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On constructing differential equations with singularities from incomplete spectral information

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Abstract. We study the inverse problem of synthesizing parameters of differential equations with singularities from incomplete spectral information. We establish properties of the spectral characteristics, obtain conditions for the solvability of such classes of inverse problems and provide algorithms for constructing the solution.

1. Introduction

Let us consider the system

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = \mathrm{i}\rho R(x) y_2 \qquad \frac{\mathrm{d}y_2}{\mathrm{d}x} = \mathrm{i}\rho \frac{1}{R(x)} y_1 \qquad x \in [0, T] \tag{1}$$

with the initial conditions

$$y_1(0, \rho) = 1 \qquad y_2(0, \rho) = -1.$$

Here $\rho = k + \mathrm{i}\tau$ is the spectral parameter, and $R(x)$ is a real function.

For a wide class of problems describing the propagation of electromagnetic waves in a stratified medium, Maxwell's equations can be reduced to the canonical form (1), where x is the variable in the direction of stratification, y_1 and y_2 are the components of the electromagnetic field, $R(x)$ is the wave resistance which describes the refractive properties of the medium and ρ is the wavenumber in a vacuum [1, 2]. System (1) also appears in radio engineering for the design of directional couplers for heterogeneous electronic lines, which constitute one of the important classes of radiophysical synthesis problems [3].

Some aspects of synthesis problems for system (1) with a positive $R(x)$ were studied in [2–4] and other works. In this paper we study the inverse problem for system (1) from incomplete spectral information in the case when $R(x)$ is non-negative and can have zeros which are called turning points. More precisely, we shall consider two classes of functions $R(x)$. We shall say that $R(x) \in B_0$ if $R(x), R'(x)$ are absolutely continuous on $[0, T]$, $R''(x) \in L_2(0, T)$, $R(x) > 0$, $R(0) = 1$, $R'(0) = 0$. We also consider the more general case when $R(x)$ has zeros $0 < x_1 < \cdots < x_p < T$, $p \geqslant 0$ inside the interval $(0, T)$. We shall say that $R(x) \in B_0^+$ if $R(x)$ has the form

$$\frac{1}{R(x)} = \sum_{j=1}^{p} \frac{R_j}{(x - x_j)^2} + R_0(x), 0 < x_1 < \cdots < x_p < T, R_j > 0$$

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and $R_0(x)$, $R_0'(x)$ are absolutely continuous on $[0, T]$, $R_0''(x) \in L_2(0, T)$, $R(x) > 0 (x \neq x_j)$, $R(0) = 1$, $R'(0) = 0$. In particular, if here $p = 0$ then $R(x) \in B_0$.

As the main spectral characteristics we introduce the amplitude reflection coefficient

$$r(\rho) = \frac{y_1(T, \rho) + R^0 y_2(T, \rho)}{y_1(T, \rho) - R^0 y_2(T, \rho)} \qquad R^0 := R(T)$$

the power reflection coefficient

$$\sigma(k) = |r(k)|^2, k := \mathrm{Re}\, \rho$$

the transmission coefficients

$$f_1(\rho) = \frac{y_1(T, \rho) - R^0 y_2(T, \rho)}{2\sqrt{R^0}} \qquad f_2(\rho) = \frac{y_1(T, \rho) + R^0 y_2(T, \rho)}{2\sqrt{R^0}}$$

and the characteristic function

$$\Delta(\rho) = \frac{1}{\sqrt{R^0}} y_1(T, \rho).$$

Clearly,

$$r(\rho) = f_2(\rho)/f_1(\rho) \tag{2}$$

$$\Delta(\rho) = f_1(\rho) + f_2(\rho). \tag{3}$$

In many cases of practical interest, the phase is difficult or impossible to measure, while the amplitude is easily accessible to measurement. Such cases lead us to the so-called incomplete inverse problems where only a part of the spectral information is available. In this paper we study one of the incomplete inverse problems, namely, the inverse problem of recovering the wave resistance from the power reflection coefficient.

Inverse problem 1. Given $\sigma(k)$, construct $R(x)$.

The lack of spectral information leads here to nonuniqueness of the solution of the inverse problem. Let us briefly describe a scheme of the solution of inverse problem 1.

Denote $\alpha_j(k) = |f_j(k)|$, $j = 1, 2$. Since $\alpha_1^2(k) - \alpha_2^2(k) \equiv 1$ (see (19) below), we get in view of (2),

$$\sigma(k) = 1 - (\alpha_1(k))^{-2} \qquad 0 \leqslant \sigma(k) < 1.$$

First, from the given power reflection coefficient $\sigma(k)$ we construct $\alpha_j(k)$. Then, using analytic properties of the transmission coefficients $f_j(\rho)$ and, in particular, information about their zeros, we reconstruct the transmission coefficients from their amplitudes. Namely, at this stage we are faced with nonuniqueness. Problems concerning the reconstruction of analytic functions from their moduli often appear in applications and have been studied in many works (see [5–7] and references therein). The last of our steps is to calculate the characteristic function $\Delta(\rho)$ by (3) and to solve the inverse problem of recovering $R(x)$ from $\Delta(\rho)$. Here we use the inverse Sturm–Liouville theory which was developed in [8–12] and other works.

To realize this scheme, in section 2 we study properties of the spectral characteristics. In section 3 we solve the synthesis problem for $R(x)$ from the characteristic function $\Delta(\rho)$, obtain necessary and sufficient conditions for its solvability and provide three algorithms for constructing the solution. In section 4 we study the problem of recovering the transmission coefficients from their moduli. In section 5, the so-called symmetrical case is considered, and in section 6 we provide an algorithm for the solution of inverse problem 1. We note that inverse problems for Sturm–Liouville equations with turning points have been studied in [13]. Some aspects of the turning point theory and a number of its applications are described in [14–16].

We shall say that $R(x) \in B_0^-$ if $(R(x))^{-1} \in B_0^+$. An investigation of the classes B_0^+ and B_0^- is completely similar because the replacement $R \to 1/R$ is equivalent to the replacement $(y_1, y_2) \to (-y_2, -y_1)$. Below it will be more convenient for us to consider the case $R(x) \in B_0^-$.

2. Properties of the spectral characteristics

We transform (1) by means of the replacement

$$y_1(x,\rho) = \sqrt{R(x)}u(x,\rho) \qquad y_2(x,\rho) = \frac{1}{\sqrt{R(x)}}v(x,\rho) \tag{4}$$

to the system

$$u' + h(x)u = \mathrm{i}\rho v \qquad v' - h(x)v = \mathrm{i}\rho u \qquad x \in [0,T] \tag{5}$$

with the initial conditions $u(0,\rho) = 1$, $v(o,\rho) = -1$, where

$$h(x) = \frac{R'(x)}{2R(x)}. \tag{6}$$

Hence the function $u(x,\rho)$ satisfies the equation

$$-u'' + q(x)u = \lambda u \qquad \lambda = \rho^2 \tag{7}$$

and the initial conditions

$$u(0,\rho) = 1 \qquad u'(0,\rho) = -\mathrm{i}\rho$$

where

$$q(x) = h^2(x) - h'(x) \tag{8}$$

or

$$q(x) = \frac{3}{4}\left(\frac{R'(x)}{R(x)}\right)^2 - \frac{1}{2}\frac{R''(x)}{R(x)}.$$

Similarly,

$$-v'' + g(x)v = \lambda v \qquad v(0,\rho) = -1 \qquad v'(0,\rho) = \mathrm{i}\rho$$

where $g(x) = h^2(x) + h'(x)$.

In view of (4), the transmission coefficients, the amplitude reflection coefficient and the characteristic function take the form

$$\begin{gathered} f_1(\rho) = \frac{u(T,\rho) - v(T,\rho)}{2} \qquad f_2(\rho) = \frac{u(T,\rho) + v(T,\rho)}{2} \\ r(\rho) = \frac{u(T,\rho) + v(T,\rho)}{u(T,\rho) - v(T,\rho)} = \frac{f_2(\rho)}{f_1(\rho)} \qquad \Delta(\rho) = u(T,\rho). \end{gathered} \tag{9}$$

Since $y_1(x,0) \equiv 1$, we have according to (4)

$$\sqrt{R(x)}u(x,0) \equiv 1. \tag{10}$$

Lemma 1. $R(x) \in B_0^-$ if and only if $q(x) \in L_2(0,T)$, $\Delta(0) \neq 0$.

Proof.

(1) Let $R(x) \in B_0^-$, i.e.

$$R(x) = \sum_{j=1}^{p} \frac{R_j}{(x-x_j)^2} + R_0(x) \qquad 0 < x_1 < \cdots < x_p < T, \;\; R_j > 0$$
$$R_0(x) \in W_2^2(0,T) \qquad R(x) > 0 (x \neq x_j) \qquad R(0) = 1 \qquad R'(0) = 0. \tag{11}$$

Using (6) and (11) we get for $x \to x_j$

$$h(x) \sim -\frac{1}{x-x_j} + h_j^*(x) \qquad h_j^*(x) \in W_2^1 \qquad h_j^*(x_j) = 0$$

and consequently $q(x) \in L_2(0,T)$. If follows from (10) that $u(T,0) \neq 0$, i.e. $\Delta(0) \neq 0$.

(2) Now let $q(x) \in L_2(0,T), u(T,0) \neq 0$, and let $0 < x_1 < \cdots < x_p < T$ be zeros of $u(x,0)$. If follows from (10) that

$$R(x) = (u(x,0))^{-2}.$$

Hence $R(x) > 0 (x \neq x_j)$, $R(0) = 1$, $R'(0) = 0$, and

$$R(x) \sim \frac{R_j}{(x-x_j)^2}, x \to x_j, R_j > 0.$$

Denote

$$R_0(x) := R(x) - \sum_{j=1}^{p} \frac{R_j}{(x-x_j)^2}.$$

It is easy to see that $R_0(x) \in W_2^2[0,T]$. $\square$

Remark 1. If $R(x) \in B_0^-$, then the functions $y_1(x,\rho)$ and $v(x,\rho)$ have singularities of order 1 at $x = x_j$, and there exist finite limits $\lim_{x \to x_j} (x-x_j) v(x,\rho), \lim_{x \to x_j} (x-x_j) y_1(x,\rho)$. In other words, we continue solutions in the neighbourhoods of the singular points with the help of generalized Bessel-type solutions (see [17]). If follows from (10) that $\lim_{x \to x_j} (x-x_j)\sqrt{R(x)} = R_j$.

Example. Let $R(x) = \frac{1}{\cos^2 x}$. Then $h(x) = tgx$, $q(x) = -1$, $u(x,0) = \frac{1}{\sqrt{R(x)}} = \cos x$, $u(x,\rho) = \cos \mu x - \mathrm{i}\rho \frac{\sin \mu x}{\mu}$, $\mu^2 = \rho^2 + 1$. If $T \in (\frac{(2p-1)\pi}{2}, \frac{(2p+1)\pi}{2})$, then $u(x,0)$ has p zeros $x_j = \frac{(2j-1)\pi}{2}$, $j = \overline{1,p}$.

Theorem 1. Let $R(x) \in B_0^-$. Then:

(i) the characteristic function $\Delta(\rho)$ is entire in ρ, and the following representation holds

$$\Delta(\rho) = \mathrm{e}^{-\mathrm{i}\rho T} + \int_{-T}^{T} \eta(t) \mathrm{e}^{-\mathrm{i}\rho t} \, \mathrm{d}t \qquad \eta(t) \in AC[-T,T], \;\; \eta'(t) \in L_2(-T,T), \eta\,(-T) = 0 \tag{12}$$

where $\eta(t)$ is a real function, and

$$\eta(T) = \frac{1}{2} \int_0^T q(t)\, \mathrm{d}t = -\frac{h(T)}{2} + \frac{1}{2} \int_0^T h^2(t)\, \mathrm{d}t$$

(ii) for real ρ, $\Delta(\rho)$ has no zeros. For $\operatorname{Im} \rho > 0$, $\Delta(\rho)$ has at most a finite number of simple zeros of the form $\rho_j = \mathrm{i}\tau_j$, $\tau_j > 0$, $j = \overline{1,m}, m \geqslant 0$;

(iii) the function $\Delta(\rho)$ has no zeros for $\operatorname{Im} \rho \geqslant 0$ if and only if $R(x) \in B_0$.

Proof.

(1) It is well known (see [9]) that $u(x,\rho)$ has the form

$$u(x,\rho) = \mathrm{e}^{-\mathrm{i}\rho x} + \int_{-x}^{x} K(x,t)\mathrm{e}^{-\mathrm{i}\rho t}\,\mathrm{d}t$$

where $K(x,t)$ is a real, absolutely continuous function, $K_t(T,t) \in L_2(-T,T)$, $K(x,-x) = 0$, $K(x,x) = \frac{1}{2}\int_0^x q(t)\,\mathrm{d}t$. Moreover,

$$\overline{u(x,\rho)} = u(x,-\bar{\rho}) \tag{13}$$

$$\langle u(x,\rho), u(x,-\rho)\rangle \equiv 2\mathrm{i}\rho \tag{14}$$

where $\langle y,z\rangle := yz' - y'z$. Hence we arrive at (12), where $\eta(t) = K(T,t)$.

(2) It follows from (13) that for real ρ, $\overline{\Delta(\rho)} = \Delta(-\rho)$, and consequently, in view of (14), for real $\rho \neq 0$, $\Delta(\rho)$ has no zeros. By virtue of lemma 1, $\Delta(0) \neq 0$. Further, denote $\Pi_+ = \{\rho : \operatorname{Im}\rho > 0\}$. Let $\rho = k + \mathrm{i}\tau \in \Pi_+$ be a zero of $\Delta(\rho)$. Since

$$-u'' + q(x)u = \rho^2 u \qquad -\overline{u}'' + q(x)\overline{u} = \overline{\rho}^2\overline{u}$$

we get

$$(\rho^2 - \bar{\rho}^2)\int_0^T |u|^2\,\mathrm{d}x = |_0^T \langle u,\overline{u}\rangle = -\mathrm{i}(\rho + \overline{\rho})$$

which is possible only if $k = 0$. Hence, all zeros of $\Delta(\rho)$ in Π_+ are pure imaginary, and by virtue of (12), the number of zeros in $\overline{\Pi}_+$ is finite.

(3) Let us show that in Π_+ all zeros of $\Delta(\rho)$ are simple. Suppose that for a certain $\rho = \mathrm{i}\tau$, $\tau > 0$, $\Delta(\rho) = \dot{\Delta}(\rho) = 0$, where $\dot{\Delta}(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\Delta(\rho)$. Denote $u_1 = \dot{u}$. Then, by virtue of (7),

$$-u_1'' + q(x)u_1 = \rho^2 u_1 + 2\rho u \qquad u_1(0,\rho) = 0 \qquad u_1'(0,\rho) = -\mathrm{i}.$$

Consequently,

$$\begin{aligned} 2\rho\int_0^T u^2(x,\rho\,\mathrm{d}x &= \int_0^T u(x,\rho)(-u_1''(x,\rho) + q(x)u_1(x,\rho) - \rho^2 u_1(x,\rho))\,\mathrm{d}x \\ &= -|_0^T\langle u,u_1\rangle = -\mathrm{i} \end{aligned}$$

i.e.

$$2\tau\int_0^\tau |u(x,\rho)|^2\,\mathrm{d}x = -1$$

which is impossible.

(4) If $R(x) \in B_0$, then in view of (10) $u(x,0) > 0$, $x \in [0,T]$. Since $u(0,\mathrm{i}\tau) = 1$, $\tau \geqslant 0$ and $u(x,\mathrm{i}\tau) > 0$, $x \in [0,T]$ for large τ, we get $u(T,\mathrm{i}\tau) > 0$ for all $\tau \geqslant 0$, i.e. $\Delta\rho)$ has no zeros in $\overline{\Pi}_+$. The inverse assertion is proved similarly. □

In an analogous manner one can prove the following theorem.

Theorem 2. Let $R(x) \in B_0^-$.

(i) The functions $f_1(\rho)$, $f_2(\rho)$ are entire in ρ, and have the form

$$f_1(\rho) = \mathrm{e}^{-\mathrm{i}\rho T} + \int_{-T}^{T} g_1(t)\mathrm{e}^{-\mathrm{i}\rho t}\,\mathrm{d}t$$

$$g_1(t) \in AC[-T,T],\ \ g_1'(t) \in L_2(-T,T),\ \ g_1(-T) = 0 \tag{15}$$

$$f_2(\rho) = \int_{-T}^{T} g_2(t)\mathrm{e}^{-\mathrm{i}\rho t}\,\mathrm{d}t \qquad g_2(t) \in AC[-T,T], g_2'(t) \in L_2(-T,T), g_2(-T) = 0 \tag{16}$$

where $g_j(t)$ are real, and

$$g_1(T) = \frac{1}{2}\int_0^T h^2(t), g_2(T) = -\frac{h}{2} \qquad h := h(T). \tag{17}$$

(ii)

$$f_1(\rho) f_1(-\rho) - f_2(\rho) f_2(-\rho) \equiv 1. \tag{18}$$

(iii) For real ρ, $f_1(\rho)$ has no zeros. In $\bar{\Pi}_+$, $f_1(\rho)$ has a finite number of simple zeros of the form $\rho_j^* = i\tau_j^*$, $\tau_j^* > 0$, $j = \overline{1, m^*}$, $m^* \geqslant 0$.

(iv) If $R(x) \in B_0$, then $f_1(\rho)$ has no zeros in $\bar{\Pi}_+$, i.e. $m^* = 0$.

We note that (18) follows from (14). Since $\overline{f_j(\rho)} = f_j(-\overline{\rho})$, we obtain from (18) that

$$\alpha_1^2(k) - \alpha_2^2(k) \equiv 1. \tag{19}$$

It follows from (9), (15), (16) and (19) that

$$\begin{aligned} &\alpha_1^2(k) = 1 + \frac{h^2}{4k^2} + \frac{\omega(k)}{k^2} \qquad \alpha_2^2(k) = \frac{h^2}{4k^2} + \frac{\omega(k)}{k^2} \\ &\sigma(k) = \frac{h^2}{4k^2 + h^2} + \frac{\omega_1(k)}{k^2} \qquad |k| \to \infty \end{aligned} \tag{20}$$

where $\omega(k), \omega_1(k) \in L_2(-\infty, \infty)$.

Lemma 2. Denote $g_j^*(t) = g_j(t - T)$. Then

$$g_1^*(\xi) + \int_0^\xi g_1^*(t) g_1^*(t + 2T - \xi)\,\mathrm{d}t = \int_0^\xi g_2^*(t) g_2^*(t + 2T - \xi)\,\mathrm{d}t \xi \in [0, 2T]. \tag{21}$$

Proof. Indeed, using (15) and (16), we calculate

$$f_1(\rho) f_1(-\rho) = 1 + \int_{-2T}^{2T} G_1(\xi) \mathrm{e}^{-\mathrm{i}\rho\xi}\,\mathrm{d}\xi \qquad f_2(\rho) f_2(-\rho) = \int_{-2T}^{2T} G_2(\xi) \mathrm{e}^{-\mathrm{i}\rho\xi}\,\mathrm{d}\xi \tag{22}$$

where $G_j(-\xi) = G_j(\xi)$,

$$\begin{aligned} &G_1(\xi) = g_1(T - \xi) + \int_{\xi - T}^T g_1(t) g_1(t - \xi)\,\mathrm{d}t \\ &G_2(\xi) = \int_{\xi - T}^\xi g_2(t) g_2(t - \xi)\,\mathrm{d}t \qquad \xi > 0. \end{aligned}$$

Substituting (22) into (18) we arrive at (21). □

Denote

$$p(x) = \begin{cases} q(T - x) & 0 \leqslant x \leqslant T \\ 0 & x > T \end{cases}$$

and consider the equation

$$-y'' + p(x) y = \lambda y, x > 0 \tag{23}$$

on the half-line. Let $e(x, \rho)$ be the Iost solution of (23) such that $e(x, \rho) \equiv \mathrm{e}^{\mathrm{i}\rho x}$ for $x \geqslant T$. Denote $\delta(\rho) = e(0, \rho)$. Clearly,

$$e(x, \rho) = \mathrm{e}^{\mathrm{i}\rho T} u(T - x, \rho) \qquad 0 \leqslant x \leqslant T \qquad \delta(\rho) = \mathrm{e}^{\mathrm{i}\rho T} \Delta(\rho).$$

Then

$$e(x,\rho) = \mathrm{e}^{\mathrm{i}\rho x} + \int_x^{2T-x} G(x,t)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}t \qquad 0 \leqslant x \leqslant T \qquad e(x,\rho) \equiv \mathrm{e}^{\mathrm{i}\rho x} \qquad x \geqslant T \tag{24}$$

$$\delta(\rho) = 1 + \int_0^{2T} \theta(t)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}t \qquad \theta(t) \in AC[0,2T] \qquad \theta'(t) \in L_2(0,2T) \qquad \theta(2T) = 0 \tag{25}$$

where $G(x,t) = K(T-x, T-t)$, $\theta(t) = \eta(T-t) = G(0,t)$, $G(x, 2T-x) = 0$, $\theta(0) = \frac{1}{2}\int_0^T p(t)\,\mathrm{d}t$, $G(x,x) = \frac{1}{2}\int_x^T p(\xi)\,\mathrm{d}\xi$.

Denote

$$\alpha_j = \left(\int_0^\infty e^2(x,\rho_j)\,\mathrm{d}x\right)^{-1} > 0 \qquad \rho_j = \mathrm{i}\tau_j \qquad \tau_j > 0,\ j = \overline{1,m}.$$

Lemma 3.

$$\alpha_j = \frac{\mathrm{i}\delta(-\rho_j)}{\dot{\delta}(\rho_j)} \tag{26}$$

where $\dot{\delta}(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\delta(\rho)$.

Proof. Since $-e'' + p(x)e = \rho^2 e$ and $\delta(\rho_j) = 0$, we have

$$(\rho^2 - \rho_j^2)\int_0^\infty e(x,\rho)e(x,\rho_j)\,\mathrm{d}x = |_0^\infty \langle e(x,\rho), e(x,\rho_j)\rangle = -e'(0,\rho_j)\delta(\rho).$$

Which yields

$$\alpha_j = -\frac{2\rho_j}{e'(0,\rho_j)\dot{\delta}(\rho_j)}. \tag{27}$$

Since $\langle e(x,\rho), e(x,-\rho)\rangle \equiv 2\mathrm{i}\rho$, we get

$$\delta(-\rho_j)e'(0,\rho_j) = 2\mathrm{i}\rho_j. \tag{28}$$

Together with (27) it gives (26). □

3. Synthesis of $R(x)$ from the characteristic function $\Delta(\rho)$

In this section we study the inverse problem of recovering the wave resistance $R(x)$ from the given $\Delta(\rho)$ in the class B_0^-. This inverse problem has a unique solution. We provide physical realization conditions, i.e. necessary and sufficient conditions on a function $\Delta(\rho)$ to be the characteristic function for a certain $R(x) \in B_0^-$. We also obtain three algorithms for the solution of the inverse problem.

Denote $s(\rho) = \frac{\delta(-\rho)}{\delta(\rho)}$. For real ρ, $s(\rho)$ is continuous and, by virtue of (25), $s(\rho) = 1 + \mathrm{O}(\frac{1}{\rho})$, $|\rho| \to \infty$. Consequently, $1 - s(\rho) \in L_2(-\infty,\infty)$. We introduce the Fourier transform

$$F_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1 - s(\rho))\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \tag{29}$$

$$1 - s(\rho) = \int_{-\infty}^{\infty} F_0(t)\mathrm{e}^{-\mathrm{i}\rho t}\,\mathrm{d}t. \tag{30}$$

Substituting (25) and (30) into the relation $\delta(\rho)s(\rho) = \delta(-\rho)$, we obtain the connection between $F_0(t)$ and $\theta(t)$:

$$\theta(t) + F_0(t) + \int_0^{2T} \theta(s)F_0(t+s)\,\mathrm{d}s = 0 \qquad t > 0 \tag{31}$$

where $\theta(t) \equiv 0$ for $t > 2T$. Further, we consider the function

$$F(t) = F_0(t) + F_1(t) \tag{32}$$

where

$$F_1(t) = \sum_{j=1}^{m} \alpha_j \mathrm{e}^{\mathrm{i}\rho_j t}. \tag{33}$$

Since

$$F_1(t) + \int_0^{2T} \theta(s)F_1(t+s)\,\mathrm{d}s = \sum_{j=1}^{m} \alpha_j \mathrm{e}^{\mathrm{i}\rho_j t}\delta(\rho_j) = 0$$

we get by virtue of (31) that

$$\theta(t) + F(t) + \int_0^{2T} \theta(s)F(t+s)\,\mathrm{d}s = 0 \qquad t > 0. \tag{34}$$

Calculating the integral (29) by Jordan's lemma for $t > 2T$ and using lemma 3 we obtain

$$F_0(t) = \mathrm{i}\sum_{j=1}^{m} \operatorname*{Res}_{\rho=\rho_j}(1 - s(\rho))\mathrm{e}^{\mathrm{i}\rho t} = -\mathrm{i}\sum_{j=1}^{m} \frac{\delta(-\rho_j)}{\dot{\delta}(\rho_j)}\mathrm{e}^{\mathrm{i}\rho_j t} = -\sum_{j=1}^{m} \alpha_j \mathrm{e}^{\mathrm{i}\rho_j t} = -F_1(t).$$

Thus,

$$F(t) \equiv 0 \qquad t > 2T. \tag{35}$$

Taking (35) into account we rewrite (34) as follows

$$\theta(t) + F(t) + \int_t^{2T} \theta(s-t)F(s)\,\mathrm{d}s = 0 \qquad t \in (0, 2T). \tag{36}$$

In particular it yields $F(t) \in AC[0, 2T]$, $F'(t) \in L_2(0, 2T)$, $F(2T) = 0$.

We note that the function $F(t)$ can be constructed from equation (36) which is usually better than calculating $F(t)$ directly by (32), (33), (29).

Acting in the same way as in [9] one can obtain that

$$G(x,t) + F(x,t) + \int_x^{2T-t} F(t+s)G(x,s)\,\mathrm{d}s = 0 \qquad 0 \leqslant x \leqslant T,\ \ x < t < 2T - x. \tag{37}$$

Equation (37) is called the Gel'fand–Levitan–Marchenko equation.

Now let us formulate physical realization conditions for the characteristic function $\Delta(\rho)$.

Theorem 3. For a function $\Delta(\rho)$ of the form (12) to be the characteristic function for a certain $R(x) \in B_0^-$, it is necessary and sufficient that all zeros of $\Delta(\rho)$ in $\bar{\Pi}_+$ are simple, have the form $\rho_j = \mathrm{i}\tau_j$, $\tau_j > 0$, $j = \overline{1, m}$, $m \geqslant 0$, and

$$\alpha_j := \frac{\mathrm{i}\delta(-\rho_j)}{\dot{\delta}(\rho_j)} > 0.$$

$R(x) \in B_0$ if and only if $\Delta(\rho)$ has no zeros in $\bar{\Pi}_+$, i.e. $m = 0$. The specification of the characteristic function $\Delta(\rho)$ uniquely determines $R(x)$.

Proof. The necessity part of theorem 1 was proved above. We prove the sufficiency. Put $\delta(\rho) = e^{i\rho T}\Delta(\rho)$. Then (25) holds, where $\theta(t) = \eta(T-t)$. Since for each fixed $x \geqslant 0$, the homogeneous integral equation

$$w(t) + \int_x^\infty F(t+s)w(s)\,ds = 0 \qquad t > x \tag{38}$$

has only the trivial solution (see [9]), then equation (37) has a unique solution $G(x,t)$. The function $G(x,t)$ is absolutely continuous, $G(x, 2T-x) = 0$ and $\frac{d}{dx}G(x,x) \in L_2(0,T)$. We construct the function $e(x,\rho)$ by (24).

Let us show that $e(0,\rho) = \delta(\rho)$. Indeed, it follows from (24) and (25) that

$$e(0,\rho) - \delta(\rho) = \int_0^{2T} (G(0,t) - \theta(t))e^{i\rho t}\,dt. \tag{39}$$

By virtue of (36) and (37),

$$(G(0,t) - \theta(t)) + \int_0^{2T-t} (G(0,s) - \theta(s))F(t+s)\,ds = 0.$$

In view of (35), it gives us that the function $w(t) := G(0,t) - \theta(t)$ satisfies (38) for $x = 0$. Hence $G(0,t) = \theta(t)$, and according to (39), $e(0,\rho) = \delta(\rho)$.

Put $p(x) := -2\frac{dG(x,x)}{dx}$, $x \in [0,T]$, and $p(x) \equiv 0$, $x > T$. It is easily shown that $-e''(x,\rho) + p(x)e(x,\rho) = \rho^2 e(x,\rho)$, $x > 0$. We construct $R(x)$ by the formula

$$R(x) = \frac{1}{u^2(x,0)} \tag{40}$$

where $u(x,\rho) = e^{-i\rho T}e(T-x,\rho)$. Since $\delta(0) \neq 0$, we have $u(T,0) \neq 0$. It follows from (40) that $R(x) \in B_0^-$, where $0 < x_1 < \cdots < x_p < T$ are zeros of $u(x,0)$. In particular, if $\delta(\rho)$ has no zeros in $\bar{\Pi}_+$, then, by virtue of theorem 1, $R(x) \in B_0$. The uniqueness of recovering $R(x)$ from $\Delta(\rho)$ was proved for example in [18]. $\square$

Theorem 3 gives us the following algorithm for constructing $R(x)$ from the characteristic function $\Delta(\rho)$.

Algorithm 1. Let a function $\Delta(\rho)$ satisfying the hypothesis of theorem 3 be given. Then:

(1) construct $F(t)$, $t \in (0, 2T)$ from the integral equation (36), where $\theta(t) = \eta(T-t)$, or directly from (26), (29), (32), (33);

(2) find $G(x,t)$ from the integral equation (37);

(3) calculate $R(x)$ by

$$R(x) = \frac{1}{e^2(T-x)} \qquad e(x) = 1 + \int_x^{2T-x} G(x,t)\,dt. \tag{41}$$

Next we provide two other algorithms for the synthesis of the wave resistance from the characteristic function, which sometimes may give advantage from the numerical point of view.

Let $S(x,\lambda)$ be the solution of (23) under the conditions $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$. Then

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_0^x Q(x,t)\frac{\sin\rho t}{\rho}\,dt \tag{42}$$

where $Q(x,t)$ is a real, absolutely continuous function, $Q(x,0) = 0$, $Q(x,x) = \frac{1}{2}\int_0^x p(t)\,dt$.

Since $\langle e(x,\rho), S(x,\lambda)\rangle \equiv \delta(\rho)$ and $\delta(\rho_j) = 0$, we get

$$e(x,\rho_j) = e'(0,\rho_j)S(x,\lambda_j) \qquad \lambda_j = \rho_j^2 \tag{43}$$

and by virtue of (27) and (28),

$$\beta_j := \left(\int_0^\infty S^2(x, \lambda_j)\,\mathrm{d}x \right)^{-1} = -\frac{2\rho_j e'(0, \rho_j)}{\dot{\delta}(\rho_j)} = -\frac{4i\rho_j^2}{\delta(-\rho_j)\dot{\delta}(\rho_j)} > 0. \quad (44)$$

Taking into account the relations

$$\dot{\delta}(\rho_j) = \mathrm{e}^{\mathrm{i}\rho_j T}\dot{\Delta}(\rho_j),\, \delta(-\rho_j) = \mathrm{e}^{-\mathrm{i}\rho_j T}\Delta(-\rho_j)$$

we obtain from (44)

$$\beta_j = -\frac{4i\rho_j^2}{\Delta(-\rho_j)\dot{\Delta}(\rho_j)}.$$

Consider the function

$$a(x) = \sum_{j=1}^m \beta_j \frac{\cos\rho_j x}{\rho_j^2} + \frac{2}{\pi}\int_0^\infty \cos\rho x \left(\frac{1}{|\Delta(\rho)|^2} - 1 \right) \mathrm{d}\rho. \quad (45)$$

By virtue of (12), $\rho(\frac{1}{|\Delta(\rho)|^2} - 1) \in L_2(0, \infty)$, and consequently $a(x)$ is absolutely continuous and $a'(x) \in L_2$. The kernel $Q(x, t)$ from (42) satisfies the equation (see [10])

$$f(x, t) + Q(x, t) + \int_0^x Q(x, s) f(s, t)\,\mathrm{d}s = 0 \qquad x \geqslant 0 \;\; 0 < t < x \quad (46)$$

where $f(x, t) = \frac{1}{2}(a(x - t) - a(x + t))$. The function $R(x)$ can be constructed by the following algorithm.

Algorithm 2. Let $\Delta(\rho)$ be given. Then:
(1) construct $a(x)$ by (45);
(2) find $Q(x, t)$ from the integral equation (46);
(3) calculate $p(x) := 2\frac{\mathrm{d}}{\mathrm{d}x}Q(x, x)$;
(4) construct

$$R(x) := \frac{1}{e^2(T - x)}$$

where $e(x)$ is the solution of the Cauchy problem $y'' = p(x)y$, $y(T) = 1$, $y'(T) = 0$.

Remark 2. We also can construct $R(x)$ using (6) and (8), namely,

$$R(x) = \exp\left(2\int_0^x h(t)\,\mathrm{d}t \right)$$

where $h(x)$ is the solution of the equation

$$h(x) = \int_0^x h^2(t)\,\mathrm{d}t + 2Q(T - x, T - x) - 2Q(T, T).$$

Remark 3. If $R(x) \in B_0$, then

$$a(x) = \frac{2}{\pi}\int_0^\infty \cos\rho x \left(\frac{1}{|\Delta(\rho)|^2} - 1 \right) \mathrm{d}\rho$$

and for constructing the solution of the inverse problem it is sufficient to specify $|\Delta(k)|$ for $k \geqslant 0$.

Now we provide an algorithm for the solution of the inverse problem which uses discrete spectral characteristics.

Let $X_k(x,\lambda)$, $k=1,2$ be solutions of (7) under the conditions $X_1(0,\lambda)=X_2'(0,\lambda)=1$, $X_1'(0,\lambda)=X_2(0,\lambda)=0$. Denote $\Delta_k(\lambda)=X_k(T,\lambda)$. Clearly,

$$u(x,\rho)=X_1(x,\lambda)-\mathrm{i}\rho X_2(x,\lambda),\ \Delta(\rho)=\Delta_1(\lambda)-\mathrm{i}\rho\Delta_2(\Lambda)$$

and consequently

$$\Delta_1(\lambda)=\frac{\Delta(\rho)+\Delta(-\rho)}{2}\qquad \Delta_2(\lambda)=\frac{\Delta(\rho)-\Delta(-\rho)}{2\mathrm{i}\rho}.$$

Let $\{\mu_n\}_{n\geqslant 1}$ be the zeros of $\Delta_2(\lambda)$, and

$$\gamma_n:=\int_0^T X_2^2(x,\mu_n)\,\mathrm{d}x>0.$$

It is easily shown that

$$\gamma_n=(\Delta_1(\mu_n))^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta_2(\lambda)\right)_{|\lambda=\mu_n}$$

and

$$\sqrt{\mu_n}=\frac{\pi n}{T}+\frac{1}{2\pi n}\int_0^T q(\xi)\,\mathrm{d}\xi+\frac{\omega_n}{n}\qquad \gamma_n=\frac{T^3}{2n^2\pi^2}+\frac{\omega_{n1}}{n^3}\qquad \{\omega_n\},\{\omega_{n1}\}\in\ell_2. \tag{47}$$

We introduce the function

$$A(x)=\frac{2}{T}\sum_{n=1}^{\infty}\left(\frac{T\cos\sqrt{\mu_n}x}{2\gamma_n\mu_n}-\cos\frac{\pi n x}{T}\right). \tag{48}$$

By virtue of (47), $A(x)$ is absolutely continuous, and $A'(x)\in L_2$. It is known (see [9]) that

$$X_2(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x D(x,t)\frac{\sin\rho t}{\rho}\,\mathrm{d}t$$

where $D(x,t)$ is a real, absolutely continuous function, and $D(x,0)=0, D(x,x)=\frac{1}{2}\int_0^x q(t)\,\mathrm{d}t$. The function $D(x,t)$ satisfies the equation (see [10])

$$D(x,t)+B(x,t)+\int_0^x D(x,s)B(s,t)\,\mathrm{d}s=0\qquad 0\leqslant x\leqslant T,\ 0<t<x \tag{49}$$

where $B(x,t)=\frac{1}{2}(A(x-t)-A(x+t))$. The function $R(x)$ can be constructed from the discrete data $\{\mu_n,\gamma_n\}_{n\geqslant 1}$ by the the following algorithm.

Algorithm 3. Let $\{\mu_n,\gamma_n\}_{n\geqslant 1}$ be given. Then

(1) Construct $A(x)$ by (48).

(2) Find $D(x,t)$ from the integral equation (49).

(3) Calculate $q(x):=2\frac{\mathrm{d}D(x,x)}{\mathrm{d}x}$.

(4) Construct $R(x):=\frac{1}{u^2(x)}$, where $u(x)$ is the solution of the Cauchy problem

$$u''=q(x)u\qquad u(0)=1\qquad u'(0)=0$$

or by

$$R(x)=\exp\left(2\int_0^x h(t)\,\mathrm{d}t\right)\qquad h(x)=\int_0^x h^2(t)\,\mathrm{d}t-2D(x,x).$$

4. Reconstruction of the transmission coefficients from their moduli

Lemma 4. Suppose that a function $\gamma(\rho)$ is regular in $\bar{\Pi}_+$, has no zeros in Π_+, and for $|\rho| \to \infty$, $\rho \in \bar{\Pi}_+$, $\gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(k) = |\gamma(k)|e^{-i\beta(k)}$, $k = \mathrm{Re}\,\rho$. Then

$$\beta(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi - k}\,d\xi. \tag{50}$$

In (50) (and everywhere below) the integral is understood in the principal value sense.

Proof. First we suppose that $\gamma(k) \neq 0$ for real k. By Cauchy's theorem, taking into account the hypothesis of the lemma, we obtain

$$\frac{1}{2\pi i}\int_{C_{r,\epsilon}} \frac{\ln\gamma(\xi)}{\xi - k}\,d\xi = 0 \tag{51}$$

where $C_{r,\epsilon}$ is the closed contour (with anticlockwise circuit) consisting of the semicircles $C_r = \{\xi : \xi = re^{i\varphi}, \varphi \in [0,\pi]\}$, $\Gamma_\epsilon = \{\xi : \xi - k = \epsilon e^{i\varphi}, \varphi \in [0,\pi]\}$ and the intervals $\xi \in [-r, r] \setminus [k-\epsilon, k+\epsilon]$ of the real axis. Since

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Gamma_\epsilon} \frac{\ln\gamma(\xi)}{\xi - k}\,d\xi = -\tfrac{1}{2}\ln\gamma(k)$$
$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{C_r} \frac{\ln\gamma(\xi)}{\xi - k}\,d\xi = 0$$

we get from (51) that

$$\ln\gamma(k) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln\gamma(\xi)}{\xi - k}\,d\xi.$$

Separating here real and imaginary parts, we arrive at (50).

Suppose now that for real ρ, the function $\gamma(\rho)$ has one zero $\rho_0 = 0$ of multiplicity s (the general case is treated in the same way). Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho)\left(\frac{\rho + i\epsilon}{\rho}\right)^s \qquad \epsilon > 0 \qquad \tilde{\gamma}(k) = |\tilde{\gamma}(k)|e^{-i\tilde{\beta}(k)}.$$

Then

$$\beta(k) = \tilde{\beta}(k) + s\,\mathrm{arctg}\,\frac{\epsilon}{k}. \tag{52}$$

For the function $\tilde{\gamma}(\rho)$, (50) has been proved. Hence, (52) takes the form

$$\beta(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi - k}\,d\xi + \frac{s}{2\pi}\int_{-\infty}^{\infty} \frac{\ln(1 + \frac{\epsilon^2}{\xi^2})}{\xi - k}\,d\xi + s\,\mathrm{arctg}\,\frac{\epsilon}{k}.$$

When $\epsilon \to 0$, it gives us (50). □

Lemma 5. Suppose that a function $\gamma(\rho)$ is regular in $\bar{\Pi}_+$, $\gamma(-\overline{\rho}) = \overline{\gamma(\rho)}$, and for $|\rho| \to \infty$, $\rho \in \bar{\Pi}_+$, $\gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(k) = |\gamma(k)|e^{-i\beta(k)}$, and let $\rho_j = k_j + i\tau_j$, $\tau_j > 0$, $j = \overline{1, s}$ be zeros of $\gamma(\rho)$ in Π_+. Then

$$\beta(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi - k}\,d\xi + 2\sum_{j=1}^{s} \mathrm{arctg}\,\frac{\tau_j}{k - k_j}. \tag{53}$$

Proof. Since $\gamma(-\overline{\rho}) = \overline{\gamma(\rho)}$, the zeros of $\gamma(\rho)$ in Π_+ are symmetrical with respect to the imaginary axis. If $\rho_j = \mathrm{i}\tau_j$, $\tau_j > 0$, then

$$\arg \frac{k+\rho_j}{k-\rho_j} = 2\,\mathrm{arctg}\,\frac{\tau_j}{k}. \tag{54}$$

If $\rho_j = k_j + \mathrm{i}\tau_j$, $k_j > 0$, $\tau_j > 0$, then

$$\arg \frac{(k+\rho_j)(k-\bar{\rho}_j)}{(k-\rho_j)(k+\bar{\rho}_j)} = 2\,\mathrm{arctg}\,\frac{\tau_j}{k-k_j} + 2\,\mathrm{arctg}\,\frac{\tau_j}{k+k_j}. \tag{55}$$

Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho)\prod_{j=1}^{s}\frac{\rho+\rho_j}{\rho-\rho_j}.$$

The function $\tilde{\gamma}(\rho)$ satisfies the hypothesis of lemma 4. Then using (50), (54) and (55) we arrive at (53). □

Using lemma 5 one can construct the transmission coefficients from their moduli and information about their zeros in Π_+. For definiteness we confine ourselves to the case $h \neq 0$.

Theorem 4. Let

$$f_1(k) = \alpha_1(k)\mathrm{e}^{-\mathrm{i}\delta_1(k)} \tag{56}$$

and let $\rho_j^* = \mathrm{i}\tau_j^*$, $\tau_j^* > 0$, $j = \overline{1, m}^*$ be zeros of $f_1(\rho)$ in Π_+. Then

$$\delta_1(k) = kT + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\alpha_1(\xi)}{\xi-k}\,\mathrm{d}\xi + 2\sum_{j=1}^{m^*}\mathrm{arctg}\,\frac{\tau_j^*}{k}. \tag{57}$$

In particular, if $R(x) \in B_0$, then

$$\delta_1(k) = kT + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\alpha_1(\xi)}{\xi-k}\,\mathrm{d}\xi. \tag{58}$$

Proof. Denote $\gamma(\rho) = \mathrm{e}^{\mathrm{i}\rho T} f_1(\rho)$. If follows from (15) that

$$\gamma(\rho) = 1 + \int_{-T}^{T} g_1(t)\mathrm{e}^{\mathrm{i}\rho(T-t)}\,\mathrm{d}t$$

and consequently, $\gamma(-\bar{\rho}) = \overline{\gamma(\rho)}$, and for $|\rho| \to \infty$, $\rho \in \bar{\Pi}_+$, $\gamma(\rho) = 1 + \mathrm{O}(\frac{1}{\rho})$. Thus, the function $\gamma(\rho)$ satisfies the hypothesis of lemma 5. Using (53) and the relations $|\gamma(k)| = \alpha_1(k)$, $\delta_1(k) = \beta(k) + kT$, we arrive at (57) . □

Similarly we prove the following theorem.

Theorem 5. Let

$$f_2(k) = \alpha_2(k)\mathrm{e}^{-\mathrm{i}\delta_2(k)} \tag{59}$$

and let $\rho_j^0 = k_j^0 + \mathrm{i}\tau_j^0$, $\tau_j^0 > 0$, $j = \overline{1, m}^0$ be zeros of $f_2(\rho)$ in Π_+. Then

$$\delta_2(k) = \frac{\pi}{2}\mathrm{sign}\left(\frac{k}{h}\right) + kT + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|\frac{2\xi}{h}\alpha_2(\xi)|}{\xi-k}\,\mathrm{d}\xi + 2\sum_{j=1}^{m_0}\mathrm{arctg}\,\frac{\tau_j^0}{k-k_j^0}. \tag{60}$$

In particular, if $f_2(\rho)$ has no zeros in Π_+, then

$$\delta_2(k) = \frac{\pi}{2}\operatorname{sign}\left(\frac{k}{h}\right) + kT + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\frac{2\xi}{h}\alpha_2(\xi)|}{\xi - k}\,\mathrm{d}\xi. \tag{61}$$

Thus, the specification of $\alpha_j(k)$ uniquely determines the transmission coefficients only when they have no zeros in Π_+. In particular, for the class B_0, $f_1(\rho)$ is uniquely determined by its modulus, and all possible transmission coefficients $f_2(\rho)$ can be constructed by means of solving the integral equation (21).

5. Symmetrical case

The wave resistance is called symmetrical if $R(T-x) = R(x)$. For the symmetrical case the transmission coefficient $f_2(k)$ is uniquely determined (up to the sign) from its modulus.

Theorem 6. For the wave resistance to be symmetrical it is necessary and sufficient that $\operatorname{Re} f_2(k) = 0$. At that, $f_2(k) = -f_2(-k)$, $g_2(t) = -g_2(-t)$, and

$$f_2(k) = -2\mathrm{i}\int_0^T g_2(t)\sin kt\,\mathrm{d}t.$$

Proof. According to (9) and (5),

$$f_2(\rho) = \frac{1}{2\mathrm{i}\rho}(u'(T,\rho) + \mathrm{i}\rho u(T,\rho) + hu(T,\rho)).$$

Since $u(x,\rho) = X_1(x,\lambda) - \mathrm{i}\rho X_2(x,\lambda)$, we calculate

$$\operatorname{Re} f_2(k) = \tfrac{1}{2}(X_1(T,\lambda) - X_2'(T,\lambda) - hX_2(T,\lambda)). \tag{62}$$

If $R(x) = R(T-x)$, then it follows from (6) and (8) that $h(x) = -h(T-x)$, $q(x) = q(T-x)$, and consequently $h = 0$, $X_1(T,\lambda) \equiv X_2'(T,\lambda)$ (see [19]), i.e. $\operatorname{Re} f_2(k) = 0$.

Conversely, if $\operatorname{Re} f_2(k) = 0$, then (62) gives us

$$X_1(T,\lambda) - X_2'(T,\lambda) - hX_2(T,\lambda) \equiv 0. \tag{63}$$

Since for $k \to \infty$,

$$X_2(T,k^2) = \frac{\sin kT}{k} + \mathrm{O}\left(\frac{1}{k^2}\right) \qquad X_1(T,\lambda) - X_2'(T,\lambda) = \mathrm{O}\left(\frac{1}{k^2}\right)$$

we obtain from (63) that $h = 0$, $X_1(T,\lambda) \equiv X_2'(T,\lambda)$. Consequently, $q(x) = q(T-x)$ (see [19]). Similarly one can prove that $g(x) = g(T-x)$. Hence $h(x) = -h(T-x)$, and $R(x) = R(T-x)$.

Further, it follows from (16) that

$$\operatorname{Re} f_2(k) = \int_{-T}^{T} g_2(t)\cos kt\,\mathrm{d}t \qquad \operatorname{Im} f_2(k) = -\int_{-T}^{T} g_2(t)\sin kt\,\mathrm{d}t.$$

For the symmetrical case $\operatorname{Re} f_2(k) = 0$, and consequently $g_2(t) = -g_2(-t)$. Then

$$f_2(k) = \mathrm{i}\operatorname{Im} f_2(k) = -2\mathrm{i}\int_0^T g_2(t)\sin kt\,\mathrm{d}t \qquad f_2(-k) = -f_2(k).$$

□

Thus, $f_2(k)$ can be constructed (up to the sign) by the formulae $\operatorname{Re} f_2(k) = 0$, $|\operatorname{Im} f_2(k)| = \alpha_2(k)$, $f_2(k) = -f_2(-k)$.

6. Synthesis of the wave resistance from the power reflection coefficient

In this section, using results obtained above, we provide a procedure for constructing $R(x)$ from the given power reflection coefficient $\sigma(k)$. For definiteness we confine ourselves to the case $h \neq 0$. Let $\sigma(k)(0 \leqslant \sigma(k) < 1,\ \sigma(k) = \sigma(-k))$ be given. Our scheme of calculation is as follows.

Step 1. Calculate $\alpha_1(k)$ and $\alpha_2(k)$ by the formula

$$\alpha_1^2(k) = 1 + \alpha_2^2(k) = \frac{1}{1-\sigma(k)}. \tag{64}$$

Step 2. Construct $f_1(k)$ by (56), (57) or for $R(x) \in B_0$ by (56), (58). Find $g_1(t)$ from the relation

$$\int_{-T}^{T} g_1(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t = f_1(k) - \mathrm{e}^{\mathrm{i}kT}.$$

Step 3. Construct $f_2(k)$ by (59), (60) or, if $f_2(\rho)$ has no zeros in Π_+, by (59), (61). Find $g_2(t)$ from the relation

$$\int_{-T}^{T} g_2(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t = f_2(k).$$

We note that $g_2(t)$ can be constructed also from the integral equation (21). In this case $f_2(k)$ is calculated by (16).

Step 4. Calculate $\eta(t) = g_1(t) + g_2(t)$ and $\Delta(\rho)$ by (12).

Step 5. Construct $R(x)$ using one of the algorithms 1–3.

Remark 4. In some concrete algorithms it is not necessary to make all the calculations mentioned above. For example, let $R(x) \in B_0$, and let us use algorithm 2. Then it is not necessary to calculate $g_1(t)$, $g_2(t)$ and $\eta(t)$, since we need only $|\Delta(k)|$.

Now we consider a concrete algorithm which realizes this scheme. For simplicity, we consider the case $R(x) \in B_0$.

For $t \in [0, T]$ we consider the functions

$$\varphi_j(t) = g_j(t) + g_j(-t) \qquad \psi_j(t) = g_j(t) - g_j(-t) \qquad j = 1, 2 \tag{65}$$

$$\varphi(t) = \eta(t) + \eta(-t) \qquad \psi(t) = \eta(t) - \eta(-t). \tag{66}$$

Since $\eta(t) = g_1(t) + g_2(t)$, we get

$$\psi(t) = \varphi_1(t) + \varphi_2(t) \qquad \psi(t) = \psi_1(t) + \psi_2(t). \tag{67}$$

Solving (65) and (66) with respect to $g_j(t)$ and $\eta(t)$ we obtain

$$g_j(t) = \begin{cases} \frac{1}{2}(\varphi_j(t) + \psi_j(t)) & t > 0 \\ \frac{1}{2}(\varphi_j(-t) - \psi_j(-t)) & t < 0 \end{cases} \qquad \eta(t) = \begin{cases} \frac{1}{2}(\varphi(t) + \psi(t)) & t > 0 \\ \frac{1}{2}(\varphi(-t) + \psi(-t)) & t < 0. \end{cases} \tag{68}$$

It follows from (17) and (65) that

$$\varphi_1(T) = \psi_1(T) = -w_1 \qquad \varphi_2(T) = \psi_2(T) = -\frac{h}{2} \qquad \psi_1(0) = \psi_2(0) = 0 \tag{69}$$

where

$$w_1 = -\tfrac{1}{2}\int_0^T h^2(\xi)\,\mathrm{d}\xi.$$

By virtue of lemma 3,

$$
\begin{aligned}
f_1(k) &= (\cos kT + C_1(k)) - \mathrm{i}(\sin kT + S_1(k)) \\
f_2(k) &= C_2(k) - \mathrm{i}S_2(k) \\
\Delta(k) &= (\cos kT + C(k)) - \mathrm{i}(\sin kT + S(k))
\end{aligned}
\tag{70}
$$

where

$$
C_j(k) = \int_0^T \varphi_j(t)\cos kt\,\mathrm{d}t \qquad S_j(k) = \int_0^T \psi_j(t)\sin kt\,\mathrm{d}t \tag{71}
$$

$$
C(k) = \int_0^T \varphi(t)\cos kt\,\mathrm{d}t \qquad S(k) = \int_0^T \psi(t)\sin kt\,\mathrm{d}t. \tag{72}
$$

Using (69) and (71) we obtain the following asymptotic formulae for $C_j(k)$ and $S_j(k)$ as $k \to \infty$:

$$
\begin{aligned}
C_1(k) &= -w_1\frac{\sin kT}{k} + \frac{\omega(k)}{k} \qquad & C_2(k) &= -\frac{h}{2}\frac{\sin kT}{k} + \frac{\omega(k)}{k} \\
S_1(k) &= w_1\frac{\cos kT}{k} + \frac{\omega(k)}{k} \qquad & S_2(k) &= \frac{h}{2}\frac{\cos kT}{k} + \frac{\omega(k)}{k}
\end{aligned}
. \tag{73}
$$

Here and below, one and the same symbol $\omega(k)$ denotes various functions from $L_2(-\infty, \infty)$.

Comparing (70) with the relations $f_j(k) = \alpha_j(k)\mathrm{e}^{-\mathrm{i}\delta_j(k)}$, we derive

$$
C_1(k) = \alpha_1(k)\cos\delta_1(k) - \cos kT \qquad S_1(k) = \alpha_1(k)\sin\delta_1(k) - \sin kT \tag{74}
$$

$$
C_2(k) = \alpha_2(k)\cos\delta_2(k) \qquad S_2(k) = \alpha_2(k)\sin\delta_2(k). \tag{75}
$$

To calculate the arguments of the transmission coefficients we use (58) and (61), i.e.

$$
\delta_1(k) = kT + \tilde{\delta}_1(k) \qquad \delta_2(k) = \frac{\pi}{2}w + kT + \tilde{\delta}_2(k) \qquad k > 0 \tag{76}
$$

where $w = \operatorname{sign} h$,

$$
\tilde{\delta}_1(k) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\alpha_1(\xi)}{\xi - k}\,\mathrm{d}\xi \tag{77}
$$

$$
\tilde{\delta}_2(k) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\tilde{\alpha}_2(\xi)}{\xi - k}\,\mathrm{d}\xi \qquad \tilde{\alpha}_2(\xi) := \left|\frac{2\xi}{h}\alpha_2(\xi)\right|. \tag{78}
$$

It follows from (73)–(75) and (20) that for $k \to +\infty$

$$
\delta_1(k) = kT + \frac{w_1}{k} + \frac{\omega(k)}{k} \qquad \delta_2(k) = \frac{\pi}{2}w + kT + \omega(k). \tag{79}
$$

Substituting (76) into (74) and (75), we calculate for $k > 0$

$$
\begin{aligned}
C_1(k) &= \cos kT(\alpha_1(k)\cos\tilde{\delta}_1(k) - 1) - \sin kT(\alpha_1(k)\sin\tilde{\delta}_1(k)) \\
S_1(k) &= \sin kT(\alpha_1(k)\cos\tilde{\delta}_1(k) - 1) + \cos kT(\alpha_1(k)\sin\tilde{\delta}_1(k))
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
C_2(k) &= -\alpha_2(k)w(\sin kT\cos\tilde{\delta}_2(k) + \cos kT\sin\tilde{\delta}_2(k)) \\
S_2(k) &= \alpha_2(k)w(\cos kT\cos\tilde{\delta}_2(k) - \sin kT\sin\tilde{\delta}_2(k)).
\end{aligned}
\tag{81}
$$

Further, consider the functions

$$
\begin{aligned}
\varphi_1^*(t) &= \varphi_1(t) + w_1 \qquad & \psi_1^*(t) &= \psi_1(t) + w_1\frac{t}{T} \\
\varphi_2^*(t) &= \varphi_2(t) + \frac{h}{2} \qquad & \psi_2^*(t) &= \psi_2(t) + \frac{h}{2}\cdot\frac{t}{T}.
\end{aligned}
\tag{82}
$$

Then in view of (69),

$$\varphi_j^*(T) = \psi_j^*(T) = \psi_j^*(0) = 0 \qquad j = 1, 2. \tag{83}$$

Denote

$$C_j^*(k) = \int_0^T \varphi_j^*(t) \cos kt \, \mathrm{d}t \qquad S_j^*(k) = \int_0^T \psi_j^*(t) \sin kt \, \mathrm{d}t. \tag{84}$$

Integrating the integrals in (84) by parts and taking (83) into account, we obtain

$$C_j^*(k) = -\int_0^T \varphi_j^{*'}(t) \frac{\sin kT}{k} \, \mathrm{d}t \qquad S_j^*(k) = \int_0^T \psi_j^{*'}(t) \frac{\cos kt}{k} \, \mathrm{d}t. \tag{85}$$

Clearly,

$$C_1(k) = C_1^*(k) - w_1 \frac{\sin kT}{k} \qquad S_1(k) = S_1^*(k) + w_1 \left(\frac{\cos kT}{k} - \frac{\sin kT}{Tk^2} \right) \tag{86}$$

$$C_2(k) = C_2^*(k) - \frac{h}{2} \frac{\sin kT}{k} \qquad S_2(k) = S_2^*(k) + \frac{h}{2} \left(\frac{\cos kT}{k} - \frac{\sin kT}{Tk^2} \right). \tag{87}$$

Now let the power reflection coefficient $\sigma(k) (0 \leqslant \sigma(k) < 1, \sigma(k) = \sigma(-k))$ be given for $|k| \leqslant B$, and put

$$\sigma(k) = \frac{h^2}{4k^2 + h^2} \qquad |k| > B \tag{88}$$

where $B > |h|$ is chosen sufficiently large, such that (see (20)) $\sigma(k)$ is sufficiently accurate for $|k| > B$.

Using (64) we calculate $\alpha_1(k)$ and $\alpha_2(k)$. Then $\alpha_1^2(k) - \alpha_2^2(k) = 1$, $\alpha_j(k) = \alpha_j(-k) > 0$, and

$$\alpha_1^2(k) = 1 + \frac{h^2}{4k^2} \qquad \alpha_2^2(k) = \frac{h^2}{4k^2} \qquad |k| > B. \tag{89}$$

In order to calculate $\tilde{\delta}_1(k)$ we use (77). First let $|k| > B + \chi$, $\chi > 0$. In view of (89), equality (77) takes the form

$$\tilde{\delta}_1(k) = \frac{1}{\pi} \int_{-B}^{B} \frac{\ln \alpha_1(\xi)}{\xi - k} \, \mathrm{d}\xi + \frac{1}{2\pi} \int_{|\xi| > B} \frac{\ln(1 + \frac{h^2}{4\xi^2})}{\xi - k} \, \mathrm{d}\xi.$$

Since

$$\begin{aligned} \frac{1}{2\pi} \int_{|\xi| > B} \frac{\ln\left(1 + \frac{h^2}{4\xi^2}\right)}{\xi - k} \, \mathrm{d}\xi &= \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{h^2}{4}\right)^j \int_{|\xi| > B} \frac{\mathrm{d}\xi}{\xi^{2j}(\xi - k)} \\ &= \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{h^2}{4}\right)^j \int_{|\xi| > B} \left\{ \frac{1}{k^{2j}} \left(\frac{1}{\xi - k} - \frac{1}{\xi} \right) - \sum_{s=1}^{2j-1} \frac{1}{k^{2j-s} \xi^{s+1}} \right\} \mathrm{d}\xi \end{aligned}$$

we get

$$\begin{aligned} \tilde{\delta}_1(k) = \frac{1}{\pi} \int_{-B}^{B} \frac{\ln \alpha_1(\xi)}{\xi - k} \, \mathrm{d}\xi + \frac{1}{2\pi} \ln\left(1 + \frac{2B}{k - B}\right) \ln\left(1 + \frac{h^2}{4k^2}\right) \\ + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left(\frac{h^2}{4}\right)^j \sum_{\mu=0}^{j-1} \frac{1}{(2j - 2\mu - 1) B^{2j-2\mu-1} k^{2\mu+1}}. \end{aligned} \tag{90}$$

Using the relation

$$\frac{1}{\pi}\int_{-B}^{B}\frac{\ln\alpha_1(\xi)}{\xi-k}\,d\xi=-\frac{1}{k\pi}\int_{-B}^{B}\ln\alpha_1(\xi)\,d\xi+\frac{1}{\pi}\int_{-B}^{B}\frac{\xi\ln\alpha_1(\xi)}{k(\xi-k)}\,d\xi$$

we separate in (90) the terms of order $1/k$, i.e.

$$\tilde{\delta}_1(k)=\frac{w_1}{k}+\tilde{\delta}_1^*(k) \tag{91}$$

where

$$\tilde{\delta}_1^*(k)=\frac{1}{\pi}\int_{-B}^{B}\frac{\xi\ln\alpha_1(\xi)}{k(\xi-k)}\,d\xi+\frac{1}{2\pi}\ln\left(1+\frac{2B}{k-B}\right)\ln\left(1+\frac{h^2}{4k^2}\right)$$
$$+\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{h^2}{4}\right)^j\sum_{\mu=1}^{j-1}\frac{1}{(2j-2\mu-1)B^{2j-2\mu-1}k^{2\mu+1}}\qquad |k|>B+\chi \tag{92}$$

$$w_1=-\frac{1}{\pi}\int_{-B}^{B}\ln\alpha_1(\xi)\,d\xi+\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{h^2}{4B^2}\right)^j\cdot\frac{B}{2j-1}<0. \tag{93}$$

It is easy to show that

$$|\tilde{\delta}_1^*(k)|\leqslant\frac{\Omega_1^*}{k^2}$$

where

$$\Omega_1^*=\frac{B+\chi}{\pi\chi}\int_0^B\xi\ln\alpha_1(\xi)\,d\xi+\frac{h^2}{8\pi}\ln\left(1+\frac{2B}{\chi}\right)+\frac{h^2}{2\pi}.$$

Now let $|k|\leqslant B+\chi$. In this case we rewrite (77) in the form

$$\tilde{\delta}_1(k)=\frac{1}{\pi}\int_0^{\infty}\frac{\ln\alpha_1(\xi+k)-\ln\alpha_1(\xi-k)}{\xi}\,d\xi. \tag{94}$$

Take $r\geqslant 2B+\chi$. Since

$$\left|\ln\left(1+\frac{h^2}{4\xi^2}\right)-\frac{h^2}{4\xi^2}\right|\leqslant\frac{h^4}{32\xi^4}$$

we get

$$\frac{1}{\pi}\int_r^{\infty}\frac{\ln\alpha_1(\xi+k)-\ln\alpha_1(\xi-k)}{\xi}\,d\xi=\frac{h^2}{8\pi}\int_r^{\infty}\frac{1}{\xi}\left(\frac{1}{(\xi+k)^2}-\frac{1}{(\xi-k)^2}\right)d\xi+\epsilon(k)$$

where

$$|\epsilon(k)|\leqslant\frac{h^4}{96\pi r(r-B-\chi)^3}.$$

Calculating this integral and substituting into (94), we obtain

$$\tilde{\delta}_1(k)=\frac{1}{\pi}\int_0^r\frac{\ln\alpha_1(\xi+k)-\ln\alpha_1(\xi-k)}{\xi}\,d\xi$$
$$+\frac{h^2}{8\pi}\left(\frac{1}{k^2}\ln\frac{r+k}{r-k}-\frac{1}{k}\left(\frac{1}{R+k}+\frac{1}{R-k}\right)\right)+\epsilon(k)$$
$$\tilde{\delta}_1(0)=0,\ |k|\leqslant B+\chi,\ r\geqslant 2B+\chi. \tag{95}$$

In order to calculate $\delta_2(k)$ we use (78). Let $|k| > B + \chi$, $\chi > 0$. According to (89) we have $\tilde{\alpha}_2(k) = 1$ for $|k| > B$. Hence

$$\tilde{\delta}_2(k) = \frac{1}{\pi}\int_{-B}^{B} \frac{\ln\tilde{\alpha}_2(\xi)}{\xi - k}\,\mathrm{d}\xi \qquad |\xi| > B + \chi. \tag{96}$$

For $|k| \leqslant B + \chi$ it is more convenient to use another formula:

$$\tilde{\delta}_2(k) = \frac{1}{\pi}\int_0^r \frac{\ln\tilde{\alpha}_2(\xi + k) - \ln\tilde{\alpha}_2(\xi - k)}{\xi}\,\mathrm{d}\xi \qquad |k| \leqslant B + \chi,\ r = 2B + \chi. \tag{97}$$

Let us calculate $\psi_j^*(t), \varphi_j^*(t)$. By virtue of (84),

$$\begin{aligned}\psi_j^*(t) &= \frac{2}{T}\sum_{n=1}^{\infty} S_j^*\left(\frac{n\pi}{T}\right)\sin\frac{n\pi}{T}t\\ \varphi_j^*(t) &= \frac{1}{T}C_j^*(0) + \frac{2}{T}\sum_{n=1}^{\infty} C_j^*\left(\frac{n\pi}{T}\right)\cos\frac{n\pi}{T}t.\end{aligned} \tag{98}$$

According to (85), the series in (98) converge absolutely and uniformly. By virtue of (80), (81), (86), (87), the coefficients $S_j^*(\frac{n\pi}{T})$ and $C_j^*(\frac{n\pi}{T})$ can be calculated via the formulae

$$\begin{aligned}S_1^*(k) &= (-1)^n\left(\alpha_1(k)\sin\tilde{\delta}_1(k) - \frac{w_1}{k}\right) && k = \frac{n\pi}{T},\ n \geqslant 1\\ S_2^*(k) &= (-1)^n\left(\alpha_2(k)w\cos\tilde{\delta}_2(k) - \frac{h}{2k}\right) && k = \frac{n\pi}{T},\ n \geqslant 1\\ C_1^*(k) &= (-1)^n(\alpha_1(k)\cos\tilde{\delta}_1(k) - 1) + \delta_{n0}w_1T && k = \frac{n\pi}{T},\ n \geqslant 0\\ C_2^*(k) &= (-1)^{n+1}w\sin\tilde{\delta}_2(k) + \delta_{n0}\frac{hT}{2} && k = \frac{n\pi}{T},\ n \geqslant 0\end{aligned} \tag{99}$$

where δ_{nj} is the Kronecker delta.

Using the formulae obtained above we arrive at the following numerical algorithm for the solution of inverse problem 1.

Algorithm 4. Let the power reflection coefficient $\sigma(k)$ be given for $|k| \leqslant B$, and take $\sigma(k)$ according to (88) for $|k| > B$. Then:

(1) construct $\alpha_1(k)$ and $\alpha_2(k)$ by (64);

(2) find the number w_1 by (93);

(3) calculate $\tilde{\delta}_1(k), \tilde{\delta}_2(k)$ by (91), (92), (96) for $|k| > B + \chi$, and by (95), (97) for $|k| \leqslant B + \chi$;

(4) construct $\varphi_j^*(t), \psi_j^*(t)$ by (98), where $S_j^*(\frac{n\pi}{T})$, $C_j^*(\frac{n\pi}{T})$ is defined by (99);

(5) find $\varphi_j(t)$, $\psi_j(t)$ using (82);

(6) calculate $\eta(t), t \in [-T, T]$ by (67), (68);

(7) find $F(t)$, $0 < t < 2T$ from the integral equation (36), where $\theta(t) = \eta(T - t)$;

(8) calculate $G(x, t)$ from the integral equation (37);

(9) construct $R(x)$, $x \in [0, T]$ via (41).

Remark 5. This algorithm is one of several possible numerical algorithms for the solution of inverse problem 1. Using the obtained results one can construct various algorithms for synthesizing $R(x)$ from spectral characteristics.

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