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Summability of eigenfunction expansions of three-point boundary value problems

1. Introduction

We consider the irregular three-point eigenvalue problem

$$l(y) = y^{(5)} + \sum_{\nu=2}^{5} f_\nu y^{(5-\nu)} = \lambda y, \tag{1}$$

$$U_\nu(y) = \sum_{j=0}^{4} \alpha_{\nu j} y^{(j)}(a_\nu) = 0,\ \nu = 1, ..., 5, \tag{2}$$

with irregular splitting three point conditions, where $f_\nu \in \mathbf{C},\ \alpha_{\nu j} \in \mathbf{C}$ and
$0 := a_1 = ... = a_{\nu_1} < a := a_{\nu_1+1} = ... = a_{\nu_1+\nu_2} < a_{\nu_1+\nu_2+1} = ... = a_5 := 1$.

This problem can have one or two infinite sequences of eigenvalues (see [2]) and we use the modification of the well known definition of Abel-summability (see [1]).

Let $\lambda_1, \lambda_2, ...$ be the eigenvalues of the operator L, generated by (1),(2). For $\sigma \in \{1, -1\}$ and $\alpha > 0$ we fix the branch of the multivalued analytic function $\lambda \mapsto (\sigma\lambda)^\alpha$ so, that $(\sigma\lambda)^\alpha > 0$ for $\sigma\lambda > 0$.

For a function $f : [0, 1] \to \mathbf{C}$ we construct a series

$$u(x,t) = \sum_{Re\lambda_k \le 0} Res_{\lambda=\lambda_k} e^{-(-\lambda)^\alpha t}(L - \lambda E)^{-1} f(x) + \sum_{Re\lambda_k > 0} Res_{\lambda=\lambda_k} e^{-\lambda^\alpha t}(L - \lambda E)^{-1} f(x)$$

where E is the identity operator.

Further we assume in what follows that the number α has been chosen so that the series $u(x,t)$ converges for every $t > 0$.

We will say that Fourier series in the system of eigen- and associated functions of the operator L for a function f is *Abel-summable* to order α uniformly with respect to $x \in [a, b] (\subset [0, 1])$ if the limit $lim_{t\to 0} u(x,t) = f(x)$ exists in the sense of the norm in $C[a, b]$.

2. Results

Theorem 1. *Let $\nu_1 = 2, \nu_2 = 1$ and $f \in D(L)$, the domain of definition of the operator L. Then the Fourier series of f is Abel-summable to order $\alpha \in (1/5, 1]$ uniformly on the whole interval $[0, 1]$*

From the estimates in [2] it follows, that there is a ray in the λ-plane, where the Greens function of the problem decreases like $(\lambda^{-4/5})$, and for the proof of the Theorem 1 we can use a modification of the Lidskii method (see [1],[3]).

If for example $\nu_1 = \nu_2 = 2$ or $\nu_1 = 3,\ \nu_2 = 1$, then it follows from [2] that the Greens function $G(x, \xi, \lambda)$ increases exponentially on the whole λ-plane for $x > a$, and we have to use a modification of the Kostyuchenko-Shkalikov method (see [4]).

Theorem 2. *Let $\nu_1 = \nu_2 = 2$ and $f \in D(L)$. Then the Fourier series of f is Abel-summable to order $\alpha \in (1/5, 1/3]$ uniformly on the whole interval $[0, 1]$.*

Theorem 3. *Let $\nu_1 = 3, \nu_2 = 1$ and $f \in D(L)$. Then the Fourier series of f is Abel-summable to order $\alpha \in (1/5, 1/3]$ uniformly on the interval $[0, a]$.*

Let us note, that in the case $\nu_1 = \nu_2 = 2$ there is only one sequence of eigenvalues, as in the two-point case.
In the case $\nu_1 = 3, \nu_2 = 1$ there are two sequences of eigenvalues and the summability can only be obtained on the smaller segment $[0, a] = [0, a_4]$. Moreover for arbitrary points $x_0 \in (a, 1]$ and $k \in \mathbf{N}$ we can construct a function from the domain of definition of the operator L^k, such that $\lim_{t \to 0} u(x_0, t)$ does not exist.

From the above mentioned results it follows that for the problem of Abel-summability there is no straightforward analogy between multipoint and two point boundary value problems (see [3]-[5]).

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3. References

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