

Coupled Matrix Riccati Equations in Minimal Cost Variance Control Problems

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Abstract: We present an algorithm for the solution of a nontrivial coupled system of algebraic Riccati equations appearing in risk sensitive control problems. Moreover we use comparison methods to derive non blow up conditions for the solutions of a corresponding terminal value problem for coupled systems of Riccati differential equations.

1. Introduction

Won, Sain and Spencer studied recently the connection between linear-quadratic Gaussian control [3], minimal cost variance control and risk-sensitive control [1], [13] via the cost cumulants (see [11], [12], [14] for details).

The main result of [12] is that in the linear-quadratic case the minimal cost variance controller has the form

$$k^*(t, x) = -R^{-1}(t)B^T(t)[M(t) + \gamma(t)V(t)]x,$$

where M and V are solutions of the following terminal value problem for coupled matrix Riccati-type differential equations (suppressing the argument t):

$$\dot{M} = -A^T M - MA - Q + MBR^{-1}B^T M - \gamma^2 VBR^{-1}B^T V, M(t_f) = Q_f, \quad (1.1)$$

$$\begin{aligned} \dot{V} = & -A^T V - VA + 2\gamma VBR^{-1}B^T V + MBR^{-1}B^T V + VBR^{-1}B^T M \\ & - 4ME\tilde{W}E^T M, V(t_f) = 0. \end{aligned} \quad (1.2)$$

It is well known that it is usually very difficult to check a priori if the solutions of such coupled systems exist on a given interval $[t_0, t_f]$ since the matrix-functions M and V may have finite escape-time. Equations (1.1) and (1.2) are strongly related to the mean and variance of the LQG performance measure and to the first two equations in a countable set of Lyapunov-type differential equations associated with all of the LQG cost cumulants; see [10].

As far as we know there exist even in the constant coefficient case no results on the existence of the solutions of (1.1), (1.2) or of the corresponding algebraic system

$$0 = -A^T M - MA - Q + MBR^{-1}B^T M - \gamma^2 VBR^{-1}B^T V, \quad (1.3)$$

$$0 = -A^T V - VA + 2\gamma VBR^{-1}B^T V + MBR^{-1}B^T V + VBR^{-1}B^T M - 4ME\tilde{W}E^T M. \quad (1.4)$$

In section 2 of this paper we give an algorithm which allows - under appropriate assumptions - to calculate a stabilizing pair of solutions of the algebraic system (1.3), (1.4) and

in section 3 we study (1.1), (1.2).

In the whole paper we assume that the scalar function γ and the matrix functions A, Q, B, R, E and \tilde{W} are at least piecewise continuous with

$$\gamma(t) \geq 0, R(t) > 0, Q(t) \geq 0, Q_f \geq 0 \text{ and } \tilde{W}(t) \geq 0. \quad (1.5)$$

We use a comparison method to derive sufficient conditions for the existence of the solutions M and V of the boundary value problem (1.1), (1.2); a preliminary version of our results can be found in section 7 of [9].

2. The coupled algebraic system

In section 2 we assume in addition that all coefficients of (1.1), (1.2) are constant and that (\sqrt{Q}, A) is detectable.

Using the abbreviation

$$K := (K(M, V, \gamma) =) - R^{-1} B^T (M + \gamma V)$$

we get by an elementary calculation that the coupled algebraic equations (1.3), (1.4) can be rewritten as

$$(A + BK)^T M + M(A + BK) + K^T R K + Q = 0 \quad (2.1)$$

$$(A + BK)^T V + V(A + BK) + 4ME\tilde{W}E^T M = 0; \quad (2.2)$$

these equations look formally like two Lyapunov equations but they are of course coupled and much more complicated since K depends on M and V . In contrast to standard algebraic Riccati equations the coupled system (2.1), (2.2) cannot be solved in an elementary way.

The main purpose of this section is to provide an algorithm for the solution of (2.1), (2.2) and sufficient conditions ensuring the existence of a solution M_∞, V_∞ of (2.1), (2.2) such that $A + BK(M_\infty, V_\infty)$ is stable - such a pair M_∞, V_∞ is called stabilizing. Here and in the sequel a $n \times n$ matrix is called stable if all of its eigenvalues have negative real part.

It is natural to try to solve (2.1), (2.2) by Lyapunov iterations but unfortunately it is very difficult to determine sufficient conditions ensuring the convergence of these iterations. For convenience we use in the sequel the abbreviations

$$S := BR^{-1}B^T \text{ and } W := E\tilde{W}E^T;$$

notice that these matrices are positive semidefinite. Moreover we suppose in the sequel that

$$\Delta W + W\Delta + \Delta W\Delta \geq 0 \text{ if } \Delta \geq 0; \quad (2.3)$$

notice that (2.3) holds if $W = \alpha I$ if $\alpha \geq 0$.

In the subsequent algorithm we present a procedure for the calculation of a pair of solutions of (2.1), (2.2); if this algorithm does not stop after a finite number of steps it always produces a stabilizing pair of solutions.

Algorithm A: (1) Choose K_0 such that $A + BK_0$ is stable, choose a positive semidefinite matrix M_0 and put $\nu = 1$.

(2) Define $M_\nu, V_\nu, K_\nu, Z_\nu$, $\nu \in \mathbb{N}$, and Γ_ν , $\nu \geq 2$, as follows:

- (i) Determine (by an adequate method) M_ν , the unique positive semidefinite solution (see Lemma 4.1) of

$$(A + BK_{\nu-1})^T M_\nu + M_\nu(A + BK_{\nu-1}) + Q + K_{\nu-1}^T R K_{\nu-1} = 0, \quad (2.4)$$

and V_ν , the unique positive semidefinite solution of

$$(A + BK_{\nu-1})^T V_\nu + V_\nu(A + BK_{\nu-1}) + 4M_{\nu-1} W M_{\nu-1} = 0. \quad (2.5)$$

- (ii) Let $Z_\nu = M_\nu + \gamma V_\nu$ and $K_\nu := K(M_\nu, V_\nu, \gamma) = -R^{-1} B^T Z_\nu$.
For $\nu > 1$ go to (iii), else put $\nu = 2$ and go to (i).

- (iii) Let

$$\Gamma_\nu := \gamma(Z_{\nu-1} - Z_\nu) S V_\nu + \gamma V_\nu S (Z_{\nu-1} - Z_\nu) + (Z_{\nu-1} - Z_\nu) S (Z_{\nu-1} - Z_\nu). \quad (2.6)$$

If $\Gamma_\nu \geq 0$ (which implies - as we shall prove - that $A + BK_\nu$ is stable) replace ν by $\nu + 1$ and go to (i), else stop.

(In practical applications, in particular if one does not need the monotonicity of the sequences (M_ν) , (V_ν) and $(M_\nu + \gamma V_\nu)$, it is often recommendable to remove this *else stop* - see Remark 2.4 (ii).)

Notice that Algorithm A does not stop if for example K_0 can be chosen such that $A + BK_i$ is stable for $i = 0, 1$ and $\Gamma_\nu \geq 0$ for $\nu \geq 2$.

Theorem 2.1 *If K_0 and M_0 can be chosen so that Algorithm A does not stop, then the limits $M_\infty := \lim_{\nu \rightarrow \infty} M_\nu$ and $V_\infty := \lim_{\nu \rightarrow \infty} V_\nu$ exist and M_∞, V_∞ is a pair of positive semidefinite solutions of (2.1), (2.2) (and hence also of (1.3), (1.4)) such that the closed loop matrix $(A - BR^{-1}B^T(M_\infty + \gamma V_\infty))$ is stable.*

Moreover the sequences $(M_\nu)_{\nu \geq 2}$ and $(M_\nu + \gamma V_\nu)_{\nu \geq 3}$ are monotonically decreasing.

Proof. a) Assume that K_0 has been chosen such that Algorithm A does not stop. We use the preceding notation and set for $\mu, \nu \in \mathbb{N}$

$$\begin{aligned} \psi_\mu(K_{\nu-1}) &= (A + BK_{\nu-1})^T (M_\mu + \gamma V_\mu) + (M_\mu + \gamma V_\mu)(A + BK_{\nu-1}) \\ &\quad + K_{\nu-1}^T R K_{\nu-1}. \end{aligned} \quad (2.7)$$

An elementary calculation (see [15], (3.2)) yields for $\nu \in \mathbb{N}$

$$\psi_\nu(K_{\nu-1}) - \psi_\nu(K_\nu) = (K_{\nu-1} - K_\nu)^T R (K_{\nu-1} - K_\nu). \quad (2.8)$$

On account of (2.4) we can rewrite (2.7) as

$$\psi_\nu(K_{\nu-1}) = \gamma[(A + BK_{\nu-1})^T V_\nu + V_\nu(A + BK_{\nu-1})] - Q \quad (2.9)$$

and, using (2.5), this yields

$$\psi_\nu(K_{\nu-1}) = -4\gamma M_{\nu-1} W M_{\nu-1} - Q. \quad (2.10)$$

Combining (2.7) - (2.9) we obtain for $\nu \in \mathbb{N}$, $\nu \geq 2$,

$$\begin{aligned}
& (A + BK_\nu)^T M_\nu + M_\nu(A + BK_\nu) + K_\nu^T R K_\nu + Q \\
& \stackrel{(2.7)}{=} \psi_\nu(K_\nu) - \gamma[(A + BK_\nu)^T V_\nu + V_\nu(A + BK_\nu)] + Q \\
& \stackrel{(2.8)}{=} [\psi_\nu(K_{\nu-1}) - (K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu)] - \gamma[(A + BK_\nu)^T V_\nu + V_\nu(A + BK_\nu)] + Q \\
& \stackrel{(2.9)}{=} \gamma[(A + BK_{\nu-1})^T V_\nu + V_\nu(A + BK_{\nu-1})] - Q - (K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu) \\
& \quad - \gamma[(A + BK_\nu)^T V_\nu + V_\nu(A + BK_\nu)] + Q \\
& = \gamma(K_{\nu-1} - K_\nu)^T B^T V_\nu + \gamma V_\nu B(K_{\nu-1} - K_\nu) - (K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu) \\
& = -\Gamma_\nu \text{ (see Algorithm A, (2), (iii))},
\end{aligned}$$

which can be written as

$$(A + BK_\nu)^T M_\nu + M_\nu(A + BK_\nu) + K_\nu^T R K_\nu + Q + \Gamma_\nu = 0. \quad (2.11)$$

Since (\sqrt{Q}, A) is detectable and $\Gamma_\nu \geq 0$ it follows from Lemma 4.3 that $(\sqrt{K_\nu^T R K_\nu + Q + \Gamma_\nu}, A + BK_\nu)$ is also detectable. Consequently we get, using Lemma 4.2 and $M_\nu \geq 0$, that $(A + BK_\nu)$ is stable. Subtracting the identity

$$(A + BK_\nu)^T M_{\nu+1} + M_{\nu+1}(A + BK_\nu) + Q + K_\nu^T R K_\nu = 0$$

(which we get by replacing in (2.4) ν by $\nu + 1$) from (2.11) we obtain

$$(A + BK_\nu)^T (M_\nu - M_{\nu+1}) + (M_\nu - M_{\nu+1})(A + BK_\nu) + \Gamma_\nu = 0. \quad (2.12)$$

Since we assumed $\Gamma_\nu \geq 0$ and since $A + BK_\nu$ is stable, Lemma 4.1 implies $M_\nu \geq M_{\nu+1} \geq 0$ for $\nu \in \mathbb{N}$, $\nu \geq 2$.

The preceding steps of the proof show that Algorithm A does not stop as long as $\Gamma_\nu \geq 0$, moreover in this case $M_\infty = \lim_{\nu \rightarrow \infty} M_\nu \geq 0$ exists (see [7]).

b) Let $\nu \geq 2$. In order to prove the remaining assertions of Theorem 2.1 we infer from (2.7) that (with $Z_\nu = M_\nu + \gamma V_\nu$) for $\nu \geq 3$

$$\begin{aligned}
& (A + BK_\nu)^T (Z_\nu - Z_{\nu+1}) + (Z_\nu - Z_{\nu+1})(A + BK_\nu) \\
& = \psi_\nu(K_\nu) - \psi_{\nu+1}(K_\nu) = \psi_\nu(K_{\nu-1}) + (\psi_\nu(K_\nu) - \psi_\nu(K_{\nu-1})) - \psi_{\nu+1}(K_\nu) \\
& \stackrel{(2.8)}{=} \psi_\nu(K_{\nu-1}) - (K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu) - \psi_{\nu+1}(K_\nu) \\
& \stackrel{(2.10)}{=} -4\gamma[M_{\nu-1} W M_{\nu-1} - M_\nu W M_\nu] - (K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu) \leq 0. \quad (2.13)
\end{aligned}$$

Here the last inequality follows from $M_{\nu-1} = M_\nu + \Delta_\nu$ with $\Delta_\nu \geq 0$ and assumption (2.3). Since $A + BK_\nu$ is stable it follows, using Lemma 4.1, that $Z_\nu \geq Z_{\nu+1}$ for $\nu \geq 3$. Consequently $Z_\infty = \lim_{\nu \rightarrow \infty} Z_\nu \geq 0$ exists and $V_\infty = \frac{1}{\gamma}(Z_\infty - M_\infty) = \lim_{\nu \rightarrow \infty} V_\nu \geq 0$ exists too; notice that $V_\nu \geq 0$ for all ν by definition.

Moreover, M_∞, V_∞ solve the coupled system (2.1), (2.2).

From (2.13) it follows, using (2.12), that

$$\begin{aligned} & (A + BK_\nu)^T(V_\nu - V_{\nu+1}) + (V_\nu - V_{\nu+1})(A + BK_\nu) \\ &= -4\gamma[M_{\nu-1}WM_{\nu-1} - M_\nu WM_\nu] - \frac{1}{\gamma}(K_{\nu-1} - K_\nu)^T R(K_{\nu-1} - K_\nu) + \frac{1}{\gamma}\Gamma_\nu =: \Theta_\nu. \end{aligned} \quad (2.14)$$

Applying again Lemma 4.1 we get on account of the stability of $A + BK_\nu$ that $V_{\nu+1} \geq V_\nu$ for $\nu \geq 3$ if *in addition* $\Theta_\nu \leq 0$ for $\nu \geq 3$. This additional assumption is only rarely fulfilled.

Adding up (2.4) with the γ -fold of (2.5) and passing to the limit $\nu \rightarrow \infty$, we obtain

$$(A - BR^{-1}BZ_\infty)^T Z_\infty + Z_\infty(A - BR^{-1}BZ_\infty)^T + G = 0 \quad (2.15)$$

where $G = Q + Z_\infty^T S Z_\infty + 4\gamma M_\infty W M_\infty \geq Q$.

Since (\sqrt{Q}, A) is detectable it follows from (2.15) with Lemma 4.3 that $(\sqrt{G}, A - SZ_\infty)$ is also detectable. Therefore we infer from (2.15), $Z_\infty \geq 0$ and Lemma 4.2 that $A - SZ_\infty = A - S(M_\infty + \gamma V_\infty)$ is stable. This proves Theorem 2.1. \square

Remark 2.2 In Theorem 2.1 we used in particular the implicit condition that K_0 and M_0 can be chosen such that for $\nu \in \mathbb{N}$, $\nu \geq 2$

$$\Gamma_\nu = \gamma(Z_{\nu-1} - Z_\nu)S V_\nu + \gamma V_\nu S(Z_{\nu-1} - Z_\nu) + (Z_{\nu-1} - Z_\nu)S(Z_{\nu-1} - Z_\nu) \geq 0 \quad (2.16)$$

. For the application of Theorem 2.1 and for the initialization of Algorithm A it turns out that usually it is sufficient to choose K_0 and M_0 such that $A + BK_0$ and $A + BK_1$ are stable and $\Gamma_2 \geq 0$.

From the proof of Theorem 2.1 it follows that $\Gamma_\nu \geq 0$, $M_\nu \geq 0$, $V_\nu \geq 0$ imply that

- $A + BK_\nu$ is stable (see (2.11) ff.),
- $M_{\nu+1} \geq 0$, $V_{\nu+1} \geq 0$ (see (2.4), (2.5)),
- $0 \leq M_{\nu+1} \leq M_\nu$, $0 \leq Z_{\nu+1} \leq Z_\nu$ (see (2.12), (2.13)),
- $\Gamma_{\nu+1}$ is a sum of products of nonnegative factors;
this yields in particular in the scalar case that $\Gamma_{\nu+1} \geq 0$ and that Algorithm A does not stop. In the matrix case there is unfortunately no simple sufficient condition ensuring that (2.16) holds for all $\nu \geq 2$.

Corollary 2.3 *In the scalar case with $B \neq 0$ (otherwise (1.3), (1.4) is a partially decoupled system) and $Q > 0$ (otherwise (1.3), (1.4) admits the trivial solution) the coupled system (1.3), (1.4) always has a pair $M_\infty \geq 0$, $V_\infty \geq 0$ of solutions with $A - S(M_\infty + \gamma V_\infty) < 0$ and K_0 and M_0 can be chosen such that Algorithm A creates convergent sequences with*

$$M_\infty = \lim_{\nu \rightarrow \infty} M_\nu, \quad V_\infty = \lim_{\nu \rightarrow \infty} V_\nu.$$

Proof. On account of $B \neq 0$ and $S > 0$ it is easy to check (see Algorithm A, (2)) that we can choose K_0 and M_0 such that $A + BK_0 < 0$, $A + BK_1 < 0$ and $\Gamma_2 > 0$. Hence, according to Remark 2.2, Algorithm A does not stop. \square

Remark 2.4 (i) In all our examples it turned out that also in the matrix-case $\Gamma_2 \geq 0$ implied $\Gamma_\nu \geq 0$ for $\nu \in \mathbb{N}$, $\nu \geq 2$. On the other hand it is not clear if and how this could be proved in the general case since $A_1, A_2, A_3 \geq 0$ does in general not imply $A_1 A_2 A_3 + A_3 A_2 A_1 \geq 0$. Therefore it is in general not known how the term $\gamma(Z_{\nu-1} - Z_\nu)S V_\nu + \gamma V_\nu S(Z_{\nu-1} - Z_\nu)$, appearing in (2.16), can be estimated.

(ii) Of course the sequences M_ν, V_ν , defined in Algorithm A may also be uniquely defined if some of the matrices Γ_ν are not positive semidefinite - namely as long as $\lambda \in \sigma(A + BK_\nu)$ implies $-\lambda \notin \sigma(A + BK_\nu)$; in this case the sequences (M_ν) and (Z_ν) may be nonmonotonic and/or unbounded, the matrices M_ν and V_ν may become indefinite and in general we cannot prove convergence. If both sequences are convergent then the limits define (like above) a pair of solutions M_∞, V_∞ of (1.3), (1.4) but $A + B(M_\infty + \gamma V_\infty)$ can be unstable.

(iii) Up to now there exist no further results on the structure of the set of all solutions of (1.3), (1.4). In the scalar case it is of course possible to plot in a (M, V) -plane the curves represented by (1.3), (1.4) and to determine either graphically, numerically or, maybe, explicitly the intersections of these curves, i.e. the solutions of (1.3), (1.4).

(iv) It is also possible to replace Algorithm A by **Algorithm B**, which is obtained from Algorithm A by replacing in Algorithm A, (2.5), $M_{\nu-1}$ by M_ν and $K_{\nu-1}$ by $-R^{-1}B^T(M_\nu + \gamma V_{\nu-1})$.

For Algorithm B the proof of the monotonicity of the (modified) sequences (M_ν) and (Z_ν) is performed as for Algorithm A.

Example 2.5 We consider the coupled system (1.3), (1.4) using three different values for γ and the following data for the matrix coefficients:

$$A := \begin{pmatrix} 1 & \frac{1}{8} \\ 0 & 4 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$Q := \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad \tilde{W} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

$$W = E \tilde{W} E^T := \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}.$$

Subsequently we reproduce the results obtained from Algorithm A and its modification for indefinite matrices $A + BK_\nu$ (see Remark 2.4, (ii)) if we initialize the algorithm by choosing

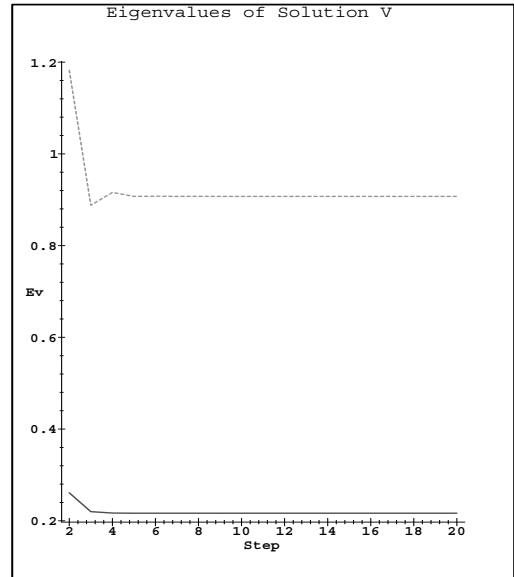
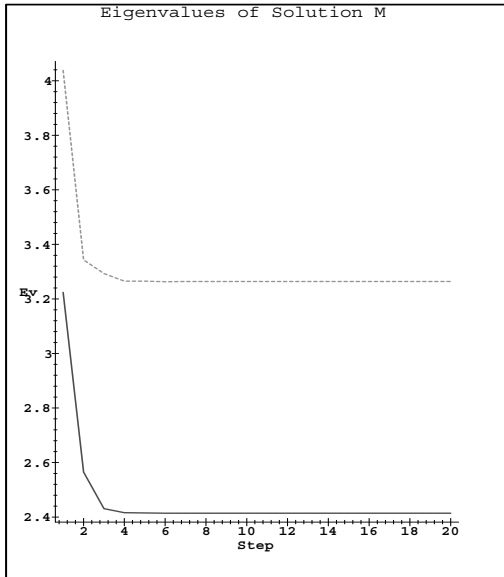
$$K_0 := \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \text{ and } M_0 := 0.$$

(i) Let $\gamma := \frac{3}{8}$, then Algorithm A yields

$$M_1 := \begin{pmatrix} 4. & .1666666667 \\ .1666666667 & 3.260416667 \end{pmatrix}, V_1 := 0.$$

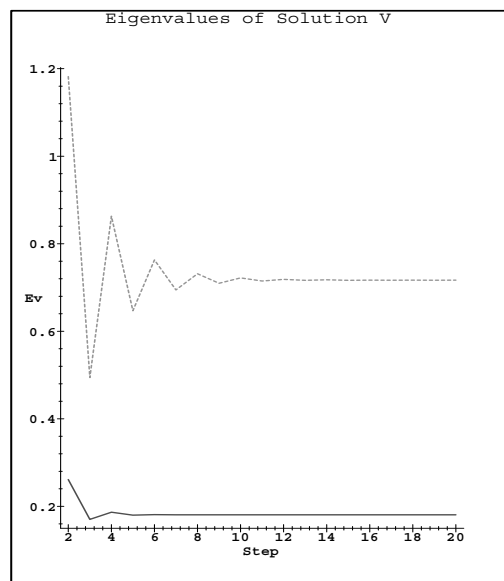
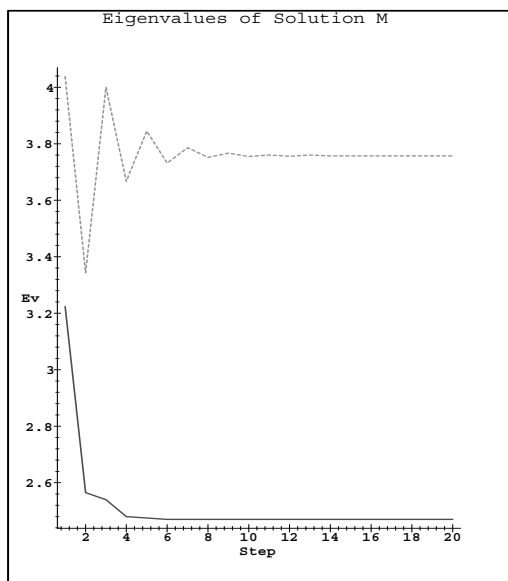
Moreover the closed loop matrices $A + BK_1$ and $A + BK_2$ are stable and the matrices Γ_ν , $\nu \geq 2$, are positive semidefinite; in particular we have $\Gamma_2 = \begin{pmatrix} .2761134730 & .2813233650 \\ .2813233650 & 1.854507011 \end{pmatrix}$. After 20 steps Algorithm A has reached (almost precisely) the pair of stabilizing solutions

$$M_{20} := \begin{pmatrix} 3.259435661 & .05190022845 \\ .05190022845 & 2.417461101 \end{pmatrix}, V_{20} := \begin{pmatrix} .9069491803 & .01643760904 \\ .01643760904 & .2169053965 \end{pmatrix}.$$

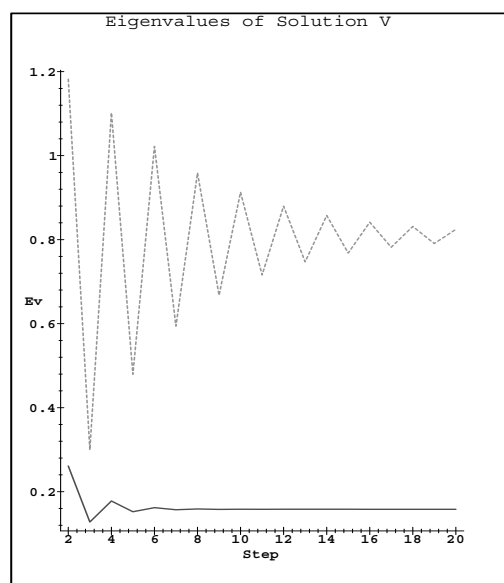
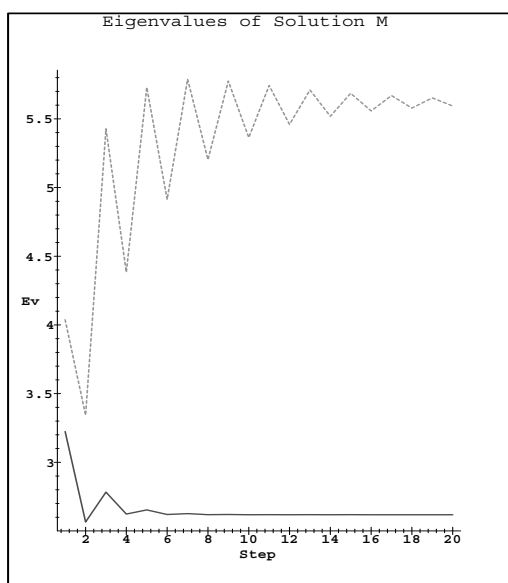


In this case the sequences (M_ν) and $(M_\nu + \gamma V_\nu)$ are monotonically decreasing.

(ii) For $\gamma := \frac{9}{4}$ we get $\Gamma_2 = \begin{pmatrix} -6.607319365 & -.00475011314 \\ -.00475011314 & .5014849641 \end{pmatrix}$ which is not positive semidefinite. If we perform analogous steps as described in Algorithm A, this yields two sequences which are both convergent but not monotonic.



(iii) Let $\gamma := 5$.



In this case Γ_2 also is not positive semidefinite, moreover (the modified) Algorithm A produces two unbounded sequences.

3. The coupled differential equations.

In this section we use the fact that the solutions of Riccati-type differential equations depend monotonically on the coefficients and the initial values in order to get information

on the global existence of the solution of the terminal value problem (1.1), (1.2), i. e. of

$$\begin{aligned}\dot{M} &= -A^T M - MA - Q + MSM - \gamma^2 VSV, \quad M(t_f) = Q_f, \\ \dot{V} &= -A^T V - VA + 2\gamma VSV + MSV + VSM - 4MWM, \quad V(t_f) = 0.\end{aligned}\quad (3.1)$$

For a pair of solutions M and V of (3.1) it follows that $M + \gamma V$ solves

$$\frac{d}{dt}(M + \gamma V) = -A^T(M + \gamma V) - (M + \gamma V)A - Q + (M + \gamma V)S(M + \gamma V) - 4\gamma MWM \quad (3.2)$$

with $(M + \gamma V)(t_f) = Q_f \geq 0$.

From Theorem 4.4 and the theory of standard Riccati equations (see [6]) it follows that

$$\begin{aligned}0 &\leq P(t) \leq M(t) \quad \text{for } t \leq t_f \\ 0 &\leq P(t) \leq M(t) + \gamma V(t) \quad \text{for } t \leq t_f\end{aligned}\quad (3.3)$$

while $M(t)$ and $V(t)$ do not blow up.

Here P is the (unique, positive semidefinite) solution of the standard Riccati equation

$$\dot{P} = -A^T P - PA - Q + PSP, \quad P(t_f) = Q_f,$$

which, in consequence of (1.5), exists for $t \leq t_f$.

This means that in order to get a non-blow-up condition for M and V it is sufficient to ensure for example the existence of upper bounds for $M(t)$ and $(M + \gamma V)(t)$ for $t \leq t_f$.

In order to construct upper bounds for $M + \gamma V$, we compare $M + \gamma V$ with L_{Q_1} and $X (= X_{Q_1, S_1})$, where L_{Q_1} and X denote the (unique) solutions of the terminal value problems

$$\dot{L}_{Q_1} = -A^T L_{Q_1} - L_{Q_1} A - (Q + Q_1), \quad L_{Q_1}(t_f) = Q_f \quad (3.4)$$

and

$$\dot{X} = -A^T X - XA - (Q + Q_1) - XS_1 X, \quad X(t_f) = Q_f. \quad (3.5)$$

Notice that $L_{Q_1}(t)$ exists for $t \leq t_f$; moreover for the existence of $X(t)$ for $t \leq t_f$ there exist sufficient conditions (see ([6])).

Theorem 3.1 *Let Q_1 and S_1 be arbitrary n -dimensional positive semidefinite matrices (which appear here as parameters); then for $t < t_f$ the following holds:*

(i) *While*

$$(M + \gamma V)(t)S(M + \gamma V)(t) + Q_1 - 4\gamma M(t)WM(t) \geq 0 \quad (3.6)$$

we have for $t \leq t_f$

$$0 \leq P(t) \leq M(t) + \gamma V(t) \leq L_{Q_1}(t). \quad (3.7)$$

(ii) *Assume that the solution X of the Riccati differential equation (3.5) exists for $t_0 \leq t \leq t_f$. While $t_0 \leq t \leq t_f$ and*

$$(M + \gamma V)(t)(S + S_1)(M + \gamma V)(t) + Q_1 - 4\gamma M(t)WM(t) \geq 0 \quad (3.8)$$

we have

$$0 \leq M(t) + \gamma V(t) \leq X(t). \quad (3.9)$$

(iii) While

$$Q_1 + (MSM - \gamma^2 VSV)(t) \geq 0 \quad (3.10)$$

we have for $t \leq t_f$

$$0 \leq P(t) \leq M(t) \leq L_{Q_1}(t). \quad (3.11)$$

Proof. The lower estimates in (3.7), (3.9) and (3.10) have already been proved above (see (3.3)). The upper bounds in (3.7) and (3.9) are obtained, if we apply Theorem 4.4 in order to compare the solutions of (3.2) with the solutions of (3.4) and (3.5) respectively; (3.11) is proved similarly, using

$$\dot{M} = -A^T M - MA - (Q + Q_1) + [Q_1 + MSM - \gamma^2 VSV], \quad M(t_f) = Q_f.$$

□

Remark 3.2 a) The assumptions (3.6), (3.8) and (3.10) used in Theorem 3.1 are implicit assumptions since they are containing the unknown functions M and V . On the other hand it can be easily seen that in many situations these assumptions are automatically fulfilled.

(i) For $W = 0$ (3.6) and (3.8) are always fulfilled with $Q_1 = 0$ and $S_1 = 0$.

(ii) For $Q_f = 0$ we have $V(t) \equiv 0$ and M coincides with the solution of the uncoupled standard Riccati equation, i.e. $V(t)$ and $M(t)$ exist for $t \leq t_f$.

(iii) Consider now the scalar case $\mathbf{n} = \mathbf{1}$ and let $W, Q_f > 0$. Consequently

$M(t) \geq P(t) \geq Q_f > 0$ for $t \leq t_f$ and on account of $\dot{V}(t_f) = -4\gamma Q_f W Q_f < 0$ it is easy to see that $V(t) > 0$ for $t \leq t_f$; notice that $V(\tau) = 0$ for $\tau < t_f$ would imply $\dot{V}(\tau) \geq 4\gamma Q_f W Q_f < 0$. Therefore Theorem 3.1 (i) or (ii) ensures that $M(t)$ and $V(t)$ cannot blow-up for $t \leq t_f$ if for example either $S \geq 4\gamma W$ or $S + S_1 \geq 4\gamma W$ for $t_0 \leq t = t_f$, where S_1 is such that (3.5) has a solution on $[t_0 t_f]$.

b) Analogously to the proof of Theorem 3.1 we can derive further variants of the obtained comparison results. For example we infer from

$$\frac{d}{dt}(M - \frac{\gamma}{2}V) =$$

$$-A^T(M - \frac{\gamma}{2}V) - (M - \frac{\gamma}{2}V)A - Q + (M - \frac{\gamma}{2}V)S(M - \frac{\gamma}{2}V) + 2\gamma MWM - \frac{\gamma^2}{4}VSV$$

that for $t \leq t_f$

$$0 \leq Q(t) \leq (M - \frac{\gamma}{2}V)(t)$$

as long as $(MWM - \frac{\gamma}{8}VSV)(t) \leq 0$.

c) Combining the estimates of Theorem 3.1 we get implicit non-blow-up conditions for M and V which are similar to the results obtained in [5] for coupled systems of Riccati equations appearing in noncooperative differential games.

4. Appendix

For convenience of the reader we summarize here without proof some well known facts from the literature which have been used in sections 2 and 3.

First we consider the Lyapunov equation

$$A^T X + X A = -Q; \quad (4.1)$$

here $A, Q \in \mathbb{R}^{n \times n}$ are given matrices with $Q = Q^T$.

Lemma 4.1 (see [8], p.101). *If A is stable (i.e. if all eigenvalues of A have negative real part) then (4.1) has the unique solution*

$$X = \int_0^\infty e^{\tau A} Q e^{\tau A^T} d\tau;$$

therefore $Q \geq 0$ implies $X \geq 0$.

Lemma 4.2 (see [15])

Let X be a solution of (4.1). If $X \geq 0$, $Q \geq 0$ and if (\sqrt{Q}, A) is detectable then A is stable.

Lemma 4.3 (see [15], proof of Theorem 3.6)

If $Q \geq 0$ and if (\sqrt{Q}, A) is observable (or detectable) then $(\sqrt{M + Q + F^T N F}, A + B F)$ is observable (or detectable) for all matrices $M \geq 0$, $N > 0$, B and F of adequate dimensions.

The following comparison theorem is used as an essential tool in section 3; the subsequent version is taken from [4], Theorem 2.1 - a preliminary version can be found in [2], pp.51/52.

Theorem 4.4 *For $i = \{1, 2\}$ let K_i be a solution of the differential equation*

$$\dot{K}_i = -A_i^T(t)K - K A_i(t) - Q_i(t) + K S_i(t)K$$

(with piecewise continuous coefficients) on some interval \mathcal{I} . If for some $t_f \in \mathcal{I}$ $K_1(t_f) \leq K_2(t_f)$ or $K_1(t_f) < K_2(t_f)$ and if

$$JH_2(t) = \begin{pmatrix} Q_2 & A_2^T \\ A_2 & -S_2 \end{pmatrix} (t) \geq JH_1(t) = \begin{pmatrix} Q_1 & A_1^T \\ A_1 & -S_1 \end{pmatrix} (t) \text{ for } t \in \mathcal{I}$$

then $K_1(t) \leq K_2(t)$ or $K_1(t) < K_2(t)$, respectively, for all $t \in \mathcal{I} \cap (-\infty, t_f]$.

5. Conclusion

Algorithm A presents a method for computing a stabilizing pair of solutions M_∞, V_∞ of the coupled system of algebraic matrix Riccati-type equations (1.3), (1.4). The idea for solving this system by Lyapunov iterations is standard but we give for the first time a version with a proof of convergence. In the second part of the paper we state sufficient conditions ensuring that the solutions of the terminal value problem (1.1), (1.2) cannot blow up on a given interval $[t_0, t_f]$.

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