Time-varying discrete Riccati equation: Some monotonicity results

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Abstract. Using a Fréchet derivative based approach some monotonicity and comparison results concerning the solutions of the time-varying discrete time Riccati equation are obtained. Connections with the existence of the semistabilizing solution are made explicit as well.

Keywords: Discrete time systems, Riccati equation, monotonicity and comparison results.

1 Introduction and problem statement

One of the most remarkable properties of the (matrix) Riccati differential equation (RDE)

$$-\dot{X} = A^T(t)X + XA(t) - XS(t)X + Q(t), \quad X(t_0) = X_0,$$

where $A(t),\ \ S(t) = S^T(t),\ \ Q(t) = Q^T(t)$ and $X_0 = X_0^T \in \mathbb{R}^{n\times n}$, is the following:
Its solutions are monotonic with respect to both initial value X_0 and input data matrix

$$E(t) = \begin{bmatrix} Q & A^T \\ A & -S \end{bmatrix}(t).$$

Stokes [?] proved the striking fact that RDE is, for $n > 1$, the only (matrix) differential equation $\dot{X} = \Phi(t, X),\ \ X(t_0) = X_0$ possessing the so-called *order-preserving* property, i.e. X depends monotonically on X_0. Notice also that under different assumptions the monotonicity of X with respect to E has been studied by many authors (see for instance [?], [?], [?], [?] and [?]).

Furthermore, various types of monotonicity results have been also obtained for both differential and difference Riccati equations as well as for the algebraic Riccati equation in both continuous and discrete versions (see [?], [?], [?], [?], [?], [?], [?], [?]).

The main purpose of the present paper is that of generalizing the aforementioned monotonicity and comparison results to the case of the time-varying discrete Riccati equation (TVDRE). A nice survey on TVDRE theory could be found in [?].
To this end a Fréchet derivative based approach combined with the bilateral shift algebra has been proposed. Notice that in connection with the continuous-time algebraic Riccati equation, the idea of using the Fréchet derivatives belongs to Delchamps who proved the analyticity of the stabilizing solution with respect to the coefficients [?]. Furthermore, similar techniques are used in a forthcoming paper in order to prove and generalize convexity results obtained in [?], [?], [?], [?] and [?].

In the sequel the following notation will be used. By $\mathbb{Z}$, $\mathbb{R}^n$ $(\mathbb{C}^n)$ and $\mathbb{R}^{n\times m}$ we denote the ring of integers, the real (complex) n–dimensional Euclidian space and the set of $n \times m$ matrices with real entries. If $M \in \mathbb{R}^{n\times m}$ then M^T stands for its transpose. By I_n we denote the $n \times n$ unit matrix. The spectrum and the spectral radius of a linear bounded operator T on a Hilbert $\mathcal{H}$ space will be denoted by $\Lambda(T)$ and $\rho(T)$, respectively, and T^* will stands for the adjoint of T. Further, if T is self-adjoint, i.e. $T = T^*$ then we shall term it *coercive* if there exists $\nu_0 > 0$ such that $\langle Tx, x\rangle \geq \nu_0 \|x\|^2\ \ \forall x \in \mathcal{H}$. Here $\langle , \rangle$ and $\|\ \ \|$ stand for the inner product and the *associated* norm on $\mathcal{H}$, respectively. The coerciveness of T will be denoted by $T >> 0$. Let $\ell^{2,m}$ be the Hilbert space of norm-square doubly infinite $\mathbb{C}^m$-valued sequences $u = (u_k)_{k\in\mathbb{Z}}$, i.e. $u_k \in \mathbb{C}^m$ and $||u||_2 := \left(\sum_{k=-\infty}^{\infty} \|u_k\|^2\right)^{\frac{1}{2}} < \infty$ is the ℓ^2-norm, here $\|\ \ \|$ stands for the usual Euclidian norm. If $B = (B_k)_{k\in\mathbb{Z}}$ with $B_k \in \mathbb{R}^{n\times m}$, is any *bounded matrix sequence*, that is, $\|B_k\| \leq c_0\ \forall k \in \mathbb{Z}$ for some $c_0 \geq 0$ then B will be interpreted as a linear bounded *multiplication or block diagonal operator* from $\ell^{2,m}$ into $\ell^{2,n}$. This means that if $u \in \ell^{2,m}$ then we shall adopt for it the doubly infinite column representation $u = col(u_k)_{k=-\infty}^{\infty}$ and the action of B on u, i.e. $y := Bu$ will be explicitly described by $y = col(y_i)_{i=-\infty}^{\infty} = mat(\delta_{ij}B_i)_{i,j=-\infty}^{\infty}\ col(u_j)_{j=-\infty}^{\infty} = diag(B_i)_{i=-\infty}^{\infty} col(u_i)_{i=-\infty}^{\infty}$. Here δ_{ij} stands for the Kronecker symbol. If $B = mat(\delta_{ij}B_i)_{i,j=-\infty}^{\infty} : \ell^{2,m} \to \ell^{2,n}$ is a block diagonal operator then we shall introduce the following four linear bounded operators from $\ell^{2,m}$ into $\ell^{2,n}$: $\sigma B = mat(\delta_{ij}B_{i+1})_{i,j=-\infty}^{\infty}$, $\sigma^{-1}B = mat(\delta_{ij}B_{i-1})_{i,j=-\infty}^{\infty}$, $B\sigma = mat(\delta_{i,j-1}B_i)_{i,j=-\infty}^{\infty}$

and $B\sigma^{-1} = mat(\delta_{i,j+1}B_i)_{i,j=-\infty}^{\infty}$. Let $I_{\ell^{2n}} = diag(A_i)_{i=-\infty}^{\infty}$, $A_i = I_n$, be the identity operator on $\ell^{2,n}$. Then $I_{\ell^{2,n}}\sigma$ and $I_{\ell^{2,n}}\sigma^{-1}$ describe the action of the *bilateral shift* σ and its inverse σ^{-1} on $\ell^{2,n}$, respectively. Thus if $x \in \ell^{2,n}$ then $(I_{\ell^{2,n}}\sigma)x = \sigma x$ and $(I_{\ell^{2,n}}\sigma^{-1})x = \sigma^{-1}x$ where $(\sigma^{\pm 1}x)_k := x_{k\pm 1}$. Let $A = \text{diag}(A_i)_{i=-\infty}^{\infty}$ be a block diagonal operator on $\ell^{2,n}$. We shall say that A defines an exponentially stable evolution (anticausal exponentially stable evolution) if $\rho(A\sigma^{-1}) < 1 \quad (\rho(A\sigma) < 1)$. If the above inequality is relaxed to $\rho(A\sigma^{-1}) \leq 1 \quad (\rho(A\sigma) \leq 1)$ then we shall say that A defines a semistable (anticausal semistable) evolution. If A defines an exponentially stable evolution (anticausal exponentially stable evolution) then clearly the operator $I_{\ell^{2,n}}\sigma - A \quad (I_{\ell^{2,n}}\sigma^{-1} - A^*)$ is boundedly invertible on $\ell^{2,n}$. For more details concerning shift operator algebra, exponential stability (both causal and anticausal) in terms of the spectral radius see [?], [?].

For any $k \in \mathbb{Z}$ let $\ell_{k,+}^{2,n}(\ell_{k,-}^{2,n})$ be the (closed) subspace of $\ell^{2,n}$ consisting of those sequences with the support in $[k,\infty)$ $((-\infty, k-1])$. Clearly $\ell^{2,n} = \ell_{k,-}^{2,n} \oplus \ell_{k,+}^{2,n}$ where $\oplus$ denotes the direct sum. Denote by $P_{k,+}^n$ $(P_{k,-}^n)$ the orthogonal projection of $\ell^{2,n}$ onto $\ell_{k,+}^{2,n}$ $(\ell_{k,-}^{2,n})$. If T is a linear bounded operator from $\ell^{2,n}$ into $\ell^{2,m}$ then for any $k \in \mathbb{Z}$, $T_k := P_{k,+}^m T P_{k,+}^n$ is called the Toeplitz operator associated with T at k. T is called causal (anticausal) if $T_k = TP_{k,+}^n$ $(T_k = P_{k,+}^m T)$. Clearly if $A = diag(A_i)_{i=-\infty}^{\infty} : \ell^{2,n} \to \ell^{2,n}$ defines an exponentially stable evolution (anticausal exponentially stable evolution) then A is causal (anticausal).
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ be the Banach space of the linear bounded operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. Let $T : M \mapsto T(M)$ be any function from $\mathcal{B}_1$ to $\mathcal{B}_2$ where M ranges the domain $\mathcal{D} \subset \mathcal{B}_1$, and let $P \in \mathcal{D}$. We shall say that T is Fréchet differentiable (with respect to M) in P if there exists $T_M(P) \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ such that $\lim_{\varepsilon \to 0} [T(P + \varepsilon N) - T(P)]/\varepsilon \to T_M(P)(N)$ uniformly with respect to all $N \in \mathcal{B}_1$. $T_M(P)(\cdot)$ is called the Fréchet derivative of T (with respect to M) in P. If T is Fréchet differentiable in each point of the domain $\mathcal{D} \subset \mathcal{B}$ then T is called Fréchet differentiable on $\mathcal{D}$. If $T_M(\cdot) : \mathcal{D} \subset \mathcal{B}_1 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is Fréchet differentiable on $\mathcal{D}$ then T is called twicely Fréchet differentiable on $\mathcal{D}$ and $T_{MM}(P)$ denotes its second Fréchet derivative (in P). For more details see [?].

The present paper is organized as follows. Section 1 has an introductory character. Section 2 deals with the evaluation of Fréchet derivatives for the TVDRE and *local* monotonicity of any solution with respect to the input data. In Section 3 some *global* monotonicity and comparison results are given. Some *limiting* cases are also studied in the context of the perturbation of the nominal data. Section 4 deals with the *global* existence and monotonicity of the *semi*stabilizing solution with respect to the input data. Some conclusions are given in Section 5.

Let us be now more specific concerning the topics in order. First introduce

$$\mathbf{G}(E, X) \;:=\; A^*\sigma X(I_{\ell^{2,n}} + S\sigma X)^{-1}A - X + Q \tag{1.1}$$

where

$$E = \begin{bmatrix} Q & A^* \\ A & -S \end{bmatrix} = E^* \tag{1.2}$$

is termed as the *input data matrix sequence.* Here $A : \ell^{2,n} \to \ell^{2,n}$, $B : \ell^{2,m} \to \ell^{2,n}$, $Q : \ell^{2,n} \to \ell^{2,n}$, $S = BR^{-1}B^*$, and $R : \ell^{2,m} \to \ell^{2,m}$ are *block diagonal operators*, with R boundedly invertible.

The TVDRE is now introduced by $\mathbf{G}(E, X) = 0$, that is,

$$A^*\sigma X(I_{\ell^{2,n}} + S\sigma X)^{-1}A - X + Q = 0. \tag{1.3}$$

Recall that any global, bounded on $\mathbb{Z}$ and selfadjoint solution X to the TVDRE (**??**) is called *stabilizing* (*semistabilizing*) if $I_{\ell^{2,n}} + S\sigma X$ has a bounded inverse and

$$A_{cl} := (I_{\ell^{2,n}} + S\sigma X)^{-1}A \tag{1.4}$$

defines an exponentially stable (semistable) evolution. In the subsequent sections we shall investigate the local and global monotonicity of any selfadjoint solution X to the TVDRE (**??**), existing on any interval $[l, s]$ with $l \geq -\infty$, with respect to the input data matrix sequence E and terminal condition X_s. The results are derived under no assumptions on the signature of S. Further, semipositive definiteness of S will be assumed for establishing some comparison results and global monotonicity of the semistabilizing solution. In the last case some existence conditions for the semistabilizing solution in terms of the generalized Popov theory are implicitly given.

2 Fréchet derivatives and local monotonicity

We shall start with the following result.

Proposition 2.1. *For each pair (E, X) with $X = X^*$ the following statements hold*

1.
$$\mathbf{G}_E(E, X)(\Delta E) = N^*(E, X)\Delta E N(E, X) \tag{2.1}$$

where

$$N(E, X) = \begin{bmatrix} I \\ \sigma X A_{cl} \end{bmatrix} \tag{2.2}$$

*with A_{cl} defined in (**??**) and*

$$\Delta E = \begin{bmatrix} \Delta\widetilde{Q} & (\Delta\widetilde{A})^* \\ \Delta\widetilde{A} & -\Delta S \end{bmatrix} = (\Delta E)^*. \tag{2.3}$$

2.
$$\mathbf{G}_X(E, X)(\Delta X) = A_{cl}^*\sigma\Delta X A_{cl} - \Delta X \tag{2.4}$$

where $\Delta X = (\Delta X)^*$.

3.
$$\mathbf{G}_{XX}(E,X)(\Delta X,\Delta Z) \;=\; -A_{cl}^*[\sigma\Delta XP\sigma\Delta Z+\sigma\Delta ZP\sigma\Delta X]A_{cl} \tag{2.5}$$

with $\Delta X=(\Delta X)^*$, $\Delta Z=(\Delta Z)^*$ *and*

$$P:=S(I+\sigma XS)^{-1}=P^* \tag{2.6}$$

Proof. By elementary direct computations that use the definition of the Fréchet derivative. □

Theorem 2.2. *Let the input data matrix sequence E together with any interval $\mathbf{I}=[l,s]\subset\mathbb{Z}$ be given. Assume that the TVDRE (??) has a selfadjoint solution X on $\mathbf{I}$. Then X is locally monotonic with respect to both terminal condition X_s and input data sequence E.*

Proof. We have $\mathbf{G}(E,X)=0$ on $\mathbf{I}$. In addition the Stein equation

$$A_{cl}^*\sigma\Delta XA_{cl}-\Delta X+W=0 \tag{2.7}$$

has, for $(\Delta X)_s=(\Delta X^*)_s$ specified, and arbitrary free term $W=W^*$, a unique solution $\Delta X=\Delta X^*$ on $\mathbf{I}$. Hence, according to (??), this means that the Fréchet derivative $\mathbf{G}_X(E,X)$ is invertible. In addition both $\mathbf{G}_X$ and $\mathbf{G}_{XX}$ (see (??), (??)) are continuous in (E,X).

Hence, according to the implicit function theorem, we can write

$$\mathbf{G}_E(E,X)(\Delta E)+\mathbf{G}_X(E,X)(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))=0, \tag{2.8}$$

with the terminal condition

$$(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s=\Delta Z.$$

Using (??) and (??), (??) receives the explicit form

$$\begin{aligned}&A_{cl}^*\sigma X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)A_{cl}-X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)\\&+N^*(E,X)\Delta EN(E,X)=0\end{aligned} \tag{2.9}$$

with

$$(X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E))_s=\Delta Z.$$

Since (??) has, for $\Delta Z\geq 0$ and $\Delta E\geq 0$, a unique positive semidefinite solution $X_{(X_s,E)}(X_s,E)(\Delta Z,\Delta E)\geq 0$ on $\mathbf{I}$, the conclusion follows. □

Corollary 2.3. *Let E be given and assume that the TVDRE (??) has a stabilizing solution X. Then X is locally monotonic with respect to E.*

Proof. The proof runs similarly like in Theorem 2.2 by ignoring the terminal condition ΔZ of equations (??), (??) and by taking into account the additional key remark that, since A_{cl} defines an exponentially stable evolution, (??) has a unique global and bounded on $\mathbb{Z}$ solution (see for instance [12]). $\square$

Remark 2.4. (i) Theorem 2.2 and its proof suggest a new Fréchet derivative based proof for well known monotonicity results (see for instance [?] and [?]) concerning the maximal (stabilizing) solution of algebraic Riccati equations both in continuous and discrete time. In the present paper we confine to TVDREs with real coefficients, however it is obvious that the results obtained also hold in the case of complex coefficients.

(ii) In both statements of Theorem 2.2 and Corollary 2.3, the existence of the solution to the TVDRE was preassumed. The following lemma which is a slight generalization of the result given in [?] gives some sufficient conditions for the above mentioned existence.

Lemma 2.5. *Assume $Q \geq 0$ and $R = I_{\ell^2,m}$. Then for any interval $\mathbf{I} = [l, s] \subset \mathbb{Z}$ and any terminal condition $X_s \geq 0$ the TVDRE (??) has a solution $X \geq 0$ on $\mathbf{I}$.*

Proof. Rewrite (??) as

$$X = A_{cl}^* \sigma X A_{cl} + Q, \quad X_s = (\sigma X)_{s-1} \geq 0 \tag{2.10}$$

and, since $I + B_{s-1}^* X_s B_{s-1} > 0$. i. e. $I + S_{s-1} X_s$ is nonsingular, the conclusion follows trivially by induction. $\square$

3 Global monotonicity and some comparison results

The main result of this section is stated in the subsequent theorem.

Theorem 3.1. *Let $\mathbf{I} = [l, s] \subset \mathbb{Z}$ be given. Let $\Gamma_1 = \Gamma_1^T$, $\Gamma_2 = \Gamma_2^T \in \mathbb{R}^{n \times n}$ with $\Gamma_1 \leq \Gamma_2$, and E_1, E_2 with $E_1 \leq E_2$ be also given. If for all Γ and E given by $\Gamma = (1-\lambda)\Gamma_1 + \lambda\Gamma_2$ and $E = (1-\lambda)E_1 + \lambda E_2$ with $0 \leq \lambda \leq 1$, the TVDRE (??) has, for the terminal condition $X_s = \Gamma$, a solution $X(\Gamma, E)$ on $\mathbf{I}$ then*

$$X(\Gamma_1, E_1) \leq X(\Gamma_2, E_2). \tag{3.1}$$

Proof. Using the mean value theorem one gets

$$X(\Gamma_2, E_2) - X(\Gamma_1, E_1) = X_{(\Gamma,E)}(\tilde{\Gamma}, \tilde{E})((\Gamma_2 - \Gamma_1, E_2 - E_1)) \tag{3.2}$$

where $\tilde{\Gamma}$ and $\tilde{E}$ are each on the segments $[\Gamma_1, \Gamma_2]$ and $[E_1, E_2]$, respectively. Since $\Delta\Gamma := \Gamma_2 - \Gamma_1 \geq 0$, $\Delta E := E_2 - E_1 \geq 0$, one gets $X_{(\Gamma,E)}(\tilde{\Gamma}, \tilde{E})(\Delta\Gamma, \Delta E) \geq 0$ as has been shown in the proof of Theorem 2.2. Hence (??) follows. $\square$

Assume now for the rest of the paper that $R = I_{\ell^{2,m}}$, that is, $S = BB^*$.
The two next theorems generalize to the time-varying case, those results obtained in [?].

Theorem 3.2. *Let the data matrix sequence E be given. Assume that $Q \geq 0$ and $E = (E_k)_{k\in\mathbb{Z}}$ is monotonically decreasing, i.e. $E_i \leq E_j$ $\forall i < j$. If for some $s \in \mathbb{Z}$ there exists $\Gamma \geq 0$ such that for the terminal condition $X_s = \Gamma$ we have $X_{s-1} \leq X_s$ then the TVDRE (??) has a monotonically decreasing positive semidefinite solution $X = (X_k)_{k\leq s}$, i.e. $0 \leq \ldots \leq X_k \leq \ldots \leq X_s$. Here $X_{s-1} := A_{s-1}^T X_s (I + S_{s-1}X_s)^{-1} A_{s-1} + Q_{s-1}$. If in addition $\lim_{k\to-\infty} E_k = \overline{E} \in \mathbb{R}^{2n\times 2n}$ then the TVDRE converges, for $k \to -\infty$, to the discrete algebraic Riccati equation associated with $\overline{E}$, i.e. to $\overline{A}^T\overline{X}(I+\overline{S}\,\overline{X})^{-1}\overline{A}-\overline{X}+\overline{Q} = 0$, fulfilled by $\overline{X} = \lim_{k\to-\infty} X_k$.*

Proof. According to Lemma 2.5 the solution $X = (X_k)_{l\leq k\leq s}$ exists for *arbitrary* $l < s$ and it is positive semidefinite, i.e. $X_k \geq 0$ for $l \leq k \leq s$. Let us show by induction that $(X_k)_{k\leq s}$ is monotonically decreasing.

As $X_{s-1} \leq X_s$ we have to show that $X_{k-1} \leq X_k \Rightarrow X_{k-2} \leq X_{k-1}$. Assume temporarily that $X_{k-1} > 0$. Then we can write

$$-X_k^{-1} \geq -X_{k-1}^{-1}. \tag{3.3}$$

By combining the monotonicity of E with (??) one gets

$$\begin{bmatrix} Q_{k-1} & A_{k-1}^T \\ A_{k-1} & -S_{k-1} - X_k^{-1} \end{bmatrix} \geq \begin{bmatrix} Q_{k-2} & A_{k-2}^T \\ A_{k-2} & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix}. \tag{3.4}$$

As $S_k \geq 0$ $\forall k$, (??) yields

$$\begin{bmatrix} I & W_{k-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{k-1} + A_{k-1}^T(S_{k-1} + X_k^{-1})^{-1}A_{k-1} & 0 \\ 0 & -S_{k-1} - X_k^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ W_{k-1}^T & I \end{bmatrix} \geq$$

$$\begin{bmatrix} I & W_{k-2} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{k-2} + A_{k-2}^T(S_{k-2} + X_{k-1}^{-1})^{-1}A_{k-2} & 0 \\ 0 & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ W_{k-2}^T & I \end{bmatrix} \tag{3.5}$$

where $W_i := -A_i^T(S_i + X_{i+1}^{-1})^{-1}$ for $i = k-2, k-1$. Let

$$\begin{bmatrix} I & \widehat{W}_{k-1} \\ 0 & I \end{bmatrix} := \begin{bmatrix} I & -W_{k-2} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & W_{k-1} \\ 0 & I \end{bmatrix}. \tag{3.6}$$

Since $X_{i-1} = A_{i-1}^T(X_i^{-1} + S_{i-1})^{-1}A_{i-1} + Q_i$ for $i = k-1, k$, as directly follows from (??),

(??) yields with (??)

$$\begin{bmatrix} I & \widehat{W}_{k-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{k-1} & 0 \\ 0 & -S_{k-1} - X_k^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \widehat{W}_{k-1}^T & I \end{bmatrix}$$

$$= \begin{bmatrix} X_{k-1} - \widehat{W}_{k-1}(S_{k-1} + X_k^{-1})\widehat{W}_{k-1}^T & -\widehat{W}_{k-1}(S_{k-1} + X_k) \\ -(S_{k-1} + X_k^{-1})\widehat{W}_{k-1}^T & -S_{k-1} - X_k^{-1} \end{bmatrix} \tag{3.7}$$

$$\geq \begin{bmatrix} X_{k-2} & 0 \\ 0 & -S_{k-2} - X_{k-1}^{-1} \end{bmatrix}.$$

Therefore $X_{k-1} - \widehat{W}_{k-1}(S_{k-1} + X_k^{-1})\widehat{W}_{k-1}^T \geq X_{k-2}$. Since $S_{k-1} + X_k^{-1} > 0$, we get eventually $X_{k-1} \geq X_{k-2}$. If now the strict positiveness of X_{k-1} is not fulfilled, i.e. $X_{k-1} \geq 0$, let $X_{k-1}^{\varepsilon} := X_{k-1} + \varepsilon I$ and $X_k^{\varepsilon} := X_k + \varepsilon I$ for $\varepsilon > 0$. Let $\tilde{X}_{k-1}^{\varepsilon}$ and $\tilde{X}_{k-2}^{\varepsilon}$ be obtained recurrently from X_k^{ε} and X_{k-1}^{ε} via (??), respectively. Since clearly (??) updated with X_k^{ε} and X_{k-1}^{ε} holds as well, we shall get as above $\tilde{X}_{k-1}^{\varepsilon} \geq \tilde{X}_{k-2}^{\varepsilon}$. Further, since both limits of $\tilde{X}_{k-1}^{\varepsilon}$ and $\tilde{X}_{k-2}^{\varepsilon}$ exist for $\varepsilon \downarrow 0$ one gets finally $X_{k-1} \geq X_{k-2}$. As $X_k \geq 0$, $\overline{X} = \lim_{k\to-\infty} X_k \geq 0$ is well defined. Taking into account that $\lim_{i\to-\infty} E_i = \overline{E}$, the last part of the theorem follows trivially. □

A direct consequence of Theorem 3.2, which is essentially based on Theorem 3.1, is the next result.

Theorem 3.3. *Let*

$$E_i = \begin{bmatrix} Q_i & \tilde{A}_i^* \\ A_i & -S_i \end{bmatrix}, \qquad i = 1, 2, \tag{3.8}$$

be two input data matrix sequences and let TVDRE1 and TVDRE2 be the Riccati equations associated with E_1 *and* E_2*, respectively. Assume that both sequences* $E_1 = (E_{1,k})_{k\in\mathbb{Z}}$ *and* $E_2 = (E_{2,k})_{k\in\mathbb{Z}}$ *are monotonically decreasing,* $0 \leq Q_{1,k} \leq Q_{2,k}$, $\forall k \in \mathbb{Z}$, *and that in addition* $\lim_{k\to-\infty} E_{i,k} = \overline{E}_i \in \mathbb{R}^{2n\times 2n}$, $i = 1, 2$. *Assume that for some* $s \in \mathbb{Z}$ *there exists* $\Gamma \geq 0$ *such that if* $X_{2,s} = \Gamma$ *then* $X_{2,s-1} \leq X_{2,s} = \Gamma$ *where* $X_{2,s-1} := A_{2,s-1}^T X_{2,s}(I + S_{2,s-1}X_{2,s})^{-1}A_{s-1} + Q_{2,s-1}$. *If* Γ *is taken as a common terminal condition at* $s \in \mathbb{Z}$ *for both TVDRE 1 and TVDRE 2 then both TVDRE 1 and TVDRE 2 have the solutions* $X_1 = (X_{1,k})_{k\leq s}$, $X_2 = (X_{2,k})_{k\leq s}$, *respectively with the following properties*

$$0 \leq X_{1,k} \leq X_{2,k} \ \ \forall k \leq s, \tag{3.9}$$

$$X_{1,k}\downarrow \ \text{ and } \ X_{2,k}\downarrow \ \text{ for } \ k \downarrow -\infty \ \ (k \leq s). \tag{3.10}$$

Furthermore, TVDRE 1 and TVDRE 2 converge for $k \to -\infty$ *to the discrete algebraic Riccati equations* $\overline{A}_1^T\overline{X}_1(I + \overline{S}_1\overline{X}_1)^{-1}\overline{A}_1 - \overline{X}_1 + \overline{Q}_1 = 0$ *and* $\overline{A}_2^T\overline{X}_2(I + \overline{S}_2\overline{X}_2)^{-1}\overline{A}_2 - \overline{X}_2 + \overline{Q}_2 = 0$, *each of them fulfilled by* $\overline{X}_1 = \lim_{k\to-\infty} X_{1,k}$, $\overline{X}_2 = \lim_{k\to-\infty} X_{2,k}$, *respectively.*

Proof. Let $E = (1-\lambda)E_1 + \lambda E_2$. As $Q \geq 0$ it follows from Lemma 2.5, that the TVDRE associated with E has a positive semidefinite solution $X = (X_k)_{l \leq k \leq s}$ for arbitrary $l < s$. Hence, by invoking Theorem 3.1, (**??**) follows. In particular

$$X_{1,s-1} \leq X_{2,s-1} \leq \Gamma, \tag{3.11}$$

that is, all the conditions stated in Theorem 3.2 hold for both TVDREs. Furthermore (**??**) as well as the remainder of the theorem are true. □

Remark 3.4. If we put in the context of the above theorem $A_1 = A_2$, $B_1 = B_2$ and $\Delta Q = Q_2 - Q_1$ then the two TVDREs can be written as $X = A^*\sigma X(I + S\sigma X)^{-1}A + Q_1$ and $X = A^*\sigma X(I + S\sigma X)^{-1}A + Q_1 + \Delta Q$, respectively, that is, the second equation can be seen as a perturbed form of the first one. In these circumstances Theorem 3.3 makes explicit the asymptotic behaviour of the associated solutions. More specific results are obtained in the time invariant case in [?] and for a modified Riccati difference equation in [?].

4 Global monotonicity of the stabilizing solution

Let

$$E = \begin{bmatrix} Q & A^* \\ A & -S \end{bmatrix}, \text{ with } S = BB^* \tag{4.1}$$

be a bounded input data matrix sequence. If A defines an exponentially stable evolution then

$$\Pi_E := I_{\ell^{2,m}} + B^*(A^* - I_{\ell^{2,n}}\sigma^{-1})^{-1}Q(A - I_{\ell^{2,n}}\sigma)^{-1}B = \Pi_E^* \tag{4.2}$$

is a linear bounded selfadjoint operator form $\ell^{2,m}$ to $\ell^{2,m}$ and is called the *Popov operator* associated with E (see [?], [?], [?], [?]).

Of major importance is the following result of the so-called generalized Popov theory (see [?], [?], [?], [?]).

Theorem 4.1 *Let E as in (4.1) be given with A defining an exponentially stable evolution. Then the following two statements are equivalent.*

1. *The TVDRE (**??**) associated with E has a stabilizing solution X and in addition $I_{\ell^{2,m}} + B^*\sigma XB >> 0$.*

2. *$\Pi_E >> 0$.*

If 2. is fulfilled then the following representation formula holds $\forall k \in \mathbb{Z}$

$$X_k = \Phi_k^* Q[I_{\ell^{2,n}} + (I_{\ell^{2,n}}\sigma - A)^{-1}P_{k,+}^n S(I_{\ell^{2,n}}\sigma^{-1} - A^*)^{-1}Q]^{-1}\Phi_k, \tag{4.3}$$

where Φ_k is the state transition operator at k, i. e. $(\Phi_k\, x)_i = A_{i-1}\cdots A_k\, x,\ \ i > k$. □

The main result of this section is given below.

Theorem 4.2. *Let*

$$E_i = \begin{bmatrix} Q_i & A_i^* \\ A_i & -S_i \end{bmatrix}, \quad \textit{with} \quad S_i = B_i B_i^* \;\; i = 1,2 \tag{4.4}$$

be two input data matrix sequences and let the following three assumptions be fulfilled:

1.

$$\Pi_{E_1} >> 0;$$

2.

$$E_2 \geq E_1;$$

3.

$$(1-\lambda)A_1 + \lambda A_2 \;\; \forall \lambda \in [0,1] \;\; \textit{is exponentially stable.}$$

Then both TVDREs associated with E_1 and E_2 have semistabilizing solutions X_1 and X_2, respectively. In addition $X_2 \geq X_1$.

Proof Invoking assumptions 1. and (for $\lambda = 0$) 3., it follows from Theorem 4.1 that the TVDRE associated with E_1 has a stabilizing solution X_1 and in addition

$$I_{l^2,m} + B_1^* \sigma X_1 B_1 >> 0. \tag{4.5}$$

Further assumption 2. yields

$$\begin{bmatrix} Q_2 & A_2^* - I_{l^2,n}\sigma^{-1} \\ A_2 - I_{l^2,n}\sigma & -S_{2,\epsilon} \end{bmatrix} \geq \begin{bmatrix} Q_1 & A_1^* - I_{l^2,n}\sigma^{-1} \\ A_1 - I_{l^2,n}\sigma & -S_{1,\epsilon} \end{bmatrix}, \tag{4.6}$$

where $S_{i,\epsilon} = S_i + \epsilon I_{l^2,n} >> 0$, $i = 1,2$, for some $\epsilon > 0$. As $S_{i,\epsilon}$ is l^2–boundedly invertible, (**??**) can also be written as

$$\begin{aligned} &\begin{bmatrix} I_{l^2,n} & W_2^* \\ O & I_{l^2,n} \end{bmatrix} \begin{bmatrix} Q_{2,\epsilon}^\times & O \\ O & -S_{2,\epsilon} \end{bmatrix} \begin{bmatrix} I_{l^2,n} & O \\ W_2 & I_{l^2,n} \end{bmatrix} \\ \geq &\begin{bmatrix} I_{l^2,n} & W_1^* \\ O & I_{l^2,n} \end{bmatrix} \begin{bmatrix} Q_{1,\epsilon}^\times & O \\ O & -S_{1,\epsilon} \end{bmatrix} \begin{bmatrix} I_{l^2,n} & O \\ W_1 & I_{l^2,n} \end{bmatrix}, \end{aligned} \tag{4.7}$$

where

$$Q_{i,\epsilon}^\times := Q_i + (A^* - I_{l^2,n}\sigma^{-1}) S_{i,\epsilon}^{-1} (A - I_{l^2,n}\sigma), \quad i = 1,2 \tag{4.8}$$

is the Schur complement of $S_{i,\epsilon}$ and $W_i := -S_{i,\epsilon}^{-1}(A - I_{l^2,n}\sigma)$, $\quad i = 1,2$. Further, (**??**) yields the inequality

$$\begin{bmatrix} I_{l^2,n} & W^* \\ O & I_{l^2,n} \end{bmatrix} \begin{bmatrix} Q_{2,\epsilon}^\times & O \\ O & -S_{2,\epsilon} \end{bmatrix} \begin{bmatrix} I_{l^2,n} & O \\ W & I_{l^2,n} \end{bmatrix} \geq \begin{bmatrix} Q_{1,\epsilon}^\times & O \\ O & -S_{1,\epsilon} \end{bmatrix} \tag{4.9}$$

where $W := W_2 - W_1$. From (??) one gets

$$Q^{\times}_{2,\epsilon} - W^* S_{2,\epsilon} W \geq Q^{\times}_{1,\epsilon},$$

which leads to the inequality

$$Q^{\times}_{2,\epsilon} \geq Q^{\times}_{1,\epsilon}. \tag{4.10}$$

But

$$\Pi_{E_1} = I_{l^2,m} + B_1^*(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_1 (A - I_{l^2,n}\sigma)^{-1} B_1 >> 0,$$

hence

$$\Lambda(I_{l^2,n} + S_1(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_1 (A - I_{l^2,n}\sigma)^{-1}) \subset (\delta, \infty),$$

for some $\delta > 0$. In order to obtain the last inclusion we recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and $U_1 : \mathcal{H}_1 \mapsto \mathcal{H}_2$, $U_2 : \mathcal{H}_2 \mapsto \mathcal{H}_1$ are two linear bounded operators, then $I_2 + U_1 U_2$ has a bounded inverse if and only if $I_1 + U_2 U_1$ has a bounded inverse. Hence $\Lambda(I_1 + U_2 U_2) \subset (\delta, \infty)$ if and only if $\Lambda(I_2 + U_1 U_1) \subset (\delta, \infty)$ where $\delta \in (0,1)$. Here I_j denotes the identity in $\mathcal{H}_j$, $1 \leq j \leq 2$. Consequently we get further

$$\Lambda(I_{l^2,n} + S_{1,\epsilon}(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_1 (A - I_{l^2,n}\sigma)^{-1}) \subset (\delta, \infty),$$

for ϵ small enough. Hence

$$I_{l^2,n} + S^{\frac{1}{2}}_{1,\epsilon}(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_1 (A - I_{l^2,n}\sigma)^{-1} S^{\frac{1}{2}}_{1,\epsilon} >> 0,$$

from which follows

$$S^{-1}_{1,\epsilon} + (A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_1 (A - I_{l^2,n}\sigma)^{-1} >> 0,$$

or, equivalently,

$$Q^{\times}_{1,\epsilon} >> 0. \tag{4.11}$$

Invoking (??) and (??) we conclude that

$$Q^{\times}_{2,\epsilon} >> 0. \tag{4.12}$$

Using (??), (??) reveals that

$$Q_2 + (A^* - I_{l^2,n}\sigma^{-1}) S^{-1}_{2,\epsilon} (A - I_{l^2,n}\sigma) >> 0,$$

from which one gets

$$I_{l^2,n} + S^{\frac{1}{2}}_{2,\epsilon}(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_2 (A - I_{l^2,n}\sigma)^{-1} S^{\frac{1}{2}}_{2,\epsilon} >> 0 \ \ \forall \epsilon > 0.$$

Hence

$$I_{l^2,n} + S^{\frac{1}{2}}_{2}(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_2 (A - I_{l^2,n}\sigma)^{-1} S^{\frac{1}{2}}_{2} \geq 0,$$

and, consequently,

$$\Lambda(I_{l^2,n} + S_2(A^* - I_{l^2,n}\sigma^{-1})^{-1} Q_2 (A - I_{l^2,n}\sigma)^{-1}) \subset [0, \infty),$$

from which we conclude that

$$\Pi_{E_2} \geq 0. \tag{4.13}$$

For some $\epsilon > 0$ let

$$E_{2,\epsilon} = \begin{bmatrix} Q_2 & A_2^* \\ A_2 & -(1+\epsilon)^{-1} S_2 \end{bmatrix}.$$

As $E_{2,\epsilon} \geq E_2$ it follows that

$$E_{2,\epsilon} \geq E_1 \tag{4.14}$$

But, since, by 3. for $\lambda = 1$, A_2 is exponentially stable,

$$\Pi_{E_{2,\epsilon}} = (1+\epsilon)^{-\frac{1}{2}} (\Pi_{E_2} + \epsilon I_{l^{2,m}})(1+\epsilon)^{-\frac{1}{2}} >> 0,$$

as can be checked easily. Hence, in accordance with Theorem 4.1, the TVDRE associated with $E_{2,\epsilon}$ has a stabilizing solution $X_{2,\epsilon}$, i.e.

$$\rho((I_{l^{2,n}} + (1+\epsilon)^{-1} S_2 \sigma X_{2,\epsilon})^{-1} A_2 \sigma^{-1}) < 1. \tag{4.15}$$

Notice now that the above developpment shows in fact that for all $E = (1-\lambda)E_1 + \lambda E_2$ (see also assumption 3. in the statement) the TVDRE associated with E has a stabilizing solution Hence by invoking Theorem 3.1, (**??**), reveals that

$$X_{2,\epsilon} \geq X_1. \tag{4.16}$$

But $X_{2,\epsilon}$ is monotonically decreasing when $\epsilon \downarrow 0$. Indeed, by invoking the representation formula (4.3) one gets for the input data $E_{2,\epsilon}$

$$X_{2,\epsilon,k} = \Phi_{2,k}^* Q_2 [I_{\ell^{2,n}} + (I_{\ell^{2,n}}\sigma - A_2)^{-1} P_{k,+}^n (1+\epsilon)^{-1} S_2 (I_{\ell^{2,n}}\sigma^{-1} - A_2^*)^{-1} Q_2]^{-1} \Phi_{2,k}, \tag{4.17}$$

which clearly shows that $X_{2,\epsilon}$ decreases when $\epsilon \downarrow 0$. Combining this conclusion with (**??**) one obtains

$$\lim_{\epsilon \downarrow 0} X_{2,\epsilon} = X_2 \geq X_1. \tag{4.18}$$

Further (**??**) implies

$$I_{l^{2,m}} + B_2^* \sigma X_2 B_2 >> 0. \tag{4.19}$$

Indeed, since $S_2 \leq S_1$ we may write $B_2 = B_1 K$ for some K with $\|K\| \leq 1$. Hence, for some $0 < \rho < 1$, (**??**) implies $< u, B_2^* \sigma X_1 B_2 u > = < Ku, B_1^* \sigma X_1 B_1 K u > \geq -(1-\rho)\|Ku\|_2^2$ $\geq -(1-\rho)\|u\|_2^2$, that is, $I_{l^{2,m}} + B_2^* \sigma X_1 B_2 >> 0$, from which (**??**) holds because of $X_2 \geq X_1$. Therefore, $I_{l^{2,m}} + S_2 \sigma X_2$ must be l^2–boundedly invertible. By combining this conclusion with (**??**) we deduce that $\rho((I + S_2 \sigma X_2)^{-1} A_2 \sigma^{-1}) \leq 1$. Recalling now that $X_2 \geq X_1$, the proof is completed. $\square$

Remark 4.3. Theorem 4.2 generalizes to the *global* situation and in the time-varying case the results obtained in [?], [?] [?], in particular global existence of the stabilizing solution is guaranteed. Recently in [?] another approach has been used for the investigation of inequalities for Riccati difference equations.

5. Conclusions

In this paper several results concerning monotonicity of the solution to the TVDRE have been proved. In a way, these results generalize and extend the former ones obtained up to now in the literature. It is worthwhile to mention that the Fréchet derivative type approach, used in this paper, works similarly in the continuous-time case. Moreover, using the second Fréchet derivative, convexity properties of the solution to the Riccati equation can be proved analogously. Comparison techniques turn out to be a very efficient tool for the proof of existence theorems both in the time-varying and time-invariant case.

References

[1] J. Ball, I. Gohberg, M.A. Kaashoek, Nevanlinna-Pick interpolation for time-varying input-output maps: The discrete case, Operator Theory, Advances and Applications 56, pp. 1-51, Birkhäuser, 1992.

[2] R. R. Bitmead, M. Gevers, I. R. Petersen and R.J. Kaye, Monotonicity and stability properties of solutions of the Riccati difference equation: Propositions, lemmas, theorems, fallacious conjectures and counterexamples, Syst. Contr. Letters 5, 309-315, 1985.

[3] R. R. Bitmead and M. Gevers, Riccati difference and differential equations: Convergence, monotonicity and stability, In: The Riccati equation (ed. S. Bittanti et. al.), Springer Verlag, Berlin, 1991.

[4] M. Bohner, O. Došlý and W. Kratz, Inequalities and asymptotics for Riccati matrix difference operators, submitted.

[5] W. A. Coppel, Disconjugacy, Lecture Notes in Mathematics 220, Springer Verlag, Berlin, 1971.

[6] R.F. Curtain and L. Rodman, Comparison theorems for infinite dimensional Riccati equations, *Syst. Contr. Letters* 15, 153-159, 1990.

[7] D. F. Delchamps, A note on the analyticity of the Riccati metric. In: Lectures in applied mathematics, Vol. 18 (ed. C.I. Byrnes and C.F. Martin), 37-41, AMS, Providence, Rhode Island, 1980.

[8] G. Freiling and G. Jank, Existence and comparison theorems for algebraic Riccati equations and Riccati differential and difference equations, *J. Dynam. Contr. Syst.* 2, 529-547, 1996.

[9] G. Freiling, G. Jank and H. Abou-Kandil, Generalized Riccati difference and differential equations, Lin.Alg. Appl. 241-243, 291-303, 1996.

[10] P. Gahinet, On the game Riccati equations arising in H_∞ control problems. *SIAM Journal of Control and Optimization*, 32, 635-647, 1994.

[11] A. Halanay, V. Ionescu, Generalized discrete-time Popov-Yakubovich theory, Systems Contr. Letters 20, pp. 1-6, 1993.

[12] A. Halanay, V. Ionescu, Time-varying discrete linear systems, Birkhäuser, 1994.

[13] A. Halanay, V. Ionescu, Anticausal stabilizing solution to discrete reverse-time Riccati equation, Computers and Mathematics with Applications, Special Issue on Difference Equations, Pergamon Press 28, no 1-3, pp. 115-125, 1994.

[14] G. Hewer, Existence theorems for positive semidefinite and sign indefinite stabilizing solution of H_∞ Riccati equations. *SIAM Journal of Control and Optimization*, 31, 16-29, 1993.

[15] V. Ionescu, Generalized time-varying Popov-Yakubovich theory: The discrete case, Proceeding of the MTNS'93 vol. II, pp. 717-720, Akademie Verlag, 1994.

[16] L. V. Kantorovich and G. P. Akilov, Functional analysis in normed spaces, Pergamon Press, Oxford, 1964.

[17] P. Lancaster, A. C. M. Ran and L. Rodman, An existence and monotonicity theorem for the discrete algebraic matrix Riccati equation, Lin. Multilin. Algebra 20, 353-361, 1987.

[18] X. P. Li and B. C. Chang, On convexity of H^∞ Riccati solutions and its applications, IEEE Trans. Autom. Contr. 38, 963-966, 1993.

[19] T. Morozan, Stability and control for linear discrete-time systems with Markov perturbations, Rev. Roum. Math. Pur. Appl. 40, 471-494, 1995.

[20] G. de Nicolao, On the time-varying Riccati difference equation of optimal filtering, SIAM J. Contr. Opt. 30, 1251-1269, 1992.

[21] A. C. M. Ran and R. Vreugdenhil, Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, Lin. Alg. Appl. 99, 63-83, 1988.

[22] C. E. de Souza, On stabilizing properties of the Riccati difference equation, IEEE Trans. Automat. Contr. 34, 1313-1316, 1989.

[23] A. N. Stokes, A special property of the matrix Riccati equation, Bull. Austr. Math. Soc. 10, 245-253, 1974.

[24] K. Takaba and T. Katayama, Discrete-time H_∞ algebraic Riccati equation and parameterization of all H_∞ filters, Int. J. Contr. 64, 1129-1149, 1996.

[25] H. K. Wimmer and M. Pavon, A comparison theorem for matrix Riccati difference equations, Syst. Contr. Letters 9, 233-239, 1992.

[26] G. F. Wredenhagen and P. R. Bélanger, Curvature properties of the algebraic Riccati equation, Syst. Constr. Letters 21, 285-287, 1993.

[27] M. K. Wimmer, Monotonicity and maximality of solutions of discrete-time algebraic Riccati equation, J. of Mathematical Systems, Estimation and Control, 2, no 2, pp. 219-235, 1992.