Existence of Open Loop Stackelberg Equilibria

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Abstract

We present existence and uniqueness results for a hierarchical or Stackelberg equilibrium in a two player differential game with open loop information structure. There is known a convexity condition ensuring existence of a Stackelberg equilibrium which was derived by Simaan and Cruz [15] . This condition applies to games with a rather non-conflicting structure of their cost criteria. By another approach we obtain new sufficient existence conditions for an open loop equilibrium in feedback synthesis in terms of solvability of two symmetric Riccati differential equations and a coupled system of Riccati matrix differential equations. The latter coupled system also appears as a necessary condition. In case that the convexity condition holds, both symmetric equations are of standard type and admit globally a positive semidefinite solution. But the conditions presented here also apply to conflicting situations like zero-sum or nearly zero-sum games. Then the corresponding Riccati differential equations may be of H_∞-type.

1 Introduction

In non-cooperative game theory the concept of hierarchical or Stackelberg games is eminently important, as it was pointed out for instance in [15] . Different hierarchical structures as well as different information patterns have been investigated (see [13], [5], [14], [12], [16], [10], [1]).

Further information on Stackelberg differential games may also be found for instance in [2], [3], [4].

The purpose of this paper is to study a two player Stackelberg differential game under open loop information pattern, where the performance criteria of the players are of *quadratic* type and where a linear differential equation describes the constraints to the state vector.

In the case of two players, which we consider here for reasons of simplicity, we are given a differential equation in a fixed time interval $[t_0, t_f]$

$$\dot{x} = A(t)x + B_1(t)u_1 + B_2(t)u_2, \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t), x_0 \in \mathbf{R}^n, A(t) \in \mathbf{R}^{n\times n}, B_i(t) \in \mathbf{R}^{n\times m_i}$, $u_i(t) \in \mathbf{R}^{m_i}, i = 1, 2$; and the performance criteria are

$$J_i = \frac{1}{2}x^T(t_f)K_{if}x(t_f) +$$

$$+\frac{1}{2}\int_{t_0}^{t_f}(x^T(t)Q_i(t)x(t) + \sum_{j=1}^{2}u_j^T(t)R_{ij}(t)u_j(t))\,dt, \quad i = 1, 2; \tag{1.2}$$

with the matrices $K_{if}, Q_i(t), R_{ij}(t), 1 \le i, j \le 2$, being symmetric and of appropriate size. Throughout this paper we assume that $R_{ii}^{-1}(t)$ exists for $t \in [t_0, t_f]$.

In order to apply the approach, as presented in [15], $x(t)$ has to be absolutely continuous, the controls $u_i \in L_2^{m_i}([t_0, t_f])$, while the matrices $A, B_i, Q_i, R_{ij}, 1 \le i, j \le 2$, can be considered continuous.

We also assume the player 2 to be the leader, i.e. he is seeking a strategy $u_2^*(t)$, a function of time only in open loop information structure, that he announces before the game starts, knowing how the follower reacts to any of his choices. The follower will then calculate his strategy $u_1^*(t)$ also as a function of time only.

More formally, if we define the sets Γ_1, Γ_2 of admissible controls for (1.1) and (1.2), i.e. sets of functions such that (1.1) is solvable for each $u_1 \in \Gamma_1$ and $u_2 \in \Gamma_2$ and that J_1, J_2 exists, respectively, then a Stackelberg equilibrium is defined as follows:

The optimal reaction set of player 1 (the follower) to a control $u_2 \in \Gamma_2$ is

$$R_1(u_2) = \{\gamma \in \Gamma_1 \mid J_1(\gamma, u_2) \le J_1(u_1, u_2) \text{ for all } u_1 \in \Gamma_1\}.$$

If player 2 is leading then $u_2^* \in \Gamma_2$ is called a Stackelberg equilibrium for player 2 if for all $u_2 \in \Gamma_2$

$$sup_{\gamma\in R_1(u_2^*)}J_2(\gamma, u_2^*) \le sup_{\gamma\in R_1(u_2)}J_2(\gamma, u_2).$$

Then

$$u_1^* \in R_1(u_2^*)$$

is an optimal Stackelberg strategy for the follower.

From [15],page 545, Proposition 4.2, we cite

Theorem 1.1 *In the differential game, as given by (1.1), (1.2), let player 2 be the leader.*
If the following convexity condition

$$(C)\begin{cases} R_{11}(t) > 0, R_{22}(t) > 0, & t \in [t_0, t_f] \\ R_{21}(t) \geq 0, Q_1(t) \geq 0, Q_2(t) \geq 0, & t \in [t_0, t_f] \\ K_{1f} \geq 0, K_{2f} \geq 0 \end{cases}$$

holds then there exists a unique open loop Stackelberg equilibrium u_1^*, u_2^* in the set of admissible controls $L_2^{m_1}([t_0, t_f]), L_2^{m_2}([t_0, t_f])$, respectively.

2 Existence of Stackelberg equilibria

To our knowledge the Hilbert space method - as applied in [15] - was so far the sole method to obtain existence conditions for a Stackelberg equilibrium. The convexity condition (C) in Theorem 1.1 is seen to be sufficient for existence. In [15] there is also obtained a general necessary and sufficient condition for existence in terms of invertibility of an operator in Hilbert space, but this condition cannot be applied directly.
In linear quadratic problems there exists also another approach using a value function which can be obtained by an appropriate guess.
This leads to sufficient conditions that turn out to be more general than (C) but are restricted to feedback controls.
For Nash games a similar approach is nicely presented in the papers [6],[7], [11].

Theorem 2.1 *Let the solution of the Riccati Differential equation*

$$\dot{E}_1 = -E_1 A - A^T E_1 - Q_1 + E_1 S_1 E_1, \; E_1(t_f) = K_{1f}, \tag{2.1}$$

with $S_1 = B_1 R_{11}^{-1} B_1^T$, exist on $[t_0, t_f]$.
For any given admissible open loop control u_2 of the leader define $e_1(t), d_1(t)$ by

$$\dot{e}_1 = E_1 S_1 e_1 - A^T e_1 - 2E_1 B_2 u_2, \quad e_1(t_f) = 0 \tag{2.2}$$

$$\dot{d}_1 = -u_2^T R_{12} u_2 - e_1^T B_2 u_2 + \frac{1}{4} e_1^T S_1 e_1, \quad d_1(t_f) = 0. \tag{2.3}$$

Then the following identity holds:

$$2\ J_1(u_1, u_2) = x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0) + \\ + \int_{t_0}^{t_f} ||z_1(t)||_{R_{11}}^2 \, dt, \tag{2.4}$$

where $||z_1(t)||_{R_{11}}^2 = z_1^T(t) R_{11} z_1(t)$ and

$$z_1(t) = u_1 + R_{11}^{-1} B_1^T (E_1 x + \frac{1}{2} e_1)$$

with x a solution of (1.1).

Proof: We try a "quadratic" guess for the value function, i.e. we start with

$$\tfrac{d}{dt}(x^T E_1 x + x^T e_1 + d_1) = 2x^T E_1 (Ax + B_1 u_1 + B_2 u_2) + \\ x^T \dot{E}_1 x + e_1^T (Ax + B_1 u_1 + B_2 u_2) + x^T \dot{e}_1 + \dot{d}_1 = \\ = x^T [E_1 A + A^T E_1 + \dot{E}_1 + Q_1 - E_1 S_1 E_1] x - x^T Q_1 x - \\ u_1^T R_{11} u_1 - u_2^T R_{12} u_2 + x^T [\dot{e}_1 + A^T e_1 - E_1 S_1 e_1 + \\ 2E_1 B_2 u_2] + \dot{d}_1 + u_2^T R_{12} u_2 - \tfrac{1}{4} e_1^T S_1 e_1 + e_1^T B_2 u_2 + ||z_1(t)||_{R_{11}}^2 \\ = -[x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2] + ||z_1(t)||_{R_{11}}^2.$$

Integrating this identity from t_0 to t_f and observing (2.1), (2.2) and (2.3) yields together with (1.2) the result in (2.4). □

From (2.4) we now infer that for any fixed u_2

$$J_1(u_1, u_2) \geq \frac{1}{2}(x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0)),$$

hence, the minimal costs are attained if and only if $z_1(t) \equiv 0$.
With this remark we obtain:

Theorem 2.2 *Let the solution E_1 of (2.1) exist on $[t_0, t_f]$. Then the unique response of the follower to the leaders open loop strategy u_2 is given by*

$$u_1(t) = -R_{11}^{-1}(t) B_1^T(t) (E_1(t) x(t) + \frac{1}{2} e_1(t)), \tag{2.5}$$

where e_1 is a solution of (2.2) and $x(t)$ is a solution of

$$\dot{x} = (A - S_1 E_1) x - \frac{1}{2} S_1 e_1 + B_2 u_2, \quad x(t_0) = x_0. \tag{2.6}$$

The corresponding minimal costs then are

$$J_{10}(u_2) := \frac{1}{2}(x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0)). \tag{2.7}$$

In open loop Stackelberg games now the leader (i.e. player 2) tries to find an optimal open loop control u_2 in order to minimize $J_2(u_1(u_2), u_2)$ while $u_1(u_2)$ is defined by (2.5).

Theorem 2.3 *Let the solution of the Riccati differential equations (2.1) and the solution of*

$$\dot{E}_2 = -E_2 H - H^T E_2 - Q + E_2 S E_2, \\ E_2(t_f) = \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.8}$$

exist in $[t_0, t_f]$, where

$$H = \begin{pmatrix} A & -S_1 \\ -Q_1 & -A^T \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix}, \\ S = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}. \\ S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T, \quad S_2 = B_2 R_{22}^{-1} B_2^T. \tag{2.9}$$

For any given admissible control u_2 of the leader define the functions $e_2(t) \in \mathbf{R}^{2n}, v_1(t), \quad x(t) \in \mathbf{R}^n$ and $d_2(t) \in \mathbf{R}$ in $[t_0, t_f]$ by the following initial or terminal value problems:

$$\dot{e}_2 = (-H^T + E_2 S)e_2, \quad e_2(t_f) = 0, \tag{2.10}$$

$$\dot{d}_2 = \frac{1}{4} e_2^T S e_2, \quad d_2(t_f) = 0, \tag{2.11}$$

$$\dot{v}_1 = -Q_1 x - A^T v_1, \quad v_1(t_0) = v_{10}, \tag{2.12}$$

$$\dot{x} = Ax - S_1 v_1 + B_2 u_2, \quad x(t_0) = x_0. \tag{2.13}$$

Then we obtain with

$$u_1 = -R_{11}^{-1} B_1^T v_1 \tag{2.14}$$

the following identity:

$$2J_2(u_1, u_2) = (x_0^T, v_{10}) E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix} +$$

$$+(x_0^T, v_{10}) e_2(t_0) + d_2(t_0) + \int_{t_0}^{t_f} ||z_2(t)||^2_{R_{22}} \, dt, \tag{2.15}$$

where

$$z_2 = u_2 + (R_{22}^{-1} B_2^T, 0_{m_2 \times n})(E_2 y + \frac{1}{2} e_2) \tag{2.16}$$

with $y = \begin{pmatrix} x \\ v_1 \end{pmatrix}$ and $0_{m_2 \times n}$ the $m_2 \times n$-dimensional zero matrix.

The proof of this theorem is similar to the proof of Theorem 2.1 and will be presented in a forthcoming paper.

Remark 2.4 *Notice that in the term*

$$J_{20} = (x_0^T, v_{10}^T) E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix} + (x_0^T, v_{10}^T) e_2(t_0) + d_2(t_0), \tag{2.17}$$

$x_0, E_2(t_0), e_2(t_0), d_2(t_0)$, do not depend on the choice of u_2. In view of (2.5), (2.10) and (2.14) we later have to choose $v_1(t_f) = K_{1f} x(t_f)$ and this fixes v_{10} uniquely but then depending on u_2.

In order to derive from Theorems 2.2, 2.3 sufficient conditions for the existence of a unique Stackelberg equilibrium we must get rid of the u_2-dependence of v_{10} in (2.15). Therefore we propose to restrict the set of admissible controls to functions representable in linear feedback form, i.e. $u_i = L_i(t)x(t), i = 1, 2$, where L_i denotes some linear operator.

Theorem 2.5 *Let the solutions $E_1(t) \in \mathbf{R}^{n \times n}, E_2(t) \in \mathbf{R}^{2n \times 2n}$ of (2.1) and (2.8) exist in $[t_0, t_f]$, respectively. Let, furthermore, the coupled system*

$$\begin{aligned} \dot{K}_1 &= -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2, \\ & \quad K_1(t_f) = K_{1f} \\ \dot{K}_2 &= -A^T K_2 - K_2 A - Q_2 + Q_1 P + K_2 S_1 K_1 + \\ & \quad K_2 S_2 K_2, \ K_2(t_f) = K_{2f} \\ \dot{P} &= AP - PA + PS_1 K_1 + PS_2 K_2 + S_1 K_2 - S_{21} K_1, \\ & \quad P(t_f) = 0 \end{aligned} \tag{2.18}$$

admit a solution in $[t_0, t_f]$.
Then there exists a unique open loop Stackelberg equilibrium in feedback synthesis which is given by

$$\begin{aligned} u_1^*(t) &= -R_{11}^{-1}(t) B_1^T(t) K_1(t) x(t), \\ u_2^*(t) &= -R_{22}^{-1}(t) B_2^T(t) K_2(t) x(t), \end{aligned} \tag{2.19}$$

where x is a solution of the closed loop equation

$$\dot{x} = (A - S_1 K_1 - S_2 K_2)x, \ \ x(t_0) = x_0. \tag{2.20}$$

The minimal costs for the follower then are $J_{10}(u_2^*)$ as given in (2.7), together with (2.1),(2.2) and (2.3). The costs for the leader are

$$\begin{aligned} J_{20}(u_1^*, u_2^*) &= \tfrac{1}{2}(x_0^T (I_n, K_1^T(t_0)) E_2(t_0) \begin{pmatrix} I_n \\ K_1(t_0) \end{pmatrix} x_0 \\ &+ \ e_2^T(t_0) \begin{pmatrix} I_n \\ K_1(t_0) \end{pmatrix} x_0 + d_2(t_0)), \end{aligned} \tag{2.21}$$

where $e_2(t_0), d_2(t_0)$ are determined by (2.10), (2.11).

The proof makes use of Theorems 2.2, 2.3 and the assumption that all admissible open loop controls can be represented by a feedback law. We also will present this proof in a forthcoming paper.

Here we first point out that from condition (C) in Theorem 1.1 it follows that S_1, S, Q_1, Q and $K_{1f} = E_1(t_f), E_2(t_f)$ are all positive semidefinite. Therefore, if the convexity condition (C) holds, by well known results on standard Riccati matrix differential equations, we then obtain global solvability of equations (2.1) and (2.8) in $(-\infty, t_f]$.

There remains the question about direct criteria for solvability of the Riccati matrix differential equations (2.1), (2.8) if (C) is not fulfilled and also for the coupled system (2.18).
Before stating a result in that direction let us remark that the coupled system in (2.18) also can be written as a single, nonsymmetric, Riccati matrix differential equation.

We obtain instead of (2.18)

$$
\begin{aligned}
\frac{d}{dt}\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} &= -\begin{pmatrix} Q_1 \\ Q_2 \\ 0 \end{pmatrix} \\
&- \begin{pmatrix} A^T & 0 & 0 \\ 0 & A^T & -Q_1 \\ S_{21} & -S_1 & -A \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} A \\
&+ \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} (S_1, S_2, 0) \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}, \quad \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}(t_f) = \begin{pmatrix} K_{1f} \\ K_{2f} \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.22}
$$

All these Riccati matrix differential equations (2.1), (2.8) and (2.22), now are of the (generally nonsymmetric) type

$$
\begin{aligned}
\dot{W} &= B_{21}(t) + B_{22}(t)W - WB_{11}(t) - WB_{12}(t)W, \\
& \quad W(t_0) = W_0,
\end{aligned} \tag{RDE}
$$

where $W(t)$ is a $m \times n$ matrix and the coefficients are of appropriate size and smothness In [9] (see also [8]) there are proved several global existence results for (RDE). We cite the following:

Theorem 2.6 *Let $B_{11}, B_{12}, B_{21}, B_{22}$ be piecewise continuous and locally integrable on $(-\infty, T]$. If for some matrices $C \in \mathbf{C}^{n\times n}$, with $C^* = C$, $D \in \mathbf{C}^{n\times m}$, and*

$$
L = \begin{pmatrix} CB_{11} + DB_{21} & CB_{12} + B_{11}^* D + DB_{22} \\ 0 & B_{12}^* D \end{pmatrix} \tag{2.23}
$$

holds

$$
L(t) + L^*(t) \leq 0, \tag{2.24}
$$

for all $t \leq t_0 (\leq T)$ and if

$$
C + DW_0 + W_0^* D^* > 0, \tag{2.25}
$$

for some $W_0 \in \mathbf{C}^{m\times n}$, then the solution $W(t, W_0)$ of (RDE) with $W(t_0, W_0) = W_0$ exists for all $t \leq t_0$.

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