$$A = A + B\widehat{F}, \;\; \widehat{Q} = Q + L\widehat{F} + \widehat{F}^*L^* + \widehat{F}^*R\widehat{F}, \;\; \widehat{L} = L + \widehat{F}^*R. \tag{4.6}$$

Clearly $\sqcap_{\widehat{D}}$ is now well defined and in addition the following proposition holds [12].

**Proposition 4.1.** *The TVDREs (1.5) associated with D and $\widehat{D}$ share the same stabilizing solution $X$ provided it exists.* □

Of crucial importance is the following result of the so called *generalized Popov theory* (see [11], [12],[13] and [15]).

**Theorem 4.2.** *Let $D$ as in (4.1) be given and assume that $A$ defines an exponentially stable evolution. Then the following two assertions are equivalent.*

1. *The TVDRE (1.5) associated with $D$ has a (unique) stablizing solution $X$.*

*2. There exists $\rho_0 > 0$ such that*

$$\| \sqcap_{D,k} u\|_2 \geq \rho_0 \|P^m_{k,+} u\|_2 \;\forall\, k \in \mathbb{Z},\; \forall\, u \in \ell^{2,m} \tag{4.6}$$

□

From Theorem 4.2 the following corollary is easily derived [12]

**Corollary 4.3.** *Let $D$ as in (4.1) be given with $A$ defining an exponentially stable evolution. Then the following two statements are equivalent.*

1. *The TVDRE (1.5) associated with $D$ has a stabilizing solution $X$ and in addition $R + B^* \sigma X B >> 0$ and $R >> 0$.*

2. *$\sqcap_D >> 0$.* □

**Remark 4.4.** *If the submatrix $A$ in $D$ does not define an exponentially stable evolution but the pair $(A, B)$ is stabilizable, then, in the light of Proposition 4.1, we shall work with any $\widehat{F}$-equivalent $\widehat{D}$ of $D$.* □

The main result of this section is

**Theorem 4.5.** *Let the input data matrix sequence*

$$D_1 = \begin{bmatrix} Q_1 & A_1^* & L_1 \\ A_1 & 0 & B_1 \\ L_1^* & B_1^* & R_1 \end{bmatrix}$$

*be given and assume that the following hypotheses a), b), c) are fulfilled:*

*a) $(A_1, B_1)$ is stablizable.*

*b) Among the block diagonal operators $\widehat{F}$ that make $A_1 + B_1\widehat{F}$ to define an exponentially stable evolution there exists one for which $\sqcap_{\widehat{D}_1}$ is coercive, i.e.*

$$\sqcap_{\widehat{D}_1} >> 0. \tag{4.7}$$

*Here $\widehat{D_1}$ stands for the $\widehat{F}$-equivalent of $D_1$. Notice that according to Corollary 4.3, condition (4.7) implies $R_1 >> 0$. Hence $S_1 = B_1 R_1^{-1} B_1^*$ is well defined and bounded. Therefore the reduced input data sequence*

$$E_1 = \begin{bmatrix} \tilde{Q}_1 & \tilde{A}_1^* \\ \tilde{A}_1 & -S_1 \end{bmatrix}$$

*(associated with $D_1$) is well defined as well. Let*

$$E_2 = \begin{bmatrix} \tilde{Q}_2 & \tilde{A}_2^* \\ \tilde{A}_2 & -S_2 \end{bmatrix}$$

*with*

$$E_1 \leq E_2. \tag{4.8}$$

*c) Assume now that for each*

$$E = \begin{bmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -S \end{bmatrix} \tag{4.9}$$

*satisfying*

$$E_1 \leq E \leq E_2, \tag{4.10}$$

*the associated pair $(A, B)$ is stabilizable. Here $S = BR^{-1}B^*$ with $R$ assumed boundedly invertible (for instance $R \geq R_1$).*
*Then the following statements are true*

1. *For each $E$ staisfying (4.10) the associated TVDRE (1.8) has a stabilizing solution $X(E)$.*
2. *$X(E_1) \leq X(E_2)$.*

**Proof.** 1. For any reduced input data sequence $E$ as shown in (4.9), introduce the linear bounded selfadjoint operator

$$\Sigma_E = \begin{bmatrix} \tilde{Q} & \tilde{A}^* - I_{\ell^{2,n}}\sigma^{-1} \\ \tilde{A} - I_{\ell^{2,n}}\sigma & -S \end{bmatrix} = \Sigma_E^* \tag{4.11}$$

from $\ell^{2,2n}$ to $\ell^{2,2n}$. Let $\Sigma_{E,k} = P_{k,+}^{2n}\Sigma_E P_{k,+}^{2n}$ be the Toeplitz operator associated with $\Sigma_E$ at $k$. Let us prove now that the following two assertions are equivalent.

$\mathbb{A}1$. $\Sigma_{E,k}$ is uniformly bounded invertible with respect to $k \in \mathbb{Z}$.
$\mathbb{A}2$. $\sqcap_{\widehat{D},k}$ is uniformly bounded invertible with respect to $k$.

Here $\widehat{D}$ is any $\widehat{F}$-equivalent of any $D$ that generated $E$. In order to prove the equivalence of $\mathbb{A}1$ and $\mathbb{A}2$ introduce the linear bounded self adjoint operators $\Sigma_D$ and $H_D$ defined from $\ell^{2,2n+m}$ to $\ell^{2,2n+m}$ by

$$\Sigma_D = \begin{bmatrix} \Sigma_E & 0 \\ 0 & R \end{bmatrix}, \quad H_D = \begin{bmatrix} Q & A^* - I_{\ell^{2,n}}\sigma^{-1} & L \\ A - I_{\ell^{2,n}}\sigma & 0 & B \\ L^* & B^* & R \end{bmatrix}$$

for $R$ boundedly invertible. Let $\Sigma_{D,k}$ and $H_{D,k}$ be the associated Toeplitz operators at $k$. Then the following chain of double implications

$$\Sigma_{E,k} \text{ u.b.i. } \Leftrightarrow \Sigma_{D,k} \text{ u.b.i. } \Leftrightarrow H_{D,k} \text{ u.b.i. } \Leftrightarrow H_{\widehat{D},k} = T_{\widehat{F}}^* H_{D,k} T_{\widehat{F}} \text{ u.b.i.} \tag{4.12}$$

holds. Here u.b.i. abbreviates "uniformly bounded invertible" with respect to $k$. Indeed, (4.12) follows trivially by performing elementary row and column operations on $\Sigma_D$ and $H_D$, respectively. But

$$
\begin{aligned}
H_{\widehat{D}} &= \begin{bmatrix} \widehat{Q} & \widehat{A}^* - I_{\ell^2,n}\sigma^{-1} & \widehat{L} \\ \widehat{A} - I_{\ell^2,n}\sigma & 0 & B \\ \widehat{L}^* & B^* & R \end{bmatrix} \\
&= \begin{bmatrix} I & \widehat{Q} & 0 \\ 0 & \widehat{A} - I_{\ell^2,n}\sigma & 0 \\ B^*(A^* - I_{\ell^2,n}\sigma^{-1}) & \widehat{L}^* & I_{\ell^2,m} \end{bmatrix} \\
&\times \begin{bmatrix} 0 & I_{\ell^2,n} & \widehat{L} - \widehat{Q}(A - I\sigma)^{-1}B \\ I_{\ell^2,n} & 0 & (\widehat{A} - I\sigma)^{-1}B \\ 0 & 0 & \sqcap_{\widehat{D}} \end{bmatrix} \\
&\times \begin{bmatrix} I_{\ell^2,n} & 0 & 0 \\ 0 & A^* - I_{\ell^2,n}\sigma^{-1} & 0 \\ 0 & 0 & I_{\ell^2,m} \end{bmatrix}.
\end{aligned}
\tag{4.13}
$$

Taking into account that due to causality we have $P^n_{k,+}(\widehat{A} - I\sigma)P^n_{k,+} = (\widehat{A} - I\sigma)P^n_{k,+}$, (4.13) reveals that $H_{\widehat{D},k}$ is uniformly bounded if and only if $\sqcap_{\widehat{D},k}$ is uniformly bounded invertible. This conclusion combined with (4.12) leads to the equivalence of $\mathbb{A}1$ and $\mathbb{A}2$. From (4.7) it follows automatically that $\sqcap_{\widehat{D}_1,k}$ is uniformly bounded invertible with respect to $k$. Hence according to the equivalence of $\mathbb{A}1$ and $\mathbb{A}2$ just proved above, it follows that $\Sigma_{E_1,k}$ is uniformly bounded invertible with respect to $k$. But (4.10) implies

$$\Sigma_{E_1,k} \leq \Sigma_{E,k} \quad \forall\, k \in \mathbb{Z}.$$

Hence, using again the equivalence of $\mathbb{A}1$ and $\mathbb{A}2$ combined with Theorem 4.2, it follows that the TVDRE (1.8) associated with $E$ satisfying (4.10) has a stablizing solution $X(E)$. Thus 1. has been proved.

2. The proof of this point follows trivially from 2. of Theorem 3.1 (ignoring the additional terminal conditions). □

**Remark 4.6.** *Theorem 4.5 generalizes to the global situation and in the time-varying case the results obtained in [9], [25] [26], in particular global existence of the stabilizing solution is guaranted. Recently in [4] another approach has been used for the investigation of inequalities for Riccati difference equations.*

**Remark 4.7.** *Apparently condition c) in Theorem 4.5 seems to be a strong one in comparison with conditions a) and b). Indeed a) and b) are "punctual" conditions while c) claims the preservation of the stabilizability with respect to a whole family of data. In fact the* *situation is much simpler as follows from the next proposition which is the time-varying counterpart of Lemma 2.5 in [26].* □

**Propositon 4.8.** *If*

$$\begin{bmatrix} \tilde{Q}_{1,k} & \tilde{A}_{1,k}^T \\ \tilde{A}_{1,k} & -B_{1,k}B_{1,k}^T \end{bmatrix} \leq \begin{bmatrix} \tilde{Q}_{2,k} & \tilde{A}_{2,k}^T \\ \tilde{A}_{2,k} & -B_{2,k}B_{2,k}^T \end{bmatrix} \tag{4.14}$$

*then*

$$Ker\, X_{1,k}^i \subset \; Ker\, X_{2,k}^i \quad \forall\, k \geq i, \quad \forall\, i \in \mathbb{Z} \tag{4.15}$$

*where* $X_{1,k}^i$ *and* $X_{2,k}^i$ *are the controllability Gramians associated with the pairs* $(\tilde{A}_{1,k}, B_{1,k})$ *and* $(\tilde{A}_{2,k}, B_{2,k})$*, respectively, that is,*

$$X_{1,k+1}^i = \tilde{A}_{1,k}X_{1,k}^i\tilde{A}_{1,k}^T + B_{1,k}B_{1,k}^T, \;\; k \geq i, \;\; X_{1,i}^i = 0, \tag{4.16}$$

$$X_{2,k+1}^i = \tilde{A}_{2,k}X_{2,k}^i\tilde{A}_{2,k}^T + B_{2,k}B_{2,k}^T, \;\; k \geq i, \;\; X_{2,i}^i = 0. \tag{4.17}$$

**Proof.** Clearly $X_{1,k}^i, \; X_{2,k}^i \geq 0 \quad \forall k \geq i$ and (4.15) is fulfilled for $k = i$. Assume that (4.15) holds for some $k > i$. Then

$$\text{Ker}\, X_{1,k+1}^i \subset \; \text{Ker}\, X_{2,k+1}^i. \tag{4.18}$$

In order to prove (4.18) let $x \in$ Ker $X_{1,k+1}^i$. Then $x^T X_{1,k+1}^i x = 0$ implies via (4.16) that

$$x^T B_{1,k}B_{1,k}^T x = 0 \tag{4.19}$$

and

$$x^T \tilde{A}_{1,k}X_{1,k}^i\tilde{A}_{1,k}^T x = 0. \tag{4.20}$$

Since $B_{1,k}B_{1,k}^T \geq B_{2,k}B_{2,k}^T$ as (4.14) shows, it follows from (4.19) that

$$x^T B_{2,k}B_{2,k}^T x = 0. \tag{4.21}$$

Combining (4.19) with (4.21) one gets

$$x^T(-B_{2,k}B_{2,k}^T + B_{1,k}B_{1,k}^T)x = 0. \tag{4.22}$$

But (4.14) reveals that

$$\begin{bmatrix} \tilde{Q}_{2,k} - \tilde{Q}_{1,k} & \tilde{A}_{2,k}^T - \tilde{A}_{1,k}^T \\ \tilde{A}_{2,k} - \tilde{A}_{1,k} & -B_{2,k}B_{2,k}^T + B_{1,k}B_{1,k}^T \end{bmatrix} \geq 0. \tag{4.23}$$

Combining now the positiveness property shown in (4.23) with (4.22) one obtains

$$(\tilde{A}_{2,k}^T - \tilde{A}_{1,k}^T)x = 0$$

or equivalently

$$\tilde{A}_{2,k}^T x = \tilde{A}_{1,k}^T x. \tag{4.24}$$

On the other hand by combining (4.20) with the induction hypothesis one gets

$$x^T \tilde{A}_{1,k} X_{2,k}^i \tilde{A}_{1,k}^T x = 0. \tag{4.25}$$

Using now (4.24) in (4.25) we have

$$x^T \tilde{A}_{2,k} X_{2,k}^i \tilde{A}_{2,k}^T x = 0. \tag{4.26}$$

Hence by using (4.21) and (4.26), (4.17) yields

$$x^T X_{2,k+1}^i x^T = 0 \tag{4.27}$$

and (4.18) follows from (4.27). □

**Remark 4.9.** *Assume that the pair $(\tilde{A}_2, B_2)$ is uniformly controllable, that is, there exist $\rho > 0$ and an integer $\nu > 0$ such that for all $i \in \mathbb{Z}$ we have*

$$X_{2,i+\nu}^i \geq \rho I_n. \tag{4.28}$$

*According to Proposition 4.8 it follows that*

$$X_{1,i+\nu}^i > 0 \tag{4.29}$$

*for all $i \in \mathbb{Z}$. Hence, for achieving the controllability property for the pair $(\tilde{A}_1, B_1)$, we have to strengthen a little bit more (4.29), i.e. to assume the existence of $\widehat{\rho} > 0$ such that*

$$X_{1,i+\nu}^i \geq \widehat{\rho} I_n \tag{4.30}$$

*instead of (4.29). Clearly this always happens in the time-invariant case. Hence by assuming the uniform controllability of the pair $(\tilde{A}_2, B_2)$, the uniform controllability of the pair $(\tilde{A}_1, B_1)$ is in fact a uniformity assumption, namely that (4.30) is fulfilled. As uniform controllabillity implies stabilizability, the comments on condition c) in Theorem 4.5 end.* □

# 5. Conclusions

In this paper several results concerning monotonicity of the solution to the TVDRE have been proved. In a way, these results generalize and extend the former ones obtained up to now in the literature. It is whorthwhile to mention that the Frechet derivative type approach, used in this paper, works similarly in the continous case. Moreover, using the second Frechet derivative, convexity properties of the solution to the Riccati equation can be proved analogously. Comparison techniques turn out to be a very efficient tool for the proof of existence theorems both in the time-varying and time-invariant case.

## Appendix

In this appendix we shall prove formulae (2.1), (2.4), (2.7) and (2.8). For this purpose let $D_Q$ be obtained from $D$ by zeroing all its entries except $Q$. Let $D_A, D_B, D_L$ and $D_R$ be obtained in a similar way but with respect to $A, B, L$ and $R$, further let $\Delta D_Q, \Delta D_A, \Delta D_B, \Delta D_L$ and $\Delta D_R$ be obtained similarly from $\Delta D$. Let also $\Phi := L + A^*\sigma XB$ and $\Gamma := R + B^*\sigma XB$ be introduced. Then for $\varepsilon > 0$ one gets

$$\mathbf{F}(D+\varepsilon\Delta D_Q, X) = \mathbf{F}(D,X) + \varepsilon\Delta D_Q \tag{A1}$$

$$\mathbf{F}(D+\varepsilon\Delta D_B, X) = A^*\sigma XA - X + Q$$

$$-(\Phi + \varepsilon A^*\sigma X\Delta B)[\Gamma + \varepsilon(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB)]^{-1}(\Phi^* + \varepsilon\Delta B\sigma XA^*) \tag{A2}$$

But

$$[\Gamma + \varepsilon(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB)]^{-1} = [I + \varepsilon\Gamma^{-1}(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB)]^{-1}\Gamma^{-1}$$

$$= \Gamma^{-1} - \varepsilon\Gamma^{-1}(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB)\Gamma^{-1} + o(\varepsilon) \tag{A3}$$

for $\varepsilon$ small enough such that $\varepsilon\|\Gamma^{-1}(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB\| < 1$. Here $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ With (A3) substituted in (A2) one obtains

$$\mathbf{F}(D+\varepsilon\Delta D_B, X) = \mathbf{F}(D,X)$$

$$+\varepsilon[\Phi\Gamma^{-1}(B^*\sigma X\Delta B + (\Delta B)^*\sigma XB)\Gamma^{-1}\Phi^* - \Phi\Gamma^{-1}(\Delta B)^*\sigma XA - A^*\sigma X\Delta B\Gamma^{-1}\Phi^*] + o(\varepsilon)$$

$$= \mathbf{F}(D,X) + \varepsilon[F^*B^*\sigma X\Delta BF + F^*(\Delta B)^*\sigma XBF + F^*(\Delta B)^*\sigma XA + A^*\sigma X\Delta BF] + o(\varepsilon)$$

$$= \mathbf{F}(D,X) + \varepsilon[A_{cl}^*\sigma X\Delta BF + F^*\Delta B^*\sigma XA_{cl}] + o(\varepsilon) \tag{A4}$$

where $F^* = -\Phi^*\Gamma^{-1}$.

$$\mathbf{F}(D+\varepsilon\Delta D_A, X) = (A+\varepsilon\Delta A)^*\sigma X(A+\varepsilon\Delta A) - X + Q$$

$$-(\Phi + \varepsilon(\Delta A)^*\sigma XB)\Gamma^{-1}(\Phi^* + \varepsilon B^*\sigma X\Delta A)$$

$$= \mathbf{F}(D,X) + \varepsilon[A^*\sigma X\Delta A + (\Delta A)^*\sigma XA + F^*B^*\sigma X\Delta A + (\Delta A)^*\sigma XBF] + o(\varepsilon)$$

$$= \mathbf{F}(D,X) + \varepsilon(A_{cl}^*\sigma X\Delta A + (\Delta A)^*\sigma XA_{cl}) + o(\varepsilon). \tag{A5}$$

$$\mathbf{F}(D+\varepsilon\Delta D_L, X) = A^*\sigma XA - X + Q - (\Phi + \varepsilon\Delta L)\Gamma^{-1}(\Phi^* + \varepsilon(\Delta L)^*)$$

$$= \mathbf{F}(D,X) + \varepsilon(\Delta LF + F^*(\Delta L)^*) + o(\varepsilon). \tag{A6}$$

$$\mathbf{F}(D+\varepsilon\Delta D_R, X) = A^*\sigma XA - X + Q - \Phi(\Gamma + \varepsilon\Delta R)^{-1}\Phi^*$$

$$= A^*\sigma XA - X + Q - \Phi(I + \varepsilon\Gamma^{-1}\Delta R)^{-1}\Gamma^{-1}\Phi^*. \tag{A7}$$

But

$$(I + \varepsilon\Gamma^{-1}\Delta R)^{-1} = I - \varepsilon\Gamma^{-1}\Delta R + o(\varepsilon) \tag{A8}$$

for $\varepsilon$ small enough such that $\varepsilon\|\Gamma^{-1}\Delta R\| < 1$. With (A8) in (A7) one gets eventually

$$\mathbf{F}(D+\varepsilon\Delta D_R, X) = \mathbf{F}(D,X) + \varepsilon F^*\Delta RF + o(\varepsilon). \tag{A9}$$

By taking $\varepsilon \downarrow 0$ in $[\mathbf{F}(D+\varepsilon\Delta, X) - \mathbf{F}(D,X)]/\varepsilon$ where $\Delta$ stands succesively for $\Delta D_Q$, $\Delta D_B$, $\Delta D_A$, $\Delta D_L$ and $\Delta D_R$, we obtain from (A1), (A4), (A5), (A6) and (A9)

$$\mathbf{F}_{D_Q}(D,X)(\Delta D_Q) = \Delta Q, \quad \mathbf{F}_{D_B}(D,X)(\Delta D_B) = A_{cl}^*\sigma X\Delta BF + F^*(\Delta B)^*\sigma XA_{cl},$$

$$\mathbf{F}_{D_A}(D,X)(\Delta D_A) = A_{cl}^*\sigma X\Delta A + (\Delta A)^*\sigma X A_{cl}, \quad \mathbf{F}_{D_L}(D,X)(\Delta L) = \Delta L F + F^*(\Delta L)^*$$
$$\mathbf{F}_{D_R}(D,X)(\Delta R) = F^*\Delta R F. \tag{A10}$$

Hence, by using (A10) one gets

$$\begin{aligned}
&\mathbf{F}_D(D,X)(\Delta D) = \mathbf{F}_{D_Q}(D,X)(\Delta D_Q) + \mathbf{F}_{D_A}(D,X)(\Delta D_A)\\
&+\mathbf{F}_{D_B}(D,X)(\Delta D_B) + \mathbf{F}_{D_L}(D,X)(\Delta D_l) + \mathbf{F}_{D_R}(D,X)(\Delta D_R)\\
&= \Delta Q + A_{cl}^*\sigma X\Delta A + (\Delta A)^*\sigma X A_{cl} + A_{cl}^*\sigma X\Delta B F + F^*(\Delta B)^*\sigma X A_{cl}\\
&+\Delta L F + F^*(\Delta L)^* + F^*\Delta R F
\end{aligned} \tag{A11}$$

Using now (2.2) and (2.3), (2.1) is easily recovered from (A11).
Let now $R$ be boundedy invertible and denote by $\Psi$ the right-hand side of (A11). Then, using the first expression in the right-hand side of (1.3), one gets the updated $\Psi$ to be

$$\begin{aligned}
&\Psi = \Delta\tilde{Q} + A_{cl}^*\sigma X\Delta\tilde{A} + (\Delta\tilde{A})^*\sigma X A_{cl}\\
&+A_{cl}^*\sigma X\Delta B\tilde{F} + \tilde{F}^*(\Delta B)^*\sigma X A_{cl} + \tilde{F}^*\Delta R\tilde{F}
\end{aligned} \tag{A12}$$

where now

$$\begin{aligned}
\tilde{F} &= F + R^{-1}L^* = (R + B^*\sigma X B)^{-1}B^*\sigma X\tilde{A} = -R^{-1}B^*\sigma X(I + S\sigma X)^{-1}\tilde{A}\\
&= -R^{-1}B^*\sigma X A_{cl}.
\end{aligned} \tag{A13}$$

In (A13) we used

$$A_{cl} = \tilde{A} - B(R + B^*\sigma X B)^{-1}B^*\sigma X\tilde{A} = (I + S\sigma X)^{-1}\tilde{A}. \tag{A14}$$

With (A14) in (A12) we have

$$\begin{aligned}
&\Psi = \Delta\tilde{Q} + A_{cl}^*\sigma X\Delta\tilde{A} + (\Delta\tilde{A})^*\sigma X A_{cl}\\
&-A_{cl}^*\sigma X\Delta B R^{-1}B^*\sigma X A_{cl} - A_{cl}^*\sigma X B R^{-1}(\Delta B)^*\sigma X A_{cl}\\
&= \Delta\tilde{Q} + A_{cl}^*\sigma X\Delta\tilde{A} + (\Delta\tilde{A})^*\sigma X A_{cl}\\
&-A_{cl}^*\sigma X[\Delta B R^{-1}B^* + B R^{-1}(\Delta B)^* - B R^{-1}\Delta R R^{-1}B^*]\sigma X A_{cl}\\
&= \Delta\tilde{Q} + A_{cl}^*\sigma X\Delta\tilde{A} + (\Delta\tilde{A})^*\sigma X A_{cl} - A_{cl}^*\sigma X\Delta S\sigma X A_{cl}.
\end{aligned} \tag{A15}$$

Using now (2.5) and (2.6), (2.4) is easily obtained from (A15). In (A15), $\Delta S$ comes from the Frechet derivative of $S$ with respect to $B$ and $R$, that is $[(B+\varepsilon\Delta B)R^{-1}(B+\varepsilon\Delta B)^* - S]/\varepsilon \to \Delta B R^{-1}B^* + (\Delta B)^*R^{-1}B$ and $[B(R+\varepsilon\Delta R)^{-1}B^* - S]/\varepsilon \to -BR^{-1}\Delta R R^{-1}B^*$, both limits taken for $\varepsilon \downarrow 0$. Notice that (2.4) can be also directly derived from the right most expession in (1.3) in a similar way as (2.1) has been obtained. However (A15) and the comments below show explicitly how $\Delta D$ becomes $\Delta E$ via the Frechet derivative of $S = BR^{-1}B^*$ with respect to $B$ and $R$ (see also Remark 2.2).

Let us prove now (2.7). We have

$$\begin{aligned}
&\mathbf{F}(D, X+\varepsilon\Delta X) = A^*(\sigma X + \varepsilon\sigma\Delta X)A - X - \varepsilon\Delta X + Q\\
&-(\Phi + \varepsilon A^*\varepsilon\sigma\Delta X B)(\Gamma + \varepsilon B^*\sigma\Delta X B)^{-1}(\Phi^* + \varepsilon B^*\sigma\Delta X A).
\end{aligned} \tag{A16}$$

But

$$(\Gamma+\varepsilon B^*\sigma\Delta XB)^{-1}=(I+\varepsilon\Gamma^{-1}B^*\sigma\Delta XB)^{-1}\Gamma^{-1}=I-\varepsilon\Gamma^{-1}B^*\sigma\Delta XB\Gamma+o(\varepsilon) \quad (A17)$$

for $\varepsilon$ small enough such that $\varepsilon\|\Gamma^{-1}B^*o\Delta XB\|<1$. With (A17) in (A16) one gets

$$\begin{aligned}
&\mathbf{F}(D,X+\varepsilon\Delta X)=\mathbf{F}(D,X)\\
&+\varepsilon[A^*\sigma\Delta XA-A^*\sigma\Delta XB\Gamma^{-1}\Phi^*-\Phi\Gamma^{-1}B^*\sigma\Delta XA+\Phi\Gamma^{-1}B^*\sigma\Delta XB\Gamma^{-1}\Phi^*-\Delta X]+o(\varepsilon)\\
&=\mathbf{F}(D,X)+\varepsilon[A^*\sigma\Delta XA+A^*\sigma\Delta XBF+F^*B^*\sigma\Delta XA+F^*B^*\sigma\Delta XBF-\Delta X]+o(\varepsilon)\\
&=\mathbf{F}(D,X)+\varepsilon[A_{cl}^*\sigma\Delta XA_{cl}-\Delta X]+o(\varepsilon). \qquad (A18)
\end{aligned}$$

By taking $\varepsilon\downarrow 0$ in $[\mathbf{F}(D,X+\varepsilon\Delta X)-\mathbf{F}(D,X)]/\varepsilon$, (A18) yields (2.7). Let us prove finally (2.8). To this end compute

$$\begin{aligned}
&\mathbf{F}_D(D,X+\varepsilon\Delta Z)(\Delta X)=\\
&[A-B(\Gamma+\varepsilon B^*\sigma\Delta ZB)^{-1}(\Phi^*+\varepsilon B^*\sigma\Delta ZA)]^*\sigma\Delta X\\
&\times[A-B(\Gamma+\varepsilon B^*\sigma\Delta ZB)^{-1}(\Phi^*+\varepsilon B^*\sigma\Delta ZA)]-X\\
&=[A-B(I-\varepsilon\Gamma^{-1}B^*\sigma\Delta ZB)\Gamma^{-1}(\Phi^*+\varepsilon B^*\sigma\Delta ZA)]^*\sigma\Delta X \qquad (A19)\\
&\times[A-B(I-\varepsilon\Gamma^{-1}B^*\sigma\Delta ZB)\Gamma^{-1}(\Phi^*+\varepsilon B^*\sigma\Delta ZA)]-X+o(\varepsilon)\\
&=\mathbf{F}_D(D,X)(\Delta X)-\varepsilon[A_{cl}^*(\sigma\Delta XB(R+B^*\sigma XB)^{-1}B^*\sigma\Delta Z\\
&+o\Delta ZB(R+B^*\sigma XB)^{-1}B^*\sigma\Delta X)A_{cl}]+o(\varepsilon).
\end{aligned}$$

For $\varepsilon\downarrow 0$ in $[\mathbf{F}_D(D,X+\varepsilon\Delta Z)(\Delta X)-\mathbf{F}_D(D,X)(\Delta X)]/\varepsilon$, (A19) yields (2.8). If $R^{-1}$ is boundedly invertible, then simple computations lead to the rightmost term in (2.9).

# References


[1] J. Ball, I. Gohberg, M.A. Kaashoek, Nevanlinna-Pick interpolation for time-varying input-output maps: The discrete case, Operator Theory, Advances and Applications 56, pp. 1-51, Birkhäuser, 1992.

[2] R. R. Bitmead, M. Gevers, I. R. Petersen and R.J. Kaye, Monotonicity and stability properties of solutions of the Riccati difference equation: Propositions, lemmas, theorems, fallacious conjectures and counterexamples, Syst. Contr. Letters 5, 309-315, 1985.

[3] R. R. Bitmead and M. Gevers, Riccati difference and differential equations: Convergence, monotonicity and stability, In: The Riccati equation (ed. S. Bittanti et. al.), Springer Verlag, Berlin, 1991.

[4] M. Bohner, O. Došlý and W. Kratz, Inequatilities and asymptotics for Riccati matrix difference operators, submitted.

[5] W. A. Coppel, Disconjugacy, Lecture Notes in Mathematics 220, Springer Verlag, Berlin, 1971.

[6] R.F. Curtain and L. Rodman, Comparison theorems for infinite dimensional Riccati equations, *Syst. Contr. Letters* 15, 153-159, 1990.

[7] D. F. Delchamps, A note on the analyticity of the Riccati metric. In: Lectures in applied mathematics, Vol. 18 (ed. C.I. Byrnes and C.F. Martin), 37-41, AMS, Providence, Rhode Island, 1980.

[8] G. Freiling and G. Jank, Existence and comparison theorems for algebraic Riccati equations and Riccati differential and difference equations, *J. Dynam. Contr. Syst.* 2, 529-547, 1996.

[9] G. Freiling, G. Jank and H. Abou-Kandil, Generalized Riccati difference and differential equations, Lin.Alg. Appl. 241-243, 291-303, 1996.

[10] P. Gahinet, On the game Riccati equations arising in $H_\infty$ control problems. *SIAM Journal of Control and Optimization*, 32, 635-647, 1994.

[11] A. Halanay, V. Ionescu, Generalized discrete-time Popov-Yakubovich theory, Systems Contr. Letters 20, pp. 1-6, 1993.

[12] A. Halanay, V. Ionescu, Time-varying discrete linear systems, Birkhäuser, 1994.

[13] A. Halanay, V. Ionescu, Anticausal stabilizing solution to discrete reverse-time Riccati equation, Computers and Mathematics with Applications, Special Issue on Difference Equations, Pergamon Press 28, no 1-3, pp. 115-125, 1994.

[14] G. Hewer, Existence theorems for positive semidefinite and sign indefinite stabilizing solution of $H_\infty$ Riccati equations. *SIAM Journal of Control and Optimization*, 31, 16-29, 1993.

[15] V. Ionescu, Generalized time-varying Popov-Yakubovich theory: The discrete case, Proceeding of the MTNS'93 vol. II, pp. 717-720, Akademie Verlag, 1994.

[16] L. V. Kantorovich and G. P. Akilov, Functional analysis in normed spaces, Pergamon Press, Oxford, 1964.

[17] P. Lancaster, A. C. M. Ran and L. Rodman, An existence and monotonicity theorem for the discrete algebraic matrix Riccati equation, Lin. Multilin. Algebra 20, 353-361, 1987.

[18] X. P. Li and B. C. Chang, On convexity of $H^\infty$ Riccati solutions and its applications, IEEE Trans. Autom. Contr. 38, 963-966, 1993.

[19] T. Morozan, Stability and control for linear discrete-time systems with Markov perturbations, Rev. Roum. Math. Pur. Appl. 40, 471-494, 1995.

[20] G. de Nicolao, On the time-varing Riccati difference equation of optimal filtering, SIAM J. Contr. Opt. 30, 1251-1269, 1992.

[21] A. C. M. Ran and R. Vreugdenhil, Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, Lin. Alg. Appl. 99, 63-83, 1988.

[22] C. E. de Souza, On stabilizing properties of the Riccati difference equation, IEEE Trans. Automat. Contr. 34, 1313-1316, 1989.

[23] A. N. Stokes, A special property of the matrix Riccati equation, Bull. Austr. Math. Soc. 10, 245-253, 1974.

[24] K. Takaba and T. Katayama, Discrete-time $H_\infty$ algebraic Riccati equation and parameterization of all $H_\infty$ filters, Int. J. Contr. 64, 1129-1149, 1996.

[25] H. K. Wimmer and M. Pavon, A comparison theorem for matrix Riccati difference equations, Syst. Contr. Letters 9, 233-239, 1992.

[26] M. K. Wimmer, Monotonicity and maximality of solutions of discrete-time algebraic Riccati equation, J. of Mathematical Systems, Estimation and Control, 2, no 2, pp. 219-235, 1992.

[27] G. F. Wredenhagen and P. R. Bélanger, Curvature properties of the algebraic Riccati equation, Syst. Constr. Letters 21, 285-287, 1993.