Inverse problems for differential equations with turning points*

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Abstract. We study inverse spectral problems for second-order differential equations on a finite interval having an arbitrary number of turning points. For three classes of inverse problems we prove that they have a *unique* solution and we provide a procedure for constructing the solution.

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1. Introduction. We consider boundary value problems L of the form

$$\ell y \equiv -y'' + q(x)y = \lambda R^2(x)y,\ x \in [0,1], \tag{1}$$

$$U(y) \equiv y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) \equiv y'(1) + h_1 y(1) = 0. \tag{3}$$

Here $\lambda = \rho^2$ is the spectral parameter; R^2 and q are real functions, and h, h_1 are real numbers. We suppose that

$$R^2(x) = \prod_{\nu=1}^{m} (x - x_\nu)^{\ell_\nu} R_0(x),$$

where $0 < x_1 < x_2 < \ldots < x_m < 1$, $\ell_\nu \in I\!N$, $R_0(x) > 0$ for $x \in I := [0,1]$, and R_0 is twice continuously differentiable on I. In the other words, R^2 has in I m zeros x_ν, $\nu = \overline{1,m}$ of order ℓ_ν. Zeros x_ν of R^2 are called turning points. We also assume that q is bounded and integrable on I.

In this paper we study the three inverse problems of recovering L from its spectral characteristics, namely

(i) from the Weyl function,

(ii) from two spectra and

(iii) from the so-called spectral data.

Differential equations with turning points play an important role in various areas of mathematics as well as in applications (see the textbooks [2], [3], [4] and [12] for details).
For example, turning points connected with physical situations in which zeros correspond to the limit of motion of a wave mechanical particle bound by a potential field. Turning points appear also in elasticity, optics, geophysics and other branches of natural sciences. Moreover, a wide class of differential equations with Bessel-type singularities and their perturbations can be reduced to differential equations having turning points. We also point out on concrete applications for inverse spectral problems considered in this paper; such inverse problems appear, for example, in electronics for constructing parameters of heterogeneous electronics lines with desirable technical characteristics. After reduction of the corresponding mathematical model we come in this case to a boundary-value problem L where R(x) reflects apriori known parameters, and q(x) must be constructed from the given spectral information which describes

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desirable amplitude and phase characteristics, for example, transmission coefficients. Further, it is well-known that inverse spectral problems play an important role for investigating some nonlinear integrable evolution equations of mathematical physics (like KdF equation). Inverse problems for equations with turning points and singularities help to study blow-up behavior of solutions for such nonlinear equations. We also note that inverse problems considered here appear in mathematics for investigating spectral properties of some classes of differential, integro-differential and integral operators; for further discussion of applications see [2], [3], [4] and [12].

Inverse problems for the case $R^2(x) > 0$ (in particular, when $R^2(x) \equiv 1$) have been studied fairly completely in many works (see, for example, [1]-[6] and the references therein). An important role there was played by the transformation operator method and the Gelfand-Levitan integral equation with respect to the kernel of the transformation operator. The presence of turning points in the differential equations produces essential qualitative difficulties in the investigation of the inverse problems for the boundary value problem L. The transformation operator method in this case is not suitable for the solution of the inverse problems.

To study the inverse problems in this paper we use another approach connected with the contour integral method. An important role in this method is played by the special fundamental system of solutions (FSS) for equation (1) constructed in [7]. This FSS gives us an opportunity to obtain the asymptotic behavior of the so-called Weyl solutions and the Weyl function for the boundary value problem L and to solve the corresponding inverse problems. In Sections 3-5 we prove uniqueness theorems, and in Section 6 we provide a constructive procedure for the solution of the inverse problem. Note that in [8], [9] the distribution of the eigenvalues and an expansion theorem for L have been studied, and in [11], [12] the contour integral method has been used for the solution of the inverse problems for arbitrary-order differential operators without turning points. Inverse problems for arbitrary-order differential operators on the half-line with singularities were studied in [13]. For the case $R^2(x) \equiv 1$, connections between the Gelfand-Levitan equation and the main equation obtained by the contour integral method were provided in [11], [12].

2. Notations and Preliminary Results. Let $\varepsilon > 0$ be fixed, sufficiently small and let $D_{0\varepsilon} = [0, x_1 - \varepsilon]$, $D_{\nu\varepsilon} = [x_\nu + \varepsilon,\ x_{\nu+1} - \varepsilon]$ for $1 \leq \nu \leq m-1$, $D_{m\varepsilon} = [x_m + \varepsilon, 1]$, $D_\varepsilon = \bigcup_{\nu=0}^{m} D_{\nu\varepsilon}$, and $I_{\nu\varepsilon} = D_{\nu-1,\varepsilon} \cup [x_\nu - \varepsilon,\ x_\nu + \varepsilon] \cup D_{\nu\varepsilon}$.

We distinguish four different types of turning points: For $1 \leq \nu \leq m$

$$T_\nu = \begin{cases} I, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x-x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ II, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x-x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu\varepsilon}, \\ III, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x-x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ IV, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x-x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu\varepsilon}, \end{cases}$$

is called type of x_ν. Further we set for $1 \leq \nu \leq m$

$$\mu_\nu = \frac{1}{2+\ell_\nu},$$

$$\theta_\nu = \begin{cases} 1 & \text{if } \mu_\nu > \dfrac{1}{4}, \\ 1-\delta_0 \text{ (with } \delta_0 > 0 \text{ arbitrary small)} & \text{if } \mu_\nu = \dfrac{1}{4}, \\ 4\mu_\nu & \text{if } \mu_\nu < \dfrac{1}{4}, \end{cases}$$

and $\theta_0 = \min\{\theta_\nu | 1 \leq \nu \leq m\}$.
We also denote

$$I_+ = \{x : R^2(x) > 0\},\ I_- = \{x : R^2(x) < 0\},$$

$$\xi(x) = \begin{cases} 0, & \text{for } x \in I_+ \\ 1, & \text{for } x \in I_-, \end{cases}$$

$$\gamma_\nu = \begin{cases} 2\sin\dfrac{\pi\mu_\nu}{2}, & \text{for } T_\nu = III, IV, \\ \sin\pi\mu_\nu, & \text{for } T_\nu = I, II, \end{cases}$$
$$K_\pm(x) = (\prod_{x_\nu \in (0,x)} \gamma_\nu^{-1}) \exp(\pm i\frac{\pi}{4}(\xi(x) - \xi(0))),$$
$$K_\pm^*(x) = (\prod_{x_\nu \in (0,x)} \gamma_\nu) \exp(\pm i\frac{\pi}{4}(\xi(x) + \xi(0))),$$
$$R_+^2(x) = \max(0, R^2(x)),\ R_-^2(x) = \max(0, -R^2(x)).$$

Clearly,

$$K_\pm(x)K_\pm^*(x) = \exp(\pm i\frac{\pi}{2}\xi(x)) = \begin{cases} 1, & \text{if } x \in I_+ \\ \pm\, i, & \text{if } x \in I_-. \end{cases}$$

Let

$$S_k = \{\rho | \arg\rho \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\},$$
$$\sigma_s^\delta = \{\rho | \arg\rho \in [\frac{s\pi}{2} - \delta, \frac{s\pi}{2} + \delta]\},\ \delta > 0,$$

$\sigma^\delta = \bigcup\limits_s \sigma_s^\delta,\ S_k^\delta = S_k \setminus \sigma^\delta,\ S^\delta = \bigcup\limits_{k=-2}^{1} S_k^\delta.$

Below it is sufficient to consider the sectors S_k and S_k^δ for $k = -2, -1, 0, 1$ only.

It is shown in [7] that for each fixed sector S_k $(k = -2, -1, 0, 1)$ there exists a FSS of (1) $\{z_1(x,\rho), z_2(x,\rho)\},\ x \in I,\ \rho \in S_k$ such that the functions $(x,\rho) \mapsto z_j^{(s)}(x,\rho)$ $(j = 1, 2; s = 0, 1)$ are continuous for $x \in I,\ \rho \in S_k$ and holomorphic for each fixed $x \in I$ with respect to $\rho \in S_k$; moreover for $|\rho| \to \infty,\ \rho \in S_k,\ x \in D_\varepsilon,\ j = 1, 2$

$$z_1^{(j)}(x,\rho) = (\pm i\rho)^j |R(x)|^{j-1/2} (\exp(\mp i\frac{\pi}{2}\xi(x))^j \exp(\rho\int_0^x |R_-(t)|dt)$$
$$\times \exp(\pm i\rho\int_0^x |R_+(t)|dt) K_\pm(x)\kappa(x,p), \tag{4}$$
$$z_2^{(j)}(x,\rho) = (\mp i\rho)^j |R(x)|^{j-1/2} (\exp(\mp i\frac{\pi}{2}\xi(x))^j \exp(-\rho\int_0^x |R_-(t)|dt)$$
$$\times \exp(\mp i\rho\int_0^x |R_+(t)|dt) K_\pm^*(x)\kappa(x,\rho), \tag{5}$$
$$\begin{vmatrix} z_1(x,\rho) & z_2(x,\rho) \\ z_1'(x,\rho) & z_2'(x,\rho) \end{vmatrix} = \mp(2i\rho)[1]. \tag{6}$$

Here and in the sequel

(i) the upper or lower signs in formulas correspond to the sectors S_{-2}, S_{-1} or S_0, S_1, respectively;

(ii) $[1] \overset{\text{def}}{=} 1 + O(\dfrac{1}{\rho^{\theta_0}})$ uniformly in $x \in D_\varepsilon$,

(iii) one and the same symbol $\kappa(x,\rho)$ denotes various functions such that:

(1) uniformly in $x \in D_\varepsilon,\ \kappa(x,\rho) = O(1)$ as $|\rho| \to \infty,\ \rho \in S_k$,

(2) for each fixed $\delta > 0,\ \kappa(x,\rho) = [1]$ as $|\rho| \to \infty,\ \rho \in S_k^\delta$.

3. Problem I. Let $\varphi(x,\lambda)$ be the solution of (1) under the initial conditions $\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = h$ and denote

$$\Delta(\lambda) = \varphi'(1,\lambda) + h_1\varphi(1,\lambda). \tag{7}$$

The function $\lambda \mapsto \Delta(\lambda)$ is entire in λ, and its zeros coincide with the eigenvalues $\{\lambda_n\}_{n\geq 0}$ of the boundary value problem L. The functions $x \mapsto \varphi(x,\lambda_n)$ are eigenfunctions of L.

Let $\Phi(x,\lambda)$ be the solution of (1) under the boundary conditions $U(\Phi)=1,\ V(\Phi)=0$. We set $\mathfrak{M}(\lambda) = \Phi(0,\lambda)$. The functions $x \mapsto \Phi(x,\lambda)$ and $\lambda \mapsto \mathfrak{M}(\lambda)$ are called the *Weyl solution* and the *Weyl function* for the boundary value problem (1)-(3), respectively. Clearly

$$\begin{vmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{vmatrix} \equiv 1 \tag{8}$$

for all x and λ.

The inverse problem is formulated as follows. Suppose that the function R^2 is known a priori. Our goal is to find $q(x), h$ and h_1 from the given Weyl function $\mathfrak{M}$.
In order to formulate and prove the uniqueness theorem for the solution of the inverse problem we agree that together with $L = L(R^2(x), q(x), h, h_1)$ we consider a boundary value problem $\tilde{L} = L(R^2(x),\ \tilde{q}(x), \tilde{h}, \tilde{h}_1)$ of the same form (1) - (3) but with different coefficients. If a certain symbol denotes an object related to L, then the corresponding symbol with tilde will denote the analogous object related to $\tilde{L}$.

Theorem 1. If $\mathfrak{M} = \tilde{\mathfrak{M}}$ then $q(x) = \tilde{q}(x)$ for $x \in I,\ h = \tilde{h}$ and $h_1 = \tilde{h}_1$.

Thus, the specification of the Weyl function $\mathfrak{M}$ uniquely determines L.

Proof. Let us define the matrix

$$P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$$

by the formula

$$P(x,\lambda) \begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{9}$$

Using (8) we calculate

$$\left.\begin{array}{l} P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi(x,\lambda)\tilde{\varphi}'(x,\lambda) \\ P_{12}(x,\lambda) = \Phi(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{\Phi}(x,\lambda) \end{array}\right\} \tag{10}$$

Let us now study the asymptotic behavior of $\varphi(x,\lambda),\ \Phi(x,\lambda)$ and $P_{1j}(x,\lambda)$ as $|\lambda| \to \infty$. For this purpose we use the above-mentioned FSS $\{z_1(x,\rho), z_2(x,\rho)\}$ in each fixed sector $S_k,\ k=-2,-1,0,1$.

From the initial conditions on $\varphi(x,\lambda)$ we calculate

$$\varphi(x,\lambda) = \frac{1}{w(\lambda)}(U(z_2(x,\rho))z_1(x,\rho) - U(z_1(x,\rho))z_2(x,\rho)), \tag{11}$$

where

$$w(\lambda) = \begin{vmatrix} z_1(0,\rho) & z_2(0,\rho) \\ z_1'(0,\rho) & z_2'(0,\rho) \end{vmatrix}.$$

Using (4) - (6) we get for $|\rho| \to \infty, \rho \in S_k$

$$\left.\begin{array}{l} U(z_1(x,\rho)) = (\pm\, i\ \rho)|R(0)|^{1/2}\exp(\mp\, i\frac{\pi}{2}\xi(0))\kappa(\rho), \\ U(z_2(x,\rho)) = (\mp\, i\rho)|R(0)|^{1/2}\kappa(\rho), \\ w(\lambda) = \mp 2i\rho[1]; \end{array}\right\} \tag{12}$$

here and in the sequel $\kappa(\rho) = O(1)$ for $\rho \in S_k$ and $\kappa(\rho) = [1]$ for $\rho \in S_k^\delta$ as $|\rho| \to \infty$.
Substituting (4), (5) and (12) into (11) we conclude that for $|\rho| \to \infty,\ \rho \in S_k$, $x \in D_\varepsilon,\ j = 0,1$

$$\begin{array}{l} \varphi^{(j)}(x,\lambda) = \frac{1}{2}(\pm\, i\rho)^j |R(0)|^{1/2}|R(x)|^{j-1/2}(\exp(\mp\, i\frac{\pi}{2}\xi(x)))^j \\ \times \exp(\rho\int_0^x |R_-(t)|dt)\exp(\pm\, i\rho\int_0^x |R_+(t)|dt)K_\pm(x)\kappa(x,\rho). \end{array} \tag{13}$$

Consequently, taking (7) into account, we have

$$\begin{aligned}&\Delta(\lambda)=\frac{1}{2}(\pm i\rho)|R(0)R(1)|^{1/2}\exp(\rho\int_0^1|R_-(t)|dt)\\ &\times\exp(\pm i\rho\int_0^1|R_+(t)|dt)K_\pm(1)\kappa(\rho),\ \rho\in S_k,\ |\rho|\to\infty.\end{aligned}\tag{14}$$

From (13) and (14) we obtain the estimates:

$$\begin{aligned}&|\varphi^{(j)}(x,\lambda)|\le C|\rho|^j|\exp(\rho\int_0^x|R_-(t)|dt)\exp(\pm i\rho\int_0^x|R_+(t)|dt)|,\\ &\rho\in S_k,\ x\in D_\varepsilon,\end{aligned}\tag{15}$$

and

$$|\Delta(\lambda)|\le C|\rho\exp(\rho\int_0^1|R_-(t)|dt)\exp(\pm i\rho\int_0^1|R_+(t)|dt)|,\ \rho\in S_k.\tag{16}$$

Here and below one and the same symbol C denotes various positive constants in estimates. Moreover, it can be shown that

$$|\Delta(\lambda)|\ge C_\delta|\rho\exp(\rho\int_0^1|R_-(t)|dt)\exp(\pm i\rho\int_0^1|R_+(t)|dt)|,\ \lambda\in G_\delta,\tag{17}$$

where

$$G_\delta=\{\lambda:|\lambda-\lambda_n|\ge\delta,\ n\ge0\},\ \delta>0.$$

It follows from (15) and (16) that the functions $\varphi^{(j)}(x,\cdot)$ and Δ are entire functions of order 1/2.
Let us go on to the Weyl solution $\Phi(x,\lambda)$. Applying the boundary conditions to $\Phi(x,\lambda)$ we calculate

$$\Phi(x,\lambda)=\frac{1}{\Delta_0(\rho)}(V(z_2(x,\rho))z_1(x,\rho)-V(z_1(x,\rho))z_2(x,\rho))\tag{18}$$

where

$$\Delta_0(\rho)=\left|\begin{array}{cc}U(z_1(x,\rho)) & U(z_2(x,\rho))\\ V(z_1(x,\rho)) & V(z_2(x,\rho))\end{array}\right|.$$

Taking (4)-(6) into account we obtain from (18) that for $|\rho|\to\infty,\ \rho\in S^\delta,\ x\in D_\varepsilon,\ j=0,1$

$$\begin{aligned}&\Phi^{(j)}(x,\lambda)=(\mp i\rho)^{j-1}|R(0)|^{-1/2}|R(x)|^{j-1/2}(\exp(\mp i\frac{\pi}{2}\xi(x)))^j\cdot\\ &\times\exp(-\rho\int_0^x|R_-(t)|dt)\exp(\mp i\rho\int_0^x|R_+(t)|dt)K^*_\pm(x)[1]\end{aligned}\tag{19}$$

It follows from the boundary conditions on $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$ that

$$\Phi(x,\lambda)=S(x,\lambda)+\mathfrak{M}(\lambda)\varphi(x,\lambda),\tag{20}$$

where $S(x,\lambda)$ is the solution of (1) satisfying the conditions $S(0,\lambda)=0,\ S'(0,\lambda)=1$. Using the FSS $\{z_2(x,\rho),\ z_2(x,\rho)\}$ we get

$$S(x,\lambda)=\frac{1}{w(\lambda)}(z_1(0,\rho)z_2(x,\rho)-z_2(0,\rho)z_1(x,\rho)).$$

By virtue of (4) - (6), we infer that for $\rho\in S_k,\ x\in D_\varepsilon,\ j=0,1$

$$\begin{aligned}&S^{(j)}(x,\lambda)=\frac{1}{2}(\pm i\rho)^{j-1}|R(0)|^{-1/2}|R(x)|^{j-1/2}(\exp(\mp i\frac{\pi}{2}\xi(x))^j\cdot\\ &\times\exp(\pm i\frac{\pi}{2}\xi(0))\exp(\rho\int_0^x|R_-(t)|dt)\exp(\pm i\rho\int_0^x|R_+(t)|dt)K_\pm(x)\kappa(x,\rho)\end{aligned}\tag{21}$$

and

$$|S^{(j)}(x,\lambda)|\le C|\rho|^{j-1}|\exp(\rho\int_0^x|R_-(t)|dt)\exp(\pm i\rho\int_0^x|R_+(t)|dt)|.\tag{22}$$

It follows from (22) that the functions $S^{(j)}(x,\cdot)$ are entire of order $1/2$.
Using (10), (13) and (19) we get

$$P_{11}(x,\lambda)=[1],\ P_{12}(x,\lambda)=O(\frac{1}{\rho^{\theta_0}}),\ |\rho|\to\infty,\ \rho\in S^{\delta},\ x\in D_{\varepsilon}. \tag{23}$$

Further, substituting (20) into (10), we calculate

$$P_{11}(x,\lambda)=\varphi(x,\lambda)\tilde{S}'(x,\lambda)-S(x,\lambda)\tilde{\varphi}'(x,\lambda)+(\tilde{\mathfrak{M}}(\lambda)-\mathfrak{M}(\lambda)\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda),$$
$$P_{12}(x,\lambda)=S(x,\lambda)\tilde{\varphi}(x,\lambda)-\varphi(x,\lambda)\tilde{S}(x,\lambda)+(\mathfrak{M}(\lambda)-\tilde{\mathfrak{M}}(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).$$

By the hypothesis of Theorem 1, $\mathfrak{M}(\lambda)\equiv\tilde{\mathfrak{M}}(\lambda)$, and consequently the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ of order $1/2$. It follows from this and (23) that $P_{11}(x,\lambda)\equiv 1,\ P_{12}(x,\lambda)\equiv 0$. Substituting into (9) we obtain $\varphi(x,\lambda)\equiv\tilde{\varphi}(x,\lambda),\ \Phi(x,\lambda)\equiv\tilde{\Phi}(x,\lambda)$ for all x and λ. Consequently, $q(x)=\tilde{q}(x)$ for $x\in I,\ h=\tilde{h}$ and $h_1=\tilde{h}_1$. Theorem 1 is proved.

4. Problem II. Let $\{\lambda_n^1\}_{n\geq 0}$ be the sequence of eigenvalues of the boundary value problem L_1 for equation (1) with the boundary conditions $y(0)=V(y)=0$. Now we consider the inverse problem of recovering $q(x), h$ and h_1 from the given two spectra $\{\lambda_n\}_{n\geq 0}$ of L and $\{\lambda_n^1\}_{n\geq 0}$ of L_1.

Theorem 2. If $\lambda_n=\tilde{\lambda}_n,\ \lambda_n^1=\tilde{\lambda}_n^1$ for all $n\geq 0$, then $q(x)=\tilde{q}(x)$ for $x\in I,\ h=\tilde{h}$ and $h_1=\tilde{h}_1$.

Proof. The function Δ is entire of order $1/2$, and $\Delta(\lambda)$ is uniquely determined by its zeros $\{\lambda_n\}_{n\geq 0}$ via the formula

$$\Delta(\lambda)=A\prod_{n=0}^{\infty}(1-\frac{\lambda}{\lambda_n})\ \text{ (if } \lambda_n\neq 0\ \forall n)$$

where the constant A is uniquely determined with the help of the asymptotic formula (14), (the case when $\Delta(0)=0$ requires minor modifications).
The eigenvalues $\{\lambda_n^1\}_{n\geq 0}$ of the boundary value problem L_1 coincide with zeros of the entire function $\Delta_1(\lambda)=S'(1,\lambda)+h_1S(1,\lambda)$ of order $1/2$. It follows from (21) that

$$\begin{aligned}\Delta_1(\lambda)&=\frac{1}{2}|R(0)|^{-1/2}|R(1)|^{1/2}\exp(\pm i\frac{\pi}{2}\xi(0))\exp(\rho\int_0^1|R_-(t)|dt)\\&\times\exp(\pm i\rho\int_0^1|R_+(t)|dt)K_{\pm}(1)\kappa(\rho).\end{aligned} \tag{24}$$

The function Δ_1 is uniquely determined by its zeros $\{\lambda_n^1\}_{n\geq 0}$ via the formula

$$\Delta_1(\lambda)=A_1\prod_{n=0}^{\infty}(1-\frac{\lambda}{\lambda_n^1}),$$

where the constant A_1 is uniquely determined with the help of (24).

Since $V(\Phi)=0$, it follows from (20) that

$$\mathfrak{M}(\lambda)=-\frac{\Delta_1(\lambda)}{\Delta(\lambda)}. \tag{25}$$

By hypothesis of Theorem 2, $\lambda_n=\tilde{\lambda}_n,\ \lambda_n^1=\tilde{\lambda}_n^1$ for all $n\geq 0$, and consequently $\Delta(\lambda)\equiv\tilde{\Delta}(\lambda),\ \Delta_1(\lambda)\equiv\tilde{\Delta}_1(\lambda)$. From this, in view of (25), we get $\mathfrak{M}(\lambda)\equiv\tilde{\mathfrak{M}}(\lambda)$. Using Theorem 1 we conclude that $q(x)=\tilde{q}(x)$ for $x\in I,\ h=\tilde{h}$ and $h_1=\tilde{h}_1$; hence Theorem 2 is proved.

5. Problem III. In this section we consider the inverse problem of recovering L from the so-called spectral data. For simplicity, we confine ourselves to the case when all zeros of $\Delta(\lambda)$ are simple.

Denote

$$\alpha_n = \int_0^1 R^2(x)\varphi^2(x,\lambda_n)dx.$$

The set $\{\lambda_n, \alpha_n\}_{n\geq 0}$ is called the *spectral data* of the boundary value problem L.

Theorem 3. If $\lambda_n = \tilde{\lambda}_n,\ \alpha_n = \tilde{\alpha}_n$ for all $n \geq 0$, then $q(x) = \tilde{q}(x)$ for $x \in I,\ h = \tilde{h}$ and $h_1 = \tilde{h}_1$.

Proof. Since

$$-\Phi''(x,\lambda) + q(x)\Phi(x,\lambda) = \lambda R^2(x)\Phi(x,\lambda),$$
$$-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n R^2(x)\varphi(x,\lambda_n)$$

we get

$$(\Phi(x,\lambda)\varphi'(x,\lambda_n) - \Phi'(x,\lambda)\varphi(x,\lambda_n))' = (\lambda-\lambda_n)R^2(x)\Phi(x,\lambda)\varphi(x,\lambda_n)$$

and hence

$$(\lambda-\lambda_n)\int_0^1 R^2(x)\Phi(x,\lambda)\varphi(x,\lambda_n)dx = \Big|_0^1(\Phi(x,\lambda)\varphi'(x,\lambda_n) - \Phi'(x,\lambda)\varphi(x,\lambda_n)) = 1.$$

In view of (20) and (25), we have

$$(\lambda-\lambda_n)\int_0^1 R^2(x)S(x,\lambda)\varphi(x,\lambda_n)dx - (\lambda-\lambda_n)\frac{\Delta_1(\lambda)}{\Delta(\lambda)}\int_0^1 R^2(x)\varphi(x,\lambda)\varphi(x,\lambda_n)dx = 1.$$

Let $\lambda \to \lambda_n$, then

$$\frac{\Delta_1(\lambda_n)}{\dot{\Delta}(\lambda_n)}\int_0^1 R^2(x)\varphi^2(x,\lambda_n)dx = -1,$$

where $\dot{\Delta}(\lambda) = \dfrac{d}{d\lambda}\Delta(\lambda)$. Hence

$$\alpha_n = -\frac{\dot{\Delta}(\lambda_n)}{\Delta_1(\lambda_n)}. \tag{26}$$

It follows from (25) that the Weyl function $\mathfrak{M}$ is meromorphic with simple poles in the points $\lambda_n, n \geq 0$. Using (26) we calculate

$$\mathrm{Res}_{\lambda=\lambda_n}\mathfrak{M}(\lambda) = \frac{1}{\alpha_n}. \tag{27}$$

Further, by virtue of (19), we derive

$$\mathfrak{M}(\lambda) = \frac{1}{(\mp i\rho)|R(0)|}\cdot \exp(\pm i\frac{\pi}{2}\xi(0)),\ |\rho| \to \infty,\ \rho \in S^\delta$$

and consequently

$$|\mathfrak{M}(\lambda)| \leq \frac{C}{|\rho|},\ |\rho| \to \infty,\ \rho \in S^\delta. \tag{28}$$

Let us consider the function $\lambda \mapsto \mathfrak{N}(\lambda) \overset{\mathrm{def}}{=} \mathfrak{M}(\lambda) - \tilde{\mathfrak{M}}(\lambda)$.
By hypothesis of Theorem 3 and (27), we have

$$\mathrm{Res}_{\lambda=\lambda_n}\mathfrak{N}(\lambda) = \frac{1}{\alpha_n} - \frac{1}{\tilde{\alpha}_n} = 0.$$

Thus, the function $\mathfrak{N}$ is an entire function of order 1/2. On the other hand, in view of (28),

$$|\mathfrak{N}(\lambda)| \leq \frac{c}{|\rho|},\ \rho \in S^\delta,\ |\rho| \to \infty.$$

Consequently, $\mathfrak{N}(\lambda) \equiv 0$, i.e. $\mathfrak{M}(\lambda) \equiv \tilde{\mathfrak{M}}(\lambda)$. Using Theorem 1 we obtain $q(x) = \tilde{q}(x)$ for $x \in I,\ h = \tilde{h}$ and $h_1 = \tilde{h}_1$. Theorem 3 is proved.

Remark 1. Theorem 2 and 3 are generalizations of results by Borg [1] and Marchenko [2] for the Sturm-Liouville equation without turning points where $R^2(x) \equiv 1$.

Remark 2. It can be shown that

$$\mathfrak{M}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{29}$$

Indeed, it follows from (17) and (24) that

$$|\mathfrak{M}(\lambda)| \leq \frac{C_\delta}{|\rho|},\ \lambda \in G_\delta. \tag{30}$$

Let $\gamma_N = \{\lambda : |\lambda| = r_N\}$ be *circles* of radii $r_N \to \infty$ situated at a positive distance from the spectrum of L. Then, in view of (30),

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\gamma_N} \frac{\mathfrak{M}(\mu)}{\lambda - \mu} d\mu = 0.$$

From this, by virtue of (27) and the residue theorem, we obtain (29), where the series converges "with brackets".

Remark 3. Analogous results are valid for other classes of separated boundary conditions.

Remark 4. We can also apply this method to investigate the inverse problems for equation (1) with turning points in 0 or (and) 1.

6. Solution of the inverse problem. Let us now go on to constructing the solution of the inverse problem. The central role for solving the inverse problem is played by the so-called *main equation* of the inverse problem which connects the spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation, which is a linear equation in the Banach space m of bounded sequences. Moreover we prove the unique solvability of the main equation. For simplicity, we confine ourselves to the most important particular case when $m = 1$ and $T_1 = IV$, i.e. the weight-function changes sign exactly once. The general case can be treated analogously.
For deriving the main equation of the inverse problem we need more precise asymptotics for the solutions of equation (1). For definiteness, everywhere below $\rho \in S_0 \cup S_1$ (the case $\rho \in S_{-1} \cup S_{-2}$ is considered in the same way). It has been shown in [7] that for $|\rho| \to \infty,\ j = 0, 1$ the following asymptotic formulas are valid

$$\left.\begin{aligned}
&z_1^{(j)}(x,\rho) = \rho^j |R(x)|^{j-1/2} \exp(\rho \int_0^x |R(t)|dt)[1],\ x < x_1 \\
&z_1^{(j)}(x,\rho) = (-i\rho)^j |R(x)|^{j-1/2} \exp(\rho \int_0^{x_1} |R(t)|dt) \frac{1}{2} csc \frac{\pi\mu_1}{2} \exp(i\frac{\pi}{4}) \\
&\times \exp(-i\rho \int_{x_1}^x |R(t)|dt)[1] + (-1)^{j+1} i \exp(i\rho \int_{x_1}^x |R(t)|dt)[1],\ x > x_1
\end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned}
&z_2^{(j)}(x,\rho) = (-\rho)^j |R(x)|^{j-1/2} (-i \exp(-\rho \int_0^x |R(t)|dt))[1] + \\
&+(-1)^j \exp(\rho \int_0^x |R(t)|dt) \exp(-2\rho \int_0^{x_1} |R(t)|dt)[1]),\ x < x_1 \\
&z_2^{(j)}(x,\rho) = (i\rho)^j |R(x)|^{j-1/2} 2 \sin \frac{\pi\mu_1}{2} \exp(-i\frac{\pi}{4}) \exp(-\rho \int_0^{x_1} |R(t)|dt) \\
&\times \exp(i\rho \int_{x_1}^x |R(t)|dt)[1],\ x > x_1
\end{aligned}\right\} \tag{32}$$

Here, as above, $[1] = 1 + O(\frac{1}{\rho^{\theta_0}})$ uniformly in $x \in D_\varepsilon$.

It follows from (11), in view of (6), (31)-(32), that

$$\left.\begin{array}{l}
\varphi^{(j)}(x,\lambda)=\frac{1}{2}\rho^{j}|R(0)|^{1/2}|R(x)|^{j-1/2}(\exp(\rho\int_{0}^{x}|R(t)|dt)[1]\\
+(-1)^{j}\exp(-\rho\int_{0}^{x}|R(t)|dt)[1]),\ x<x_1\\
\varphi^{(j)}(x,\lambda)=\frac{1}{2}(i\rho)^{j}|R(0)|^{1/2}|R(x)|^{j-1/2}(A_1(\rho)\exp(i\rho\int_{x_1}^{x}|R(t)|dt)[1]\\
+(-1)^{j}A_2(\rho)\exp(-i\rho\int_{x_1}^{x}|R(t)|dt)[1]),\ x>x_1
\end{array}\right\}\tag{33}$$

where

$$\left.\begin{array}{l}
A_2(\rho)=\frac{1}{2}csc\frac{\pi\mu_1}{2}\exp(i\frac{\pi}{4})(\exp(\rho\int_{0}^{x_1}|R(t)|dt)[1]\\
-i\exp(-\rho\int_{0}^{x_1}|R(t)|dt)[1]),\\
A_1(\rho)=-iA_2(\rho)+2\sin\frac{\pi\mu_1}{2}\exp(i\frac{\pi}{4})\exp(-\rho\int_{0}^{x_1}|R(t)|dt)[1]
\end{array}\right\}\tag{34}$$

Remark 5. Let $\xi=\int_{x_1}^{x}|R(t)|dt$. It follows from results of [7] that (31)-(33) are also valid uniformly for $|\rho\xi|\geq 1$ with $[1]=1+O(\frac{1}{(\rho\xi)^{\theta_0}})$; moreover for $|\rho\xi|\leq 1$ we have the estimates

$$\left.\begin{array}{l}
|\varphi(x,\lambda)\leq C|R(x)|^{-1/2}|\exp(\rho\int_{0}^{x}|R(t)|dt)|,\ x\leq x_1\\
|\varphi(x,\lambda)|\leq C|R(x)|^{-1/2}|\exp(\rho\int_{0}^{x_1}|R(t)|dt)|,\ x\geq x_1
\end{array}\right\}\tag{35}$$

Lemma 1.

(i) The spectrum $\{\lambda_k\}$ of the boundary value problem L consists of two sequences of eigenvalues: $\{\lambda_k\}=\{\lambda_k^{+}\}\cup\{\lambda_k^{-}\}, k\in I\!N$, such that

$$\rho_k^{\pm}=\sqrt{\lambda_k^{\pm}}=\rho_{k,0}^{\pm}+O(\frac{1}{k^{\theta_0}}),\ k\to+\infty\tag{36}$$

where

$$\rho_{k,0}^{+}=(\int_{x_1}^{1}|R(t)|dt)^{-1}(k+\frac{1}{4})\pi,\ \ \rho_{k,0}^{-}=(\int_{0}^{x_1}|R(t)|dt)^{-1}(k+\frac{1}{4})\pi i$$

(ii) Denote $\alpha_k^{\pm}=\int_{0}^{1}R^2(x)\varphi^2(x,\lambda_k^{\pm})dx$, i.e. $\{\alpha_k\}=\{\alpha_k^{+}\}\cup\{\alpha_k^{-}\}$.
Then

$$\left.\begin{array}{l}
\alpha_k^{+}=\frac{1}{2}|R(0)|\int_{x_1}^{1}|R(t)|dt(\frac{1}{2}(csc\frac{\pi\mu_1}{2})^2\omega_k^2(1+O(\frac{1}{k^{\sigma}})),\ k\to+\infty\\
\alpha_k^{-}=-\frac{1}{2}|R(0)|\int_{0}^{x_1}|R(t)|dt(1+O(\frac{1}{k^{\sigma}})),\ k\to+\infty
\end{array}\right\}\tag{37}$$

where

$$\omega_k=\exp(\rho_{k,0}^{+}\int_{0}^{x_1}|R(t)|dt),\ \sigma=\min(1-\delta_0,\theta_0).$$

Proof. Substituting (33) into (7) we calculate

$$\begin{array}{l}
\Delta(\lambda)=\frac{1}{2}(i\rho)|R(0)R(1)|^{1/2}(A_1(\rho)\exp(i\rho\int_{x_1}^{x}|R(t)|dt)[1]-\\
A_2(\rho)\exp(-i\rho\int_{x_1}^{x}|R(t)|dt)[1]).
\end{array}\tag{38}$$

Let $\rho \in S_0$. It follows from (34) that

$$\begin{aligned} A_1(\rho) &= \frac{1}{2} csc \frac{\pi \mu_1}{2} \exp(-i\frac{\pi}{4}) \exp(\rho \int_0^{x_1} |R(t)|dt)[1] \\ A_2(\rho) &= \frac{1}{2} csc \frac{\pi \mu_1}{2} \exp(i\frac{\pi}{4}) \exp(\rho \int_0^{x_1} |R(t)|dt)[1]. \end{aligned}$$

Hence, by virtue of (38), the equation $\Delta(\lambda) = 0$ can be rewritten in the form

$$\exp(2i\rho \int_{x_1}^{1} |R(t)|dt) = i[1].$$

This equation has a countable set of roots ρ_k^+ such that $\rho_k^+ = \rho_{k,0}^+ + O(\frac{1}{k^{\theta_0}})$. Analogously, if $\rho \in S_1$ then the equation $\Delta(\lambda) = 0$ can be transformed to

$$\exp(2\rho \int_0^{x_1} |R(t)|dt) = i[1];$$

therefore this equation has a countable set of roots ρ_k^- such that $\rho_k^- = \rho_{k,0}^- + O(\frac{1}{k^{\theta_0}})$.
Let us now consider α_k^-. Put $\alpha_k^- = \alpha_{k,1}^- + \alpha_{k,2}^-$, where

$$\alpha_{k,1}^- = \int_0^{x_1} R^2(x)\varphi^2(x,\lambda_k^-)dx, \ \alpha_{k,2}^- = \int_{x_1}^{1} R^2(x)\varphi^2(x,\lambda_k^-)dx.$$

Denote $I_{k,1} = \{x \in [0, x_1] : |\xi\rho_k^-| \geq 1\}$, $I_{k,2} = \{x \in [0, x_1] : |\xi\rho_k^-| \leq 1\}$, where $\xi = \int_{x_1}^{x} |R(t)|dt$. According to (33),

$$\int_{I_{k,1}} R^2(x)\varphi^2(x,\lambda_k^-)dx = -\frac{1}{4}|R(0)| \int_{I_{k,1}} |R(x)|(\exp(\rho \int_0^x |R(t)|dt)[1]+ \\ \exp(-\rho \int_0^x |R(t)|dt)[1])^2_{/\rho=\rho_k^-} dx.$$

The change of variables gives

$$\int_{I_{k,1}} R^2(x)\varphi^2(x,\lambda_k^-)dx = -\frac{1}{4}|R(0)| \int_{1/|\rho|}^{\xi_1} (\exp(\rho(\xi_1 - \xi))(1 + O((\rho\xi)^{-\theta_0})) \\ + \exp(-\rho(\xi_1 - \xi))(1 + O((\rho\xi)^{-\theta_0}))^2 d\xi|_{\rho=\rho_k^-},$$

where $\xi_1 = \int_0^{x_1} |R(t)|dt$. Hence

$$\int_{I_{k,1}} R^2(x)\varphi^2(x,\lambda_k^-)dx = -\frac{1}{2}|R(0)| \int_0^{x_1} |R(x)|dx(1 + O(\frac{1}{k^\sigma})).$$

Further, it follows from (35) that

$$|\int_{I_{k,2}} R^2(x)\varphi^2(x,\lambda_k^-)dx| \leq \int_{I_{k,2}} |R(x)\exp(2\rho \int_0^x |R(t)|dt)_{|_{\rho=\rho_k^-}}|dx.$$

In view of (36), the exponential is bounded, and consequently,

$$|\int_{I_{k,2}} R^2(x)\varphi^2(x,\lambda_k^-)dx| \leq \int_{I_{k,2}} |R(x)|dx = \int_0^{1/|\rho|} d\xi_{|\rho=\rho_k^-} = O(\frac{1}{k}).$$

Thus, we arrive at

$$\alpha_{k,1}^{-} = -\frac{1}{2}|R(0)| \int_0^{x_1} |R(x)|dx(1 + O(\frac{1}{k^\sigma})). \tag{39}$$

Let us now estimate $\alpha_{k,2}^{-}$. Denote $J_{k,1} = \{x \in [x_1, 1] : |\xi\rho_k^-| \geq 1\}$, $J_{k,2} = \{x \in [x_1, 1] : |\xi\rho_k^-| \leq 1\}$. In the same way as above one can show that

$$|\int_{J_{k,2}} R^2(x)\varphi^2(x, \lambda_k^-)dx| = O(\frac{1}{k}).$$

Then

$$\alpha_{k,2}^{-} = \int_{J_{k,1}} R^2(x)\varphi^2(x, \lambda_k^-)dx + O(\frac{1}{k}).$$

Taking (33) into account, we get

$$\begin{aligned}\alpha_{k,2}^{-} = -\frac{|R(0)|}{4} \int_{J_{k,1}} &(A_1(\rho) \exp(i\rho \int_{x_1}^{x} |R(t)|dt)[1] \\ &+ A_2(\rho) \exp(-i\rho \int_{x_1}^{x} |R(t)|dt)[1])^2_{\rho=\rho_k^-} dx + O(\frac{1}{k}).\end{aligned}$$

The integral seems to be unbounded as $k \to +\infty$. But it follows from (38) and (34) that

$$\frac{A_2(\rho)}{A_1(\rho)}_{|\rho=\rho_k^-} = \exp(2i\rho_k^- \int_{x_1}^{1} |R(t)|dt)[1], \tag{40}$$

$$A_1(\rho_k^-) = 2\exp(i\frac{\pi}{4}) \sin\frac{\pi\mu_1}{2} \exp(-\rho_k^- \int_0^{x_1} |R(t)|dt)[1]. \tag{41}$$

Then

$$\begin{aligned}\alpha_{k2}^{-} = -\frac{|R(0)|}{4} A_1(\rho_k^-) \int_{J_{k,1}} &|R(x)|(\exp(i\rho \int_{x_1}^{x} |R(t)|dt)[1] + \\ + \exp(2i\rho \int_{x_1}^{1} |R(t)|dt) &\exp(-i\rho \int_{x_1}^{x} |R(t)|dt)[1])^2_{/\rho=\rho_k^-} dx + O(\frac{1}{k}).\end{aligned}$$

After changing variables $x \mapsto \xi = \int_{x_1}^{x} |R(t)|dt$ and integrating we arrive at the estimate

$$\alpha_{k,2}^{-} = O(\frac{1}{k^\sigma}).$$

Combining this with (39), we obtain (37) for α_k^-. For α_k^+ the proof is analogous; hence Lemma 1 is proved.

Now we go on to the derivation of the main equation of the inverse problem. We assume that the spectral data $\{\lambda_k, \alpha_k\}_{k\geq 0}$ of L are given. Let $\tilde{L} = L(R^2(x),\ \tilde{q}(x),\ \tilde{h},\ \tilde{h}_1)$ be a certain known model boundary value problem with the same weight-function $R^2(x)$ and with the spectral data $\{\tilde{\lambda}_k, \tilde{\alpha}_k\}_{k\geq 0}$.

Lemma 2. For each fixed $x \neq x_1$, and $k \to +\infty$ the following estimates hold:

(i) If $x < x_1$ then

$$\varphi(x, \lambda_k^-) = O(1),\ \varphi(x, \tilde{\lambda}_k^-) = O(1), \tag{42}$$

$$\varphi(x, \lambda_k^-) - \varphi(x, \tilde{\lambda}_k^-) = O(\rho_k^- - \tilde{\rho}_k^-), \tag{43}$$

$$\left.\begin{aligned}\varphi(x, \lambda_k^+) &= O(\exp(\rho_{k,0}^+ \int_0^x |R(t)|dt)), \\ \varphi(x, \tilde{\lambda}_k^+) &= O(\exp(\rho_{k,0}^+ \int_0^x |R(t)|dt)),\end{aligned}\right\} \tag{44}$$

$$\varphi(x, \lambda_k^+) - \varphi(x, \tilde{\lambda}_k^+) = O(|\rho_k^+ - \tilde{\rho}_k^+| \exp(\rho_{k,0}^+ \int_0^x |R(t)|dt)). \tag{45}$$

(ii) If $x > x_1$ then

$$\varphi(x, \lambda_k^-) = O(\exp(i\rho_{k,0}^- \int_{x_1}^{x} |R(t)|dt)), \tag{46}$$

$$\varphi(x, \tilde{\lambda}_k^-) = O(k^{-\theta_0} \exp(-i\rho_{k,0}^- \int_{x_1}^{x} |R(t)|dt)), \tag{47}$$

$$\varphi(x, \lambda_k^+) = O(\omega_k),\ \varphi(x, \tilde{\lambda}_k^+) = O(\omega_k), \tag{48}$$

$$\varphi(x, \lambda_k^+) - \varphi(x, \tilde{\lambda}_k^+) = O(|\rho_k^+ - \tilde{\rho}_k^+|\omega_k). \tag{49}$$

We mark the difference in the estimates (46) and (47) for the functions $\varphi(x, \lambda_k^-)$ and $\varphi(x, \tilde{\lambda}_k^-)$; the real part of the exponent in (46) is negative but the real part of the exponent in (47) is positive.

Proof. Let $x < x_1$. The estimates (42) and (44) follow from (33) and the asymptotics of λ_k^+ and λ_k^-. Using (42), (44) and Schwarz's lemma we obtain (43) and (45).

Let now $x > x_1$. For $\lambda = \lambda_k^-$ and $\lambda = \tilde{\lambda}_k^-$ we rewrite (33) as follows

$$\begin{aligned}\varphi(x, \lambda) = \frac{1}{2}|R(0)|^{1/2}|R(x)|^{-1/2}A_1(\rho)(\exp(i\rho \int_{x_1}^{x} |R(t)|dt)[1]+ \\ +\frac{A_2(\rho)}{A_1(\rho)} \exp(-i\rho \int_{x_1}^{x} |R(t)|dt)[1]).\end{aligned} \tag{50}$$

From this, using (40) and (41), we obtain (46). Further, it is easy to show that

$$\frac{A_2(\rho)}{A_1(\rho)}_{|\rho=\tilde{\rho}_k^-} = O(\frac{1}{k^{\theta_0}}),\ \ A_1(\tilde{\rho}_k^-) = O(1).$$

Substituting into (50) we arrive at (47). The estimates (48) and (49) are proved in the same way as (44) and (45). Lemma 2 is proved.

In the sequel we use the abbreviation

$$\langle y, z\rangle := yz' - y'z.$$

Lemma 3. For each fixed $x \in I$, the following relations hold

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \sum_k \Big(\frac{\langle\tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \lambda_k)\rangle}{\alpha_k(\lambda - \lambda_k)}\varphi(x, \lambda_k) - \frac{\langle\tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \tilde{\lambda}_k)\rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)}\varphi(x, \tilde{\lambda}_k)\Big) \tag{51}$$

$$\begin{aligned}&\frac{\langle\varphi(x, \lambda), \varphi(x, \mu)\rangle}{\lambda - \mu} - \frac{\langle\tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu)\rangle}{\lambda - \mu} + \\ &\sum_k \Big(\frac{\langle\tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \lambda_k)\rangle \cdot \langle\varphi(x, \lambda_k), \varphi(x, \mu)\rangle}{\alpha_k(\lambda - \lambda_k)(\lambda_k - \mu)} - \\ &-\frac{\langle\tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \tilde{\lambda}_k)\rangle \cdot \langle\varphi(x, \tilde{\lambda}_k), \varphi(x, \mu)\rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)(\tilde{\lambda}_k - \mu)}\Big) = 0.\end{aligned} \tag{52}$$

Proof. Let $P(x, \lambda)$ be the matrix introduced above via (9). In the λ-plane we consider a oriented contour $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^\pm = \{\lambda : \pm Im\lambda = C_0,\ -\infty < \mp Re\lambda < \infty\}$; here $C_0 > 0$ is chosen such that $|Im\lambda_k| < C_0$ and $|Im\tilde{\lambda}_k| < C_0$ for all k. Put $J_\gamma = \{\lambda : |Im\lambda| > C_0\}$, and the contour $\gamma_N = (\gamma \cap \{\lambda : |\lambda| \le R_N\}) \cup \{\lambda : |\lambda| = R_N,\ \lambda \notin J_\gamma\}$, $R_N \to \infty$ with clockwise orientation. Since

$$\frac{1}{\lambda - \mu}\Big(\frac{1}{\lambda - \xi} - \frac{1}{\mu - \xi}\Big) = \frac{1}{(\lambda - \xi)(\xi - \mu)}$$

we have by Cauchy's theorem that for $\lambda, \mu \in J_\gamma \cap \{\xi : |\xi| < R_N\}$

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\gamma_N} \frac{P_{1k}(x,\xi) - \delta_{1k}}{\lambda - \xi}d\xi,$$

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda - \mu} = \frac{1}{2\pi i}\int_{\gamma_N} \frac{P_{jk}(x,\xi)d\xi}{(\lambda - \xi)(\xi - \mu)},$$

where δ_{jk} is the Kronecker delta. Using (10), (11) and (18) it is easy to show that $P_{1k}(x,\lambda) - \delta_{1k} = O(\rho^{-\theta_0})$, $\lambda \in G_\delta \cap \tilde{G}_\delta$. Therefore

$$\lim_{N\to\infty}\int_{|\xi|=R_N} \frac{P_{1k}(x,\xi) - \delta_{1k}}{\lambda - \xi}d\xi = 0, \ \lim_{N\to\infty}\int_{|\xi|=R_N} \frac{P_{jk}(x,\xi)d\xi}{(\lambda - \xi)(\xi - \mu)} = 0,$$

and consequently,

$$P_{1k}(x,\lambda) = \delta_{1k} + \frac{1}{2\pi i}\int_{\gamma} \frac{P_{1k}(x,\xi)}{\lambda - \xi}d\xi, \ \lambda \in J_\gamma, \tag{53}$$

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda - \mu} = \frac{1}{2\pi i}\int_{\gamma} \frac{P_{jk}(x,\xi)d\xi}{(\lambda - \xi)(\xi - \mu)}, \ \lambda, \mu \in J_\gamma, \tag{54}$$

where here and in the sequel $\displaystyle\int_\gamma := \lim_{N\to\infty}\int_{\gamma_N}$.

It follows from (9) that

$$\varphi(x,\lambda) = P_{11}(x,\xi)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda).$$

Then, in view of (53), we get

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma (\tilde{\varphi}(x,\lambda)P_{11}(x,\xi) - \tilde{\varphi}'(x,\lambda)P_{12}(x,\xi))\frac{d\xi}{\lambda - \xi}.$$

From this, by virtue of (10), we infer

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma (\tilde{\varphi}(x,\lambda)(\varphi(x,\xi)\tilde{\Phi}'(x,\xi) - \Phi(x,\xi)\tilde{\varphi}'(x,\xi)) -$$
$$-\tilde{\varphi}'(x,\lambda)(\Phi(x,\xi)\tilde{\varphi}(x,\xi) - \varphi(x,\xi)\tilde{\Phi}(x,\xi))\frac{d\xi}{\lambda - \xi}.$$

Using (20), (27) and the residue theorem we arrive at (51).

From (10) and the identity $\langle\varphi(x,\lambda), \Phi(x,\lambda)\rangle \equiv 1$ it follows that

$$\begin{aligned} P_{11}(x,\lambda)\varphi'(x,\lambda) - P_{21}(x,\lambda)\varphi(x,\lambda) &= \tilde{\varphi}'(x,\lambda), \\ P_{22}(x,\lambda)\varphi(x,\lambda) - P_{12}(x,\lambda)\varphi'(x,\lambda) &= \tilde{\varphi}(x,\lambda), \end{aligned} \tag{55}$$

$$P(x,\lambda)\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \langle y(x), \tilde{\Phi}(x,\lambda)\rangle \begin{bmatrix} \varphi(x,\lambda) \\ \varphi'(x,\lambda) \end{bmatrix} - \langle y(x), \tilde{\varphi}(x,\lambda)\rangle \begin{bmatrix} \Phi(x,\lambda) \\ \Phi'(x,\lambda) \end{bmatrix} \tag{56}$$

for any $y(x) \in C^1[0,1]$. Taking (54) and (56) into account, we calculate

$$\begin{aligned} \frac{P(x,\lambda) - P(x,\mu)}{\lambda - \mu}\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} &= \frac{1}{2\pi i}\int_\gamma \Big(\langle y(x), \tilde{\Phi}(x,\xi)\rangle \begin{bmatrix} \varphi(x,\xi) \\ \varphi'(x,\xi) \end{bmatrix} - \\ -\langle y(x),\ \tilde{\varphi}(x,\xi)\rangle \begin{bmatrix} \Phi(x,\xi) \\ \Phi'(x,\xi) \end{bmatrix}\Big)&\frac{d\xi}{(\lambda - \xi)(\xi - \mu)}. \end{aligned} \tag{57}$$

Using (55) we get

$$\det\Big((P(x,\lambda) - P(x,\mu))\begin{bmatrix} \tilde{\varphi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) \end{bmatrix}, \begin{bmatrix} \varphi(x,\mu) \\ \varphi'(x,\mu) \end{bmatrix}\Big) =$$
$$\langle\varphi(x,\lambda), \varphi(x,\mu)\rangle - \langle\tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\mu)\rangle,$$

and consequently, in view of (57), we obtain

$$\frac{\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle}{\lambda-\mu}-\frac{\langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\mu)\rangle}{\lambda-\mu}=$$
$$\frac{1}{2\pi i}\int_\gamma\Big(\frac{\langle\tilde\varphi(x,\lambda),\tilde\Phi(x,\xi)\rangle\langle\varphi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}$$
$$-\frac{\langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\xi)\rangle\langle\Phi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}\Big)d\xi.$$

By virtue of (20), (27) and the residue theorem, we arrive at (52); hence Lemma 3 is proved.

Denote $\lambda_{k0}=\lambda_k,\ \lambda_{k1}=\tilde\lambda_k,\ \alpha_{k0}=\alpha_k,\ \alpha_{k1}=\tilde\alpha_k,\ \varphi_{kj}(x)=\varphi(x,\lambda_{kj}),\ \tilde\varphi_{kj}(x)=\tilde\varphi(x,\lambda_{kj}),$

$$\tilde P_{ni,kj}(x)=\frac{\langle\tilde\varphi_{ni}(x),\tilde\varphi_{kj}(x)\rangle}{\alpha_{kj}(\lambda_{nj}-\lambda_{kj})}=\frac{1}{\alpha_{kj}}\int_0^x R^2(t)\tilde\varphi_{ni}(t)\tilde\varphi_{kj}(t)dt,$$
$$P_{ni,kj}(x)=\frac{\langle\varphi_{ni}(x),\varphi_{kj}(x)\rangle}{\alpha_{kj}(\lambda_{ni}-\lambda_{kj})}=\frac{1}{\alpha_{kj}}\int_0^x R^2(t)\varphi_{ni}(t)\varphi_{kj}(t)dt.$$

It follows from (51) and (52) that

$$\tilde\varphi_{ni}(x)=\varphi_{ni}(x)+\sum_k(\tilde P_{ni,ko}(x)\varphi_{ko}(x)-\tilde P_{ni,k1}(x)\varphi_{k1}(x)),\tag{58}$$

$$P_{ni,\ell j}(x)-\tilde P_{ni,\ell j}(x)+\sum_k(\tilde P_{ni,ko}(x)P_{ko,\ell j}(x)-\tilde P_{ni,k1}(x)P_{k1,\ell j}(x))=0.\tag{59}$$

Let $x<x_1$. Denote

$$\gamma_k(x)=\begin{cases}\exp(\rho^+_{k,0}\int_0^x|R(t)|dt), & \text{for } \lambda_k=\lambda_k^+\\ 1, & \text{for } \lambda_k=\lambda_k^-,\end{cases}$$
$$\gamma_k=\gamma_k(x_1),\ \xi_k=|\rho_k-\tilde\rho_k|+\frac{|\alpha_k-\tilde\alpha_k|}{\gamma_k^2}.$$

Clearly, $\xi_k=O(k^{-\sigma})$. It follows from (42) - (45) that

$$\left.\begin{array}{l}|\varphi_{kj}(x)|\le C\gamma_k(x),\ |\varphi_{ko}(x)-\varphi_{k1}(x)|\le C\xi_k\gamma_k(x),\\[2mm] |P_{ni,kj}(x)|\le\dfrac{C\gamma_n(x)\gamma_k(x)}{(|n-k|+1)\gamma_k^2}\end{array}\right\}.\tag{60}$$

Let V be a set of indices $v=(k,j),\ k\in I\!N,\ j=0,1$. Define the vector $\psi(x)=[\psi_v(x)]_{v\in V}:=[\psi_{00},\psi_{01},\psi_{10},\psi_{11},\ldots]^T$ and similarly the block matrix

$$H(x)=[H_{u,v}(x)]_{u,v\in V}=\begin{bmatrix}H_{n0,k0}(x) & H_{no,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix}_{n,k\in I\!N}$$

$u=(n,i),\ v=(k,j)$, where

$$\psi_{ko}(x)=(\varphi_{ko}(x)-\varphi_{k1}(x))(\xi_k\gamma_k(x))^{-1},\ \psi_{k1}(x)=\varphi_{k1}(x)(\gamma_k(x))^{-1},$$
$$H_{n0,k0}(x)=(P_{no,ko}(x)-P_{n1,ko}(x))\xi_k\gamma_k(x)(\xi_n\gamma_n(x))^{-1},$$
$$H_{n1,k0}(x)=P_{n1,k0}(x)\xi_k\gamma_k(x)(\gamma_n(x))^{-1},$$
$$H_{no,k1}(x)=(P_{no,ko}(x)-P_{n1,k0}(x)-P_{n0,k1}(x)+P_{n1,k1}(x))\gamma_k(x)(\xi_n\gamma_n(x))^{-1},$$
$$H_{n1,k1}(x)=(P_{n1,k0}(x)-P_{n1,k1}(x))\gamma_k(x)(\gamma_n(x))^{-1}.$$

Analogously we define $\tilde{\psi}(x),\ \tilde{H}(x)$. It follows from (60) and Schwarz's lemma that for each fixed $x \in (0, x_1)$

$$|\psi_{ni}(x)| \le C,\ \ |H_{ni,kj}(x)| \le \frac{C\xi_k}{(|n-k|+1)}. \tag{61}$$

Similarly,

$$|\tilde{\psi}_{ni}(x)| \le C,\ \ |\tilde{H}_{ni,kj}(x)| \le \frac{C\xi_k}{(|n-k|+1)}. \tag{62}$$

Let us consider the Banach space m of bounded sequences $\alpha = [\alpha_v]_{v\in V}$ with the norm $\|\alpha\|_m = \sup\limits_{v\in V} |\alpha_v|$. It follows from (61) and (62) that for each fixed $x \in (0, x_1)$ the operators $E + \tilde{H}(x)$ and $E - H(x)$ (here E is the identity operator), acting from m to m, are linear bounded operators, and

$$\|H(x)\|,\ \|\tilde{H}(x)\| \le C \sup_n \sum_k \frac{\xi_k}{(|n-k|+1}) < \infty.$$

Theorem 4. For each fixed $x \in (0, x_1)$, the vector $\psi(x) \in m$ is the solution of the equation

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x) \tag{63}$$

in the Banach space m. The operator $E + \tilde{H}(x)$ has a bounded inverse operator, i.e. equation (63) is uniquely solvable.

Indeed, taking into account our notations, we can rewrite (58) and (59) in the form

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x),\ (E + \tilde{H}(x))(E - H(x)) = E.$$

Interchanging places for L and $\tilde{L}$ we obtain analogously

$$\psi(x) = (E - H(x))\tilde{\psi}(x),\ (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence the operator $(E + \tilde{H}(x))^{-1}$ exists, and it is a linear bounded operator.

Equation (63) is called the main equation of the inverse problem. Solving (63) we find the vector $\psi(x)$, and consequently, the functions $\varphi_{ni}(x)$. Since $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ are the solutions of (1), we can construct the function $q(x)$ for $x \in (0, x_1)$ and the coefficient h. Thus, the inverse problem has been solved for the interval $x \in (0, x_1)$.

For $x > x_1$ we can act analogously, starting from (58)-(59) and using (46)-(49) instead of (42)-(45), and construct $q(x)$ for $x \in (x_1, 1)$ and the coefficient h_1.

Remark 6. To construct $q(x)$ for $x \in (x_1, 1)$ we can act also in another way. Suppose that, using the main equation of the inverse problem, we have constructed $q(x)$ for $x \in (0, x_1)$ and h. Consequently, the solutions $\varphi(x, \lambda)$ and $S(x, \lambda)$ are known for $x \in [0, x_1]$. By virtue of (20) and (29), the solution $\Phi(x, \lambda)$ is known for $x \in [0, x_1]$ too. Let $\psi(x, \lambda)$ be the solution of (1) under the conditions $\psi(1, \lambda) = 1,\ \psi'(1, \lambda) = -h_1$. Clearly, $\Phi(x, \lambda) = -\dfrac{\psi(x, \lambda)}{\Delta(\lambda)}$. Thus, using $q(x)$ for $x \in [0, x_1]$, we can constant the functions

$$\delta_j(\lambda) = \psi^{(j-1)}(x_1, \lambda),\ j = 1, 2.$$

The functions $\delta_j(\lambda)$ are characteristic functions of the boundary value problems Q_j for equation (1) on $x \in (x_1, 1)$ with the conditions $y^{(j-1)}(x_1) = y'(1) + h_1 y(1) = 0$. Thus, we can reduce our problem to the inverse problem of recovering $q(x),\ x \in (x_1, 1)$ from two spectra of boundary value problems Q_j on the interval $(x_1, 1)$. In this problem the weight-function $R^2(x)$ does not change sign. We can treat this inverse problem by the same method as above. In this case the main equation will be simpler. We note that the case when the weight-function does not change sign were studied in more general case in [11] and other papers.

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