Monotonicity and convexity results for time-varying Riccati equations

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Abstract

Using a Fréchet derivative based approach some monotonicity and comparison results concerning the solutions of the time-varying discrete time Riccati equation are obtained. Connections with the existence of the semistabilizing solution are made explicit as well.

1 Introduction and problem statement

One of the most remarkable properties of the (matrix) Riccati differential equation (RDE)

$$-\dot{X} = A^T(t)X + XA(t) - XS(t)X + Q(t), \quad X(t_0) = X_0,$$

where $A(t)$, $S(t) = S^T(t)$, $Q(t) = Q^T(t)$ and $X_0 = X_0^T \in \mathbb{R}^{n\times n}$, is that concerning *monotonicity* with respect to both initial value X_0 and input data matrix

$$E(t) = \begin{bmatrix} Q & A^T \\ A & -S \end{bmatrix}(t).$$

Stokes [20] proved the striking fact that RDE is, for $n > 1$, the only (matrix) differential equation $\dot{X} = \Phi(t, X)$, $X(t_0) = X_0$ possessing the so-called *order-preserving* property, i.e. X depends monotonically on X_0. Notice also that under different assumptions the monotonicity of X with respect to E has been studied by many authors (see for instance [5], [6], [7], [8] and [18]).

Furthermore, various types of monotonicity results have been also obtained for both differential and difference Riccati equations as well as for the algebraic Riccati equation in both continuous and discrete versions (see [2], [3], [19], [8], [15], [18], [22], [24]).

The main purpose of the present paper is that of generalizing the aforementioned monotonicity and comparison results to the case of the time-varying discrete Riccati equation (TVDRE). A nice survey on TVDRE theory could be found in [17].
To this end a Fréchet derivative based approach combined with the bilateral shift algebra has been proposed.

In the sequel we shall use the same notations as in [1] and [11].

Let us be now more specific concerning the topics in order. First introduce

$$\mathbf{G}(E, X) := A^*\sigma X(I_{\ell^{2,n}} + S\sigma X)^{-1}A - X + Q \tag{1.1}$$

where

$$E = \begin{bmatrix} Q & A^* \\ A & -S \end{bmatrix} = E^* \tag{1.2}$$

is termed as the *input data matrix sequence* and σ is the bilateral shift operator. Here $A : \ell^{2,n} \to \ell^{2,n}$, $B : \ell^{2,m} \to \ell^{2,n}$, $Q : \ell^{2,n} \to \ell^{2,n}, S = BR^{-1}B^*$, and $R : \ell^{2,m} \to \ell^{2,m}$ are *block diagonal operators*, with R boundedly invertible.

The TVDRE is now introduced by $\mathbf{G}(E, X) = 0$, that is,

$$A^*\sigma X(I_{\ell^{2,n}} + S\sigma X)^{-1}A - X + Q = 0. \tag{1.3}$$

Recall that any global, bounded on $\mathbb{Z}$ and selfadjoint solution X to the TVDRE (1.3) is called *stabilizing* (*semistabilizing*) if $I_{\ell^{2,n}} + S\sigma X$ has a bounded inverse and

$$A_{cl}(X) := (I_{\ell^{2,n}} + S\sigma X)^{-1}A \tag{1.4}$$

defines an exponentially stable (semistable) evolution, i. e. $\rho(A_{cl}(X)\sigma^{-1}) < 1$ ($\rho(A_{cl}(X)\sigma^{-1}) \leq 1$), ρ is the spectral radius. In the subsequent sections we shall investigate the local and global monotonicity of any selfadjoint solution X to the TVDRE (1.3), existing on any interval $[l, s]$ with $l \geq -\infty$, with respect to the input data matrix sequence E and terminal condition X_s. The results are derived under no assumptions on the signature of S. Further, semipositive definiteness of S will be assumed for establishing some comparison results and global monotonicity of the semistabilizing solution. In the last case some existence conditions for the semistabilizing solution in terms of the generalized Popov theory are implicitly given.

2 Fréchet derivatives and local monotonicity

Proposition 2.1. *For each pair (E, X) with $X = X^*$ the Fréchet derivatives of* $\mathbf{G}$ *are*

$$\mathbf{G}_E(E,X)(\Delta E) = N^*(E,X)\Delta E N(E,X) \tag{2.1}$$

where

$$N(E,X) = \begin{bmatrix} I \\ \sigma X A_{cl}(X) \end{bmatrix} \tag{2.2}$$

with $A_{cl}(X)$ defined in (1.4) and

$$\Delta E = \begin{bmatrix} \Delta \widetilde{Q} & (\Delta \widetilde{A})^* \\ \Delta \widetilde{A} & -\Delta S \end{bmatrix} = (\Delta E)^*. \tag{2.3}$$

$$\mathbf{G}_X(E,X)(\Delta X) = A_{cl}(X)^*\sigma\Delta X A_{cl}(X) - \Delta X \tag{2.4}$$

where $\Delta X = (\Delta X)^*$.

$$\begin{array}{c} \mathbf{G}_{XX}(E,X)(\Delta X, \Delta Z) = \\ -A_{cl}(X)^*[\sigma\Delta X P \sigma \Delta Z + \sigma \Delta Z P \sigma \Delta X] A_{cl}(X) \end{array} \tag{2.5}$$

with $\Delta X = (\Delta X)^*$, $\Delta Z = (\Delta Z)^*$ and

$$P := S(I + \sigma X S)^{-1} = P^* \tag{2.6}$$

Proof. By elementary direct computations that use the definition of the Fréchet derivative. □

Theorem 2.2. *Let the input data matrix sequence E together with any interval $\mathbf{I} = [l, s] \subset \mathbb{Z}$ be given. Assume that the TVDRE (1.3) has a selfadjoint solution X on $\mathbf{I}$. Then X is locally monotonic increasing with respect to both terminal condition X_s and input data sequence E.*

Proof. We have $\mathbf{G}(E,X) = 0$ on $\mathbf{I}$. In addition the Stein equation

$$A_{cl}(X)^*\sigma\Delta X A_{cl}(X) - \Delta X + W = 0 \tag{2.7}$$

has, for $(\Delta X)_s = (\Delta X^*)_s$ specified, and arbitrary free term $W = W^*$, a unique solution $\Delta X = \Delta X^*$ on $\mathbf{I}$. Hence, according to (2.4), this means that the Fréchet derivative $\mathbf{G}_X(E,X)$ is invertible. In addition both $\mathbf{G}_X$ and $\mathbf{G}_{XX}$ (see (2.4), (2.5)) are continuous in (E, X).

Hence, according to the implicit function theorem, we can write

$$\mathbf{G}_E(E,X)(\Delta E) +$$

$$\mathbf{G}_X(E,X)(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)) = 0, \tag{2.8}$$

with the terminal condition

$$(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s = \Delta Z.$$

Using (2.1) and (2.4), (2.8) receives the explicit form

$$\begin{array}{l} A_{cl}(X)^*\sigma X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)A_{cl}(X) \\ -X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E) \\ +N^*(E,X)\Delta E N(E,X) = 0 \end{array} \tag{2.9}$$

with

$$(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s = \Delta Z.$$

Since (2.9) has, locally, for $\Delta Z \geq 0$ and $\Delta E \geq 0$, a unique positive definite solution $X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E) \geq 0$ on $\mathbf{I}$, the conclusion follows. □

Corollary 2.3. *Let E be given and assume that the TVDRE (1.3) has a stabilizing solution X. Then X is locally monotonic with respect to E.*

Remark 2.4. (i) Theorem 2.2 and its proof suggest a new Fréchet derivative based proof for well known monotonicity results (see for instance [6] and [18]) concerning the maximal (stabilizing) solution of algebraic Riccati equations both in continuous and discrete time.
In the present paper we confine to TVDREs with real coefficients, however it is obvious that the results obtained also hold in the case of complex coefficients.

(ii) In both statements of Theorem 2.2 and Corollary 2.3, the existence of the solution to the TVDRE was preassumed. The following lemma which is a slight generalization of the result given in [8] gives some sufficient conditions for the above mentioned existence.

Lemma 2.5. *Assume $Q \geq 0$ and $R = I_{\ell^{2,m}}$. Then for any interval $\mathbf{I} = [l, s] \subset \mathbb{Z}$ and any terminal condition $X_s \geq 0$ the TVDRE (1.3) has a solution $X \geq 0$ on $\mathbf{I}$.*

3 Global monotonicity and existence of the semistabilizing solution

Theorem 3.1. *Let $\mathbf{I} = [l, s] \subset \mathbb{Z}$ be given. Let $\Gamma_1 = \Gamma_1^T$, $\Gamma_2 = \Gamma_2^T \in \mathbb{R}^{n\times n}$ with $\Gamma_1 \leq \Gamma_2$, and E_1, E_2 with $E_1 \leq E_2$ be also given. If for all Γ and E given by $\Gamma = (1-\lambda)\Gamma_1 + \lambda\Gamma_2$ and $E = (1-\lambda)E_1 + \lambda E_2$ with $0 \leq \lambda \leq 1$, the TVDRE (1.3) has, for the terminal condition $X_s = \Gamma$, a solution $X(\Gamma, E)$ on $\mathbf{I}$ then*

$$X(\Gamma_1, E_1) \leq X(\Gamma_2, E_2). \tag{3.1}$$

Proof. Using the mean value theorem one gets

$$X(\Gamma_2,E_2) - X(\Gamma_1,E_1) = X_{(\Gamma,E)}(\tilde{\Gamma},\tilde{E})((\Gamma_2-\Gamma_1, E_2-E_1)) \tag{3.2}$$

where $\tilde{\Gamma}$ and $\tilde{E}$ are each on the segments $[\Gamma_1, \Gamma_2]$ and $[E_1, E_2]$, respectively. Since $\Delta\Gamma := \Gamma_2 - \Gamma_1 \geq 0$, $\Delta E := E_2 - E_1 \geq 0$, one gets $X_{(\Gamma,E)}(\tilde{\Gamma}, \tilde{E})(\Delta\Gamma, \Delta E) \geq 0$ as has been shown in the proof of Theorem 2.2. Hence (3.1) follows. □

Assume now for the rest of the paper that $R = I_{\ell^{2,m}}$, *that is,* $S = BB^*$.
The two next theorems generalize to the time-varying case, those results obtained in [8].

Theorem 3.2. *Let the data matrix sequence E be given. Assume that $Q \geq 0$ and $E = (E_k)_{k\in\mathbb{Z}}$ is monotonically decreasing, i.e. $E_i \leq E_j \; \forall i < j$. If for some $s \in \mathbb{Z}$ there exists $\Gamma \geq 0$ such that for the terminal condition $X_s = \Gamma$ we have $X_{s-1} \leq X_s$ then the TVDRE (1.3) has a monotonically decreasing positive semidefinite solution $X = (X_k)_{k\leq s}$, i.e. $0 \leq \ldots \leq X_k \leq \ldots \leq X_s$. Here $X_{s-1} := A_{s-1}^T X_s (I + S_{s-1} X_s)^{-1} A_{s-1} + Q_{s-1}$. If in addition $\lim_{k\to-\infty} E_k = \overline{E} \in \mathbb{R}^{2n\times 2n}$ then the TVDRE converges, for $k \to -\infty$, to the discrete algebraic Riccati equation associated with $\overline{E}$, i.e. to $\overline{A}^T \overline{X}(I + \overline{S}\,\overline{X})^{-1}\overline{A} - \overline{X} + \overline{Q} = 0$, fulfilled by $\overline{X} = \lim_{k\to-\infty} X_k$.*

A direct consequence of Theorem 3.2, which is essentially based on Theorem 3.1, is the next result.

Theorem 3.3. *Let*

$$E_i = \begin{bmatrix} Q_i & \tilde{A}_i^* \\ A_i & -S_i \end{bmatrix}, \qquad i = 1, 2, \tag{3.3}$$

be two input data matrix sequences and let TVDRE1 and TVDRE2 be the Riccati equations associated with E_1 and E_2, respectively. Assume that both sequences $E_1 = (E_{1,k})_{k\in\mathbb{Z}}$ and $E_2 = (E_{2,k})_{k\in\mathbb{Z}}$ are monotonically decreasing, $0 \leq Q_{1,k} \leq Q_{2,k}$, $\forall k \in \mathbb{Z}$, and that in addition $\lim_{k\to-\infty} E_{i,k} = \overline{E}_i \in \mathbb{R}^{2n\times 2n}$, $i = 1, 2$. Assume that for some $s \in \mathbb{Z}$ there exists $\Gamma \geq 0$ such that if $X_{2,s} = \Gamma$ then $X_{2,s-1} \leq X_{2,s} = \Gamma$ where $X_{2,s-1} := A_{2,s-1}^T X_{2,s}(I + S_{2,s-1}X_{2,s})^{-1}A_{s-1} + Q_{2,s-1}$. If Γ is taken as a common terminal condition at $s \in \mathbb{Z}$ for both TVDRE 1 and TVDRE 2 then both TVDRE 1 and TVDRE 2 have the solutions $X_1 = (X_{1,k})_{k\leq s}$, $X_2 = (X_{2,k})_{k\leq s}$, respectively with the following properties

$$0 \leq X_{1,k} \leq X_{2,k} \;\; \forall k \leq s, \tag{3.4}$$

$$X_{1,k} \downarrow \text{ and } X_{2,k} \downarrow \text{ for } k \downarrow -\infty \;\; (k \leq s). \tag{3.5}$$

Furthermore, TVDRE 1 and TVDRE 2 converge for $k \to -\infty$ to the discrete algebraic Riccati equations $\overline{A}_1^T \overline{X}_1 (I + \overline{S}_1 \overline{X}_1)^{-1}\overline{A}_1 - \overline{X}_1 + \overline{Q}_1 = 0$ and $\overline{A}_2^T \overline{X}_2 (I + \overline{S}_2 \overline{X}_2)^{-1}\overline{A}_2 - \overline{X}_2 + \overline{Q}_2 = 0$, each of them fulfilled by $\overline{X}_1 = \lim_{k\to-\infty} X_{1,k}$, $\overline{X}_2 = \lim_{k\to-\infty} X_{2,k}$, respectively.

Let

$$E = \begin{bmatrix} Q & A^* \\ A & -S \end{bmatrix}, \text{ with } S = BB^* \tag{3.6}$$

be a bounded input data matrix sequence. If A defines an exponentially stable evolution then

$$\Pi_E = \Pi_E^*$$

$$:= I_{\ell^{2,m}} + B^*(A^* - I_{\ell^{2,n}}\sigma^{-1})^{-1} Q (A - I_{\ell^{2,n}}\sigma)^{-1} B \tag{3.7}$$

is a linear bounded selfadjoint operator form $\ell^{2,m}$ to $\ell^{2,m}$ and is called the *Popov operator* associated with E (see [10], [11], [12], [14]).

Of major importance is the following result of the so-called generalized Popov theory (see [10], [11], [12], [14]).

Theorem 3.4 *Let E as in (3.6) be given with A defining an exponentially stable evolution. Then the following two statements are equivalent.*

1. *The TVDRE (1.3) associated with E has a stabilizing solution X and in addition $I_{\ell^{2,m}} + B^*\sigma X B >> 0$.*
2. *$\Pi_E >> 0$.*

If 2. is fulfilled then the following representation formula holds $\forall k \in \mathbb{Z}$

$$X_k = \tag{3.8}$$

$$\Phi_k^* Q[I_{\ell^{2,n}} + (I_{\ell^{2,n}}\sigma - A)^{-1} P_{k,+}^n S (I_{\ell^{2,n}}\sigma^{-1} - A^*)^{-1} Q]^{-1}\Phi_k,$$

where Φ_k is the state transition operator at k, i. e. $(\Phi_k\, x)_i = A_{i-1}\cdots A_k\, x$, $i > k$. □

Here $T >> 0$ means that there exists $\nu_0 > 0$ such that $\langle Tx, x\rangle \geq \nu_0 \|x\|^2 \;\; \forall x$.

Theorem 3.5. *Let*

$$E_i = \begin{bmatrix} Q_i & A_i^* \\ A_i & -S_i \end{bmatrix}, \quad \text{with} \quad S_i = B_i B_i^* \;\; i = 1, 2 \tag{3.9}$$

be two input data matrix sequences and assume that $\Pi_{E_1} >> 0$, $E_2 \geq E_1$ and $(1-\lambda)A_1 + \lambda A_2 \;\; \forall\lambda \in [0,1]$ is exponentially stable. Then both TVDREs associated with E_1 and E_2 have semistabilizing solutions X_1 and X_2, respectively. In addition $X_2 \geq X_1$.

Remark 3.6 The Fréchet derivative based approach proposed in the present paper has been used by the authors in order to prove and generalize some convexity/concavity properties of the solutions of Riccati equations obtained in [9], [13], [16], [21] and [23]. The results concern in particular convexity/concavity with respect to the coefficients Q and S both in discrete and continuous cases; details will be presented elsewhere.

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