Inverse spectral problems for differential equations on the half-line with turning points

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G. Freiling and V. Yurko

Abstract. Boundary value problems for second-order differential equations on the half-line having an arbitrary number of turning points are investigated. We establish properties of the spectra, prove an expansion theorem and study inverse problems of recovering the boundary value problem from given spectral characteristics. For these inverse problems we prove uniqueness theorems and provide a procedure for constructing the solution.

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1. Introduction. This paper deals with the boundary value problem $\mathcal{L}$ for the differential equation

$$\ell y := -y'' + q(x)y = \lambda R^2(x)y, \ x > 0 \tag{1}$$

on the half-line with the boundary condition

$$U(y) := y'(0) - hy(0) = 0. \tag{2}$$

Here R^2 and q are real functions, and h is a real number. We suppose that

$$R^2(x) = \prod_{\nu=1}^{m} (x - x_\nu)^{\ell_\nu} R_0(x),$$

where $0 < x_1 < x_2 < \ldots < x_m$, $\ell_\nu \in I\!N$; $R_0(x) > 0$ for $x \in I := [0, \infty)$ and is twice continuously differentiable on I. In other words, R^2 has in I m zeros x_ν, $\nu = \overline{1, m}$ of order ℓ_ν. Zeros x_v of R^2 are called turning points. Other restrictions will be introduced later, they correspond to the condition of summability of the potential for the classical Sturm-Liouville problem on the half-line when $R^2(x) \equiv 1$.

In the present paper we establish properties of the spectrum of $\mathcal{L}$, prove an expansion theorem and study inverse problems of recovering $\mathcal{L}$ from its spectral characteristics.

Differential equations with turning points arise in various problems of mathematical physics as well as in applications (see [1-4] for details). Furthermore it is known that wide class of differential equations with Bessel-type singularities and also perturbations of such equations can be reduced to differential equations having turning points.
For the special case $R^2(x) > 0$ (in particular, when $R^2(x) \equiv 1$) inverse problems have been studied fairly completely in many works (see, for example, [5-10] and references therein). An important role there was played by the transformation operator method. The presence of turning points in the differential equation produces essential qualitative difficulties in the investigation of the inverse problems for the

boundary value problem $\mathcal{L}$. The transformation operator method in this case is not suitable for the solution of the inverse problems.

In order to study the inverse problems in this paper we use another approach connected with the contour integral method. An important role in this method is played by the special fundamental system of solutions (FSS) of equation (1) constructed in [11]. This FSS gives us an opportunity to obtain the asymptotic behavior of the so-called Weyl solutions and Weyl function of $\mathcal{L}$ and to solve the corresponding inverse problems. In Section 2 we study properties of the spectrum. For the boundary value problem with turning points, the behavior of the spectrum is more complicated than for classical Sturm-Liouville operator when $R^2(x) > 0$. In particular, the discrete spectrum can be unbounded, and when $m > 1$ the main part of the characteristic function may have an unbounded set of zeros on the continuous spectrum. In Section 3-4 we provide uniqueness theorems for the solution of two inverse problems: from the Weyl function and from the so-called spectral data. In Section 5 an expansion theorem is proved, and in Section 6 we give a constructive procedure for the solution of the inverse problem.

Note that in [12-17] some aspects of direct problems for second-order differential equations with turning points have been studied, and in [18-20] the contour integral method has been used for the solution of the inverse problem for arbitrary order differential operators without turning points. Inverse problems for equation (1) on a finite interval have been solved in [21].

2. Properties of the spectrum. Denote

$$\xi(x) = \int_{x_m}^{x} R(x)dt, \ x > x_m,$$

where $R(x) = (R^2(x))^{\frac{1}{2}} > 0$ for $x > x_m$ and let

$$b(x) = \frac{R''(x)}{2R^2(x)} - \frac{3(R'(x))^2}{4R^3(x)} + \frac{q(x)}{R(x)}. \tag{3}$$

We assume that $R'(x)(R^2(x))^{-1} = O(1)$, $x \to \infty$; $q(x) \in L^1_{\ell oc}(I)$,

$$\lim_{x\to\infty} \xi(x) = \infty \tag{4}$$

and

$$b(x) \in L(T, \infty), \tag{5}$$

where $T > x_m$. Condition (4) makes $\mathcal{L}$ singular. When $\xi(x)$ is bounded we have a qualitatively different situation requiring a separate investigation. Condition (5) corresponds to the summability assumption for the potential for the classical Sturm-Liouville problem when $R^2(x) \equiv 1$.

Remark 1. Consider the Hilbert space $L^2_{|R|^2} = \{y(x) : \int_0^\infty |R^2(x)| \cdot |y(x)|^2 dx < \infty\}$ with scalar product $(y, z) = \int_0^\infty |R^2(x)\,| \, y(x)\overline{z(x)} dx$. We identify the boundary value problem (1)-(2) and the operator

$$\mathcal{L} : D(\mathcal{L}) \to L^2_{|R|^2},\ y \mapsto \frac{1}{R^2(x)}(-y'' + q(x)y)$$

with the domain of definition $\mathcal{D}(\mathcal{L}) = \{y : y \in L^2_{|R|^2}(I) \cap AC_{\ell oc}(I),\ y' \in AC_{\ell oc}(I),$ $\mathcal{L}y \in L^2_{|R|^2}(I),\ U(y) = 0\}$.

Let $\lambda = \rho^2$, and let for definiteness $Im\rho \geq 0$. Denote by Π the λ-plane with the cut $\lambda \geq 0$. Then Π corresponds to the domain $\Pi_+ = \{\rho : Im\rho > 0\}$. Put $\mu_\nu = (\ell_\nu + 2)^{-1}$.

Following R. Langer [12] we introduce the functions

$$\Omega(x) = \frac{\xi(x)}{R(x)} \text{ and } q_0(x) = \frac{1}{4\Omega^2(x)}((\Omega^2(x))'' - 3(\Omega'(x))^2 + 4\mu_m^2).$$

Clearly,

$$q_0(x) = (\mu_m^2 - \frac{1}{4})\frac{R^2(x)}{\xi^2(x)} - \frac{R''(x)}{2R(x)} + \frac{3(R'(x))^2}{4R^2(x)}. \tag{6}$$

Further consider the functions

$$v_j(x,\rho) = \frac{A_j}{\sqrt{R(x)}}\sqrt{\rho\xi(x)}H^{(j)}_{\mu_m}(\rho\xi(x)),\ x > x_m, j = 1, 2,$$

where $A_j = (\frac{\pi}{2})^{\frac{1}{2}} \exp((-1)^{j-1} i(\frac{\pi\mu_m}{2} + \frac{\pi}{4}))$, and $H^{(j)}_\nu$ are Hankel functions [22]. It is easy to verify that the functions $v_j(x,\rho)$ are solutions of the equation

$$-y'' + q_0(x)y = \lambda R^2(x)y,$$

and

$$\langle v_1(x,\rho),\ v_2(x,\rho)\rangle \equiv -2i\rho,$$

where

$$\langle y, z\rangle := yz' - y'z.$$

For $Im\ \rho \geq 0$, $|\rho\xi(x)| \geq 1$, $x > x_m$

$$v_1(x,\rho) = \frac{1}{\sqrt{R(x)}}\exp(i\rho\xi(x))(1 + O(\frac{1}{\rho\xi(x)})),$$
$$v_2(x,\rho) = \frac{1}{\sqrt{R(x)}}\exp(-i\rho\xi(x)) \cdot O(1).$$

It follows from (3) and (6) that

$$\frac{q(x) - q_0(x)}{R(x)} = b(x) + (\frac{1}{4} - \mu_m^2)\frac{R(x)}{\xi^2(x)}.$$

By virtue of (4),

$$\frac{R(x)}{\xi^2(x)} \in L(T,\infty),\ T > x_m.$$

Together with (5) it gives

$$\frac{q(x) - q_0(x)}{R(x)} \in L(T,\infty). \tag{7}$$

Let $e(x,\rho)$, $x > x_m$ be a solution of the integral equation

$$e(x,\rho) = v_1(x,\rho) + \frac{1}{2i\rho}\int_x^\infty (v_1(x,\rho)v_2(t,\rho) - v_2(x,\rho)v_1(t,\rho))(q(t) - q_0(t))e(t,\rho)dt. \tag{8}$$

In view of (7), Eq. (8) has a unique solution for each $\rho \in \overline{\Pi}_+ \setminus \{0\}$. For each fixed x, the functions $e^{(s)}(x,\rho)$, $s = 0,1$ are holomorphic in Π_+ and continuous in $\overline{\Pi}_+ \setminus \{0\}$. For $x \to \infty$

$$e^{(s)}(x,\rho) = (i\rho)^s (R(x))^{s-\frac{1}{2}} \exp(i\rho\xi(x))(1 + o(1)) \tag{9}$$

and for $|\rho| \to \infty$ uniformly in $x \geq x_m + \varepsilon$

$$e^{(s)}(x,\rho) = (i\rho)^s (R(x))^{s-\frac{1}{2}} \exp(i\rho\xi(x))(1 + O(\frac{1}{\rho})). \tag{10}$$

For each fixed $\rho \in \Pi_+$, $e(x,\rho) \in L^2_{|R|^2}(T,\infty)$, $x > x_m$.
The function $e(x,\rho)$ satisfies Eq. (1) for $x > x_m$. We continue $e(x,\rho)$ for $x < x_m$ as a solution of (1) and consider the function

$$\Delta(\rho) := e'(0,\rho) - he(0,\rho) \tag{11}$$

which is called the characteristic function for $\mathcal{L}$. The function $\Delta(\rho)$ is regular in Π_+ and is continuous in $\overline{\Pi_+} \setminus \{0\}$. For real $\rho \neq 0$ the functions $e(x,\rho)$ and $e(x,-\rho) = \overline{e(x,\rho)}$ form the FSS of (1), and

$$\langle e(x,\rho), \overline{e(x,\rho)} \rangle \equiv -2i\rho. \tag{12}$$

Let $\varphi(x,\lambda)$ and $S(x,\lambda)$ be solutions of (1) under the initial conditions
$\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$, $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$.
Denote

$$\Phi(x,\lambda) = \frac{e(x,\rho)}{\Delta(\rho)}, \tag{13}$$

then we have

$$\langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle \equiv 1, \tag{14}$$

$$\langle \varphi(x,\lambda), e(x,\rho) \rangle \equiv \Delta(\rho). \tag{15}$$

We set $M(\lambda) := \Phi(0,\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for $\mathcal{L}$ respectively. Clearly,

$$M(\lambda) = \frac{e(0,\rho)}{\Delta(\rho)} \tag{16}$$

and

$$\Phi(x, \lambda) = S(x, \lambda) + M(\lambda)\varphi(x, \lambda). \tag{17}$$

Theorem 1. Let ρ be real, $\rho \neq 0$. Then

(i) $\Delta(\rho) \neq 0$,

(ii) $\lambda = \rho^2$ is not an eigenvalue of $\mathcal{L}$.

Proof. The functions $e(x, \rho)$ and $\overline{e(x, \rho)}$ form a FSS of (1). Then, in view of (12), we have

$$\varphi(x, \lambda) = \frac{1}{2i\rho}(\Delta(\rho)\overline{e(x, \rho)} - \overline{\Delta(\rho)}e(x, \rho)). \tag{18}$$

If $\Delta(\rho_0) = 0$ for a certain real $\rho_0 \neq 0$, then $\overline{\Delta(\rho_0)} = 0$, i.e. $\varphi \equiv 0$; but this is impossible since $\varphi(0, \lambda) = 1$.
On the other hand, if $\rho_0 \neq 0$ is real and $\lambda_0 = \rho_0^2$ is an eigenvalue of $\mathcal{L}$, then $\varphi(x, \lambda_0)$ must be an eigenfunction. This contradicts to (18) because, in view of (9), any nontrivial linear combinations of $e(x, \rho)$ and $\overline{e(x, \rho)}$ does not belong to $L^2_{|R|^2}(I)$. □

Theorem 2. In the domain Π_+ the eigenvalues $\lambda_k = \rho_k^2$ of $\mathcal{L}$ coincide with the zeros of $\Delta(\rho)$. The functions $\varphi(x, \lambda_k)$, $e(x, \rho_k)$ are eigenfunctions, and

$$e(x, \rho_k) = \beta_k \varphi(x, \lambda_k), \ \beta_k \neq 0. \tag{19}$$

Proof.

1) Let $\Delta(\rho_o) = 0$, $\rho_0 \in \Pi_+$. Then, by virtue of (15), we have $e(x, \rho_0) = \beta_0 \varphi(x, \lambda_0)$, $\beta_0 \neq 0$, $\lambda_0 = \beta_0^2$. Since $e(x, \rho_0) \in L^2_{|R|^2}(I)$, we get $\varphi(x, \lambda_0) \in L^2_{|R|^2}(I)$, and consequently $\lambda_0 = \rho_0^2$ is an eigenvalue, and $e(x, \rho_0), \varphi(x, \lambda_0)$ are eigenfunctions.

2) Let $\lambda_0 = \rho_0^2$ be an eigenvalue of $\mathcal{L}$. Then $\varphi(x, \lambda_0)$ is an eigenfunction, and consequently $\varphi(x, \lambda_0) \in L^2_{|R|^2}(I)$.
It is known (see, for example, [12]) that for each fixed $\rho \in \Pi_+$ there exists a solution $E(x, \rho)$ of (1) such that

$$E(x, \rho) = \frac{1}{\sqrt{R(x)}} \exp(-i\rho\xi(x))(1 + \circ(1)), \ x \to \infty,$$

and $\langle e(x, \rho), \, E(x, \rho) \rangle \equiv -2i\rho$. Therefore

$$\varphi(x, \lambda_0) = \beta_0 e(x, \rho_0) + \gamma_0 E(x, \rho_0)$$

with certain constants β_0 and γ_0. Since $E(x, \rho_0) \notin L^2_{|R|^2}(I)$, we infer that $\gamma_0 = 0$, i.e. $\varphi(x, \lambda_0) = \beta_0 e(x, \rho_0)$. From this and (15) it follows that $\Delta(\rho_0) = 0$.

□

Denote

$$\alpha_k := \int_0^\infty R^2(x)\varphi^2(x,\lambda_k)dx. \tag{20}$$

Lemma 1. Let $\Delta_1(\rho) := \frac{d}{d\lambda}\Delta(\rho)$, then the following relation holds:

$$\beta_k\alpha_k = \Delta_1(\rho_k). \tag{21}$$

Proof. Since

$$\begin{array}{l} -e''(x,\rho) + q(x)e(x,\rho) = \lambda R^2(x)e(x,\rho), \\ -\varphi''(x,\lambda_k) + q(x)\varphi(x,\lambda_k) = \lambda_k R^2(x)\varphi(x,\lambda_k), \end{array}$$

we get

$$\frac{d}{dx}\langle e(x,\rho),\varphi(x,\lambda_k)\rangle = (\lambda-\lambda_k)R^2(x)e(x,\rho)\varphi(x,\lambda_k),$$

hence

$$(\lambda-\lambda_k)\int_0^\infty R^2(x)e(x,\rho)\varphi(x,\lambda_k)dx = |_0^\infty \langle e(x,\rho),\varphi(x,\lambda_k)\rangle.$$

It follows from (9) and (19) that

$$\lim_{x\to\infty}\langle e(x,\rho),\varphi(x,\lambda_k)\rangle = 0,$$

and consequently, in view of (11),

$$(\lambda-\lambda_k)\int_0^\infty R^2(x)e(x,\rho)\varphi(x,\lambda_k)dx = \Delta(\rho).$$

If $\lambda\to\lambda_k$, then

$$\int_0^\infty R^2(x)e(x,\rho_k)\varphi(x,\lambda_k)dx = \Delta_1(\rho_k).$$

Using (19) and (20) we arrive at (21). □

For simplicity, we shall assume in the sequel that the operator $\mathcal{L}$ is normal (see [13], [17], [23]), and $M(\lambda) = O(\rho^{-1}), \rho\to 0$, this is fulfilled, for example, when $q(x)\geq 0$ and $b(x)\exp(\varepsilon\xi(x))\in L(T,\infty)$ for a certain $\varepsilon>0$. It is also fulfilled if ℓ_ν are even and $b(x)\xi(x)\in L(T,\infty)$.

Theorem 3. The set of eigenvalues $\Lambda = \{\lambda_k\}_{k\geq 1}$ and the eigenfunctions $\varphi(x,\lambda_k)$, $e(x,\rho_k)$ of $\mathcal{L}$ are real. Moreover, $\lambda_k<0$, and $\lambda=0$ is not an accumulation point of Λ. All zeros of $\Delta(\rho)$ are simple, i.e. $\Delta_1(\rho_k)\neq 0$.

Proof. Let $\lambda^* = u+iv$, $v\neq 0$ be a non-real eigenvalue with an eigenfunction $y^*(x)$. Since $q(x)$, $R^2(x)$ and h are real, we get that $\overline{\lambda}^* = u-iv$ is also the eigenvalue with the eigenfunction $\overline{y^*(x)}$. Then

$$\begin{aligned} \lambda^*\int_0^\infty R^2(x)y^*(x)\overline{y^*}(x)dx &= \int_0^\infty \ell y^*(x)\cdot\overline{y^*}(x)dx = \int_0^\infty y^*(x)\ell\overline{y^*(x)}dx = \\ &= \overline{\lambda^*}\int_0^\infty R^2(x)y^*(x)\overline{y^*(x)}dx. \end{aligned}$$

Since $\lambda^* \neq \bar{\lambda^*}$, we derive

$$\int_0^\infty R^2(x)\,|\,y^*(x)|^2 dx = 0$$

which is impossible for normal operators. Hence the eigenvalues λ_k are real, and taking Theorem 1 into account, we conclude that $\lambda_k < 0$. The functions $\varphi(x,\lambda)$ and $e(x,\rho)$ are real for $\lambda < 0$ (i.e. for $\rho = i\tau, \tau > 0$), and hence the eigenfunctions $\varphi(x,\lambda_k)$ and $e(x,\rho_k)$ are real.

In view of (20) and (21), we have

$$\Delta_1(\rho_k) = \beta_k \int_0^\infty R^2(x)\,|\,\varphi(x,\lambda_k)|^2 dx \neq 0.$$

At last, since the function $\rho M(\lambda)$ is bounded near the origin, it follows from (16) that there is a neighborhood of the origin without zeros of $\Delta(\rho)$, i.e. $\lambda = 0$ is not accumulation point for Λ. □

Using (16), Theorem 1-3 and the analyticity properties of $\Delta(\rho)$ and $e(0,\rho)$, we obtain the following properties of the Weyl function $M(\lambda)$.

Theorem 4. The Weyl function $M(\lambda)$ is holomorphic in $\Pi \setminus \Lambda$ and continuous in $\overline{\Pi} \setminus (\Lambda \cup \{0\})$. In the points $\lambda = \lambda_k$ the Weyl function $M(\lambda)$ has simple poles, and

$$\mathrm{Res}_{\lambda=\lambda_k} M(\lambda) = \frac{\beta_k}{\Delta_1(\rho_k)} = \frac{1}{\alpha_k}.$$

For $\lambda > 0$, there exist finite limits

$$M^{\pm}(\lambda) = \lim M(\lambda \pm iz),\ z \to 0,\ Re\ z\ > 0.$$

Denote

$$V(\lambda) := \frac{1}{2\pi i}(M^-(\lambda) - M^+(\lambda)),\ \lambda > 0.$$

Then

$$V(\lambda) = \frac{\rho}{\pi |\Delta(\rho)|^2},\ \lambda > 0. \tag{22}$$

Equality (22) follows from (11), (12) and (16), since

$$M^-(\lambda) - M^+(\lambda) = \frac{\overline{e(0,\rho)}}{\overline{\Delta(\rho)}} - \frac{e(0,\rho)}{\Delta(\rho)} = \frac{2i\rho}{|\Delta(\rho)|^2}.$$

Now we consider the resolvent of $\mathcal{L}$:

$$Y(x,\lambda) = \int_0^\infty G(x,t,\lambda) R^2(t) f(t) dt, \tag{23}$$

where

$$G(x,t,\lambda)=\begin{cases}\varphi(x,\lambda)\Phi(t,\lambda), x\le t\\ \varphi(t,\lambda)\Phi(x,\lambda), x\ge t\end{cases}=\frac{1}{\Delta(\rho)}\begin{cases}\varphi(x,\lambda)e(t,\rho), x\le t\\ \varphi(t,\lambda)e(x,\rho), x\ge t\end{cases} \tag{24}$$

is the Green's function. Then

$$-Y''+q(x)Y-\lambda R^2(x)Y+R^2(x)f(x)=0,\ U(Y)=0.$$

The spectrum of $\mathcal{L}$ coincides with the set of singularities for the resolvent, i.e. with the set of singularities of the Weyl function $M(\lambda)$. The spectrum of $\mathcal{L}$ is discrete on the negative half-line, and is continuous on the positive half-line. Thus, $sp(\mathcal{L})=\{\lambda:\lambda\ge 0\}\cup\Lambda$. The set $S=(V(\lambda),\{\lambda_k,\alpha_k\})$ is called the spectral data of $\mathcal{L}$. We also introduce the so-called spectral function $\sigma(\lambda)$ via

$$\sigma(\lambda)=\int_0^\lambda V(z)dz,\ \lambda\ge 0;\ \sigma(\lambda)=-\sum_{\lambda_k>\lambda}\frac{1}{\alpha_k},\ \lambda<0. \tag{25}$$

3. Inverse Problem 1. In this section we prove the uniqueness theorem for the solution of the inverse problem from the Weyl function. Moreover, additional properties of the characteristic function $\Delta(\rho)$ and the spectrum for $\mathcal{L}$ are provided. In particular, the discrete spectrum can be unbounded in contrast to the classical Sturm-Liouville operator when $R^2(x)\equiv 1$. We shall also show that when $m>1$, the main part of the asymptotics of $\Delta(\rho)$ for $\lambda>0$ can have a countable set of zeros, in spite of $\Delta(\rho)\ne 0$ for $\lambda>0$.

The inverse problem is formulated as follows. Suppose that the function R^2 is known a priori. Our goal is to find $q(x)$ and h from the given Weyl function M.

In order to formulate and prove the uniqueness theorem for the solution of the inverse problem we agree that together with $\mathcal{L}=\mathcal{L}(R^2(x),q(x),h)$ we consider a boundary value problem $\tilde{\mathcal{L}}=\mathcal{L}(R^2(x),\tilde{q}(x),\tilde{h})$ of the same form (1) - (2) but with different coefficients. If a certain symbol denotes an object related to $\mathcal{L}$, then the corresponding symbol with tilde will denote the analogous object related to $\tilde{\mathcal{L}}$.

Theorem 5. If $M=\tilde{M}$ then $q(x)=\tilde{q}(x)$ for $x\in I$, and $h=\tilde{h}$.

Proof. First we study the asymptotic behavior of the solutions $\varphi(x,\lambda), e(x,\rho)$ and the characteristic function $\Delta(\rho)$ for large $|\lambda|$.
Let $\varepsilon>0$ be fixed, sufficiently small and let $\mathcal{D}_{0\varepsilon}=[0,x_1-\varepsilon]$, $\mathcal{D}_{\nu\varepsilon}=[x_\nu+\varepsilon,x_{\nu+1}-\varepsilon]$ for $1\le\nu\le m-1$, $\mathcal{D}_{m\varepsilon}=[x_m+\varepsilon,\infty)$, $\mathcal{D}_\varepsilon=\bigcup_{\nu=0}^{m}\mathcal{D}_{\nu\varepsilon}$, and $I_{\nu\varepsilon}=\mathcal{D}_{\nu-1,\varepsilon}\cup[x_\nu-\varepsilon,x_\nu+\varepsilon]\cup\mathcal{D}_{\nu\varepsilon}$.

We distinguish four different types of turning points:
For $1 \le \nu \le m$

$$T_\nu = \begin{cases} I, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x-x_\nu)^{-\ell_\nu} < O \text{ in } I_{\nu\varepsilon}, \\ II, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x-x_\nu)^{-\ell_\nu} > O \text{ in } I_{\nu\varepsilon}, \\ III, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x-x_\nu)^{-\ell_\nu} < O \text{ in } I_{\nu\varepsilon}, \\ IV, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x-x_\nu)^{-\ell_\nu} > O \text{ in } I_{\nu\varepsilon}, \end{cases}$$

is called type of x_ν. Further we set for $1 \le \nu \le m$

$$\Theta_\nu = \begin{cases} 1 & \text{if } \mu_\nu > \frac{1}{4}, \\ 1-\delta_0 \text{ (with } \delta_0 > 0 \text{ arbitrary small)} & \text{if } \mu_\nu = \frac{1}{4}, \\ 4\mu_\nu & \text{if } \mu_\nu < \frac{1}{4}, \end{cases}$$

and $\Theta_0 = \min\limits_{1\le\nu\le m} \Theta_\nu$. We also denote

$$I_+ = \{x : R^2(x) > 0\},\ I_- = \{x : R^2(x) < 0\},$$

$$\theta(x) = \begin{cases} 0, & \text{for } x \in I_+, \\ 1, & \text{for } x \in I_-, \end{cases}$$

$$\gamma_\nu = \begin{cases} 2\sin\dfrac{\pi\mu_\nu}{2}, & \text{for } T_\nu = III, IV, \\ \sin\pi\mu_\nu, & \text{for } T_\nu = I, II, \end{cases}$$

$$K(x) = (\prod_{x_\nu\in(0,x)} \gamma_\nu^{-1}) \exp(-i\frac{\pi}{4}(\theta(x)-\theta(0))),$$

$$K^*(x) = (\prod_{x_\nu\in(0,x)} \gamma_\nu) \exp(-i\frac{\pi}{4}(\theta(x)+\theta(0))),$$

$$R_+^2(x) = \max(0, R^2(x)),\ R_-^2(x) = \max(0, -R^2(x)),$$

$$S_k = \{\rho : \arg\rho \in [\frac{k\pi}{4},\ \frac{(k+1)\pi}{4}]\},$$

$$\sigma_s^\delta = \{\rho : \arg\rho \in [\frac{s\pi}{2} - \delta,\ \frac{s\pi}{2} + \delta]\},\ \delta > 0,$$

$$\sigma^\delta = \bigcup_s \sigma_s^\delta,\ S_k^\delta = S_k \setminus \sigma^\delta,\ S^\delta = \bigcup_k S_k^\delta.$$

Clearly,

$$K(x)K^*(x) = \exp(-i\frac{\pi}{2}\theta(x)) = \begin{cases} 1, & \text{if } x \in I_+, \\ -i, & \text{if } x \in I_-. \end{cases}$$

Let for definiteness, $\rho \in S_0 \cup S_1$ (for other sectors estimates are similar). It was shown in [11] that for each sector S_k there exists a FSS of (1) $\{z_1(x,\rho),\ z_2(x,\rho)\}$, $x \in I$, $\rho \in S_k$ such that the functions $z_j^{(s)}(x,\rho)$; $j = 1,2$; $s = 0,1$ are continuous for $x \in I$, $\rho \in S_k$ and holomorphic with respect to ρ for each fixed $x \in I$.
For $|\rho| \to \infty$, $\rho \in S_k$, $x \in \mathcal{D}_\varepsilon$; $j,k = 0,1$

$$z_1^{(j)}(x,\rho) = (-i\rho)^j |R(x)|^{j-\frac{1}{2}} (\exp(i\frac{\pi}{2}\theta(x)))^j \exp(\rho\int_0^x |R_-(t)|dt)$$

$$\exp(-i\rho\int_0^x |R_+(t)|dt)K(x)\mathcal{H}(x,\rho), \tag{26}$$

$$z_z^{(j)}(x,\rho) = (i\rho)^j |R(x)|^{j-\frac{1}{2}} (\exp(i\frac{\pi}{2}\theta(x)))^j \exp(-\rho\int_0^x |R_-(t)|dt)$$
$$\exp(i\rho\int_0^x |R_+(t)|dt)K^*(x)\mathcal{H}(x,\rho), \tag{27}$$

$$\langle z_1(x,\rho),\, z_2(x,\rho)\rangle = 2i\rho[1]. \tag{28}$$

Here and in the sequel

(i) $[1] = 1 + O(\rho^{-\Theta_0})$ uniformly in $x \in D_\varepsilon$,

(ii) one and the same symbol $\mathcal{H}(x,\rho)$ denotes various functions such that:
- uniformly in $x \in D_\varepsilon$ $\mathcal{H}(x,\rho) = O(1)$ as $|\rho| \to \infty$, $\rho \in S_k$,
- for each fixed $\delta > 0$, $\mathcal{H}(x,\rho) = [1]$ as $|\rho| \to \infty$, $\rho \in S_k^\delta$.

The solutions $z_1(x,\rho)$, $z_2(x,\rho)$ were constructed and studied in [11] for a finite interval. For the half-line arguments are essentially the same.

Using the FSS $\{z_1(x,\rho),\, z_2(x,\rho)\}$ one can study the asymptotic behavior of the solutions $\varphi(x,\lambda)$ and $e(x,\rho)$.
From the initial conditions on $\varphi(x,\lambda)$ we get

$$\varphi(x,\lambda) = \frac{1}{w(\rho)}(U(z_2(x,\rho))z_1(x,\rho) - U(z_1(x,\rho))z_2(x,\rho)), \tag{29}$$

where $w(\rho) = \langle z_1(x,\rho), z_2(x,\rho)\rangle_{x=0}$. Using (26) - (28) we calculate for $|\rho| \to \infty$, $\rho \in S_k$,

$$\left.\begin{array}{l} U(z_1(x,\rho)) = (-i\rho)|R(0)|^{\frac{1}{2}} \exp(i\frac{\pi}{2}\theta(0))\mathcal{H}(\rho), \\ U(z_2(x,\rho)) = (i\rho)|R(0)|^{\frac{1}{2}}\mathcal{H}(\rho), \\ w(\rho) = 2i\rho[1], \end{array}\right\} \tag{30}$$

here and in the sequel $\mathcal{H}(\rho) = O(1)$ for $\rho \in S_k$, and $\mathcal{H}(\rho) = [1]$ for $\rho \in S_k^\delta$ as $|\rho| \to \infty$.
Substituting (26), (27) and (30) into (29) we conclude that for $|\rho| \to \infty$, $\rho \in S_k$, $x \in D_\varepsilon;\ j,k = 0,1$,

$$\begin{array}{l} \varphi^{(j)}(x,\lambda) = \dfrac{1}{2}(-i\rho)^j |R(0)|^{\frac{1}{2}} |R(x)|^{j-\frac{1}{2}} (\exp(i\dfrac{\pi}{2}\theta(x)))^j \exp(\rho\displaystyle\int_0^x |R_-(t)|dt) \\ \exp(-i\rho\displaystyle\int_0^x |R_+(t)|dt)K(x)\mathcal{H}(x,\rho). \end{array} \tag{31}$$

We note that the solution $z_2(x,\rho)$ is constructed such that

$$e(x,\rho) = s(\rho)z_2(x,\rho), \tag{32}$$

where, in accordance with (10) and (27),

$$s(\rho) := (\prod_{\nu=1}^m \gamma_\nu)^{-1} \exp(i\frac{\pi}{4}\theta(0)) \exp(\rho\int_0^{x_m} |R_-(t)|dt) \exp(-i\rho\int_0^{x_m} |R_+(t)|dt). \tag{33}$$

Hence for $|\rho| \to \infty$, $\rho \in S_k$, $x \in D_\varepsilon$; $j, k = 0, 1$,

$$\begin{aligned} e^{(j)}(x,\rho) = (i\rho)^j |R(x)|^{j-\frac{1}{2}} (\prod_{x_\nu > x} \gamma_\nu)^{-1} \exp(-i\frac{\pi}{4}\theta(x))(\exp(i\frac{\pi}{2}\theta(x)))^j \\ \exp(-\rho\int_{x_m}^{x} |R_-(t)|dt) \exp(i\rho\int_{x_m}^{x} |R_+(t)|dt)\,\mathcal{H}(x,\rho). \end{aligned} \tag{34}$$

Remark 2. Let $\xi_\nu(x) = \int_\nu^x |R(t)|dt$. It follows from the results of [11] that (31) and (34) are also valid uniformly for $|\rho\xi_\nu(x)| \ge 1$, $|x - x_\nu| \le \varepsilon$ with $[1] = 1 + O((\rho\xi_\nu(x))^{-\Theta_0})$. Moreover, for $x \ge 0$, $Im\ \rho \ge 0$, $|\rho| \ge \rho^*$ we have the estimates

$$\left.\begin{aligned} |\varphi^{(j)}(x,\lambda)| \le C|\rho|^j |R(x)|^{j-\frac{1}{2}} \exp(|Re\ \ \rho|\int_0^x |R_-(t)|dt) \\ \exp(Im\ \rho\int_0^x |R_+(t)|dt), \\ |e^{(j)}(x,\rho)| \le C|\rho|^j |R(x)|^{j-\frac{1}{2}} \exp(-|Re\ \rho|\int_{x_m}^x |R_-(t)|dt) \\ \exp(-Im\ \rho\int_{x_m}^x |R_+(t)|dt). \end{aligned}\right\} \tag{35}$$

Denote

$$a_\nu = \int_{x_{\nu-1}}^{x_\nu} |R(t)|dt,\ \nu = \overline{1,m},$$

$$J^+ = \{\nu : (x_{\nu-1}, x_\nu) \in I_+\} = \{\nu_1^+, \dots, \nu_p^+\},$$

$$J^- = \{\nu : (x_{\nu-1}, x_\nu) \in I_-\} = \{\nu_1^-, \dots, \nu_q^-\},\ p + q = m,$$

$$\omega_\nu = \begin{cases} -i, & \text{if } \nu \in J^+, \\ 1, & \text{if } \nu \in J^-. \end{cases}$$

From (32) it follows that

$$\Delta(\rho) = s(\rho)U(z_2).$$

By virtue of (30) and (33) we have for $|\rho| \to \infty$, $\rho \in S_k$, $k = 0, 1$,

$$\begin{aligned} \Delta(\rho) = A\rho\exp(\rho\int_0^{x_m} |R_-(t)|dt)\exp(-i\rho\int_0^{x_m} |R_+(t)|dt)\mathcal{H}(\rho) = \\ A\rho\prod_{\nu=1}^m \exp(\rho\omega_\nu a_\nu)\mathcal{H}(\rho), \end{aligned} \tag{36}$$

where

$$A = i|R(0)|^{\frac{1}{2}} (\prod_{\nu=1}^m \gamma_\nu)^{-1} \exp(i\frac{\pi}{4}\theta(0)) \ne 0.$$

Using the more precise asymptotics of $\mathcal{H}(x,\rho)$ for $z_1(x,\rho)$, $z_2(x,\rho)$ from [11] one can obtain the asymptotics for $\mathcal{H}(\rho)$ from (36) and consequently the more precise asymptotics for $\Delta(\rho)$ as $|\rho| \to \infty$, $Im\ \rho \ge 0$:

$$\Delta(\rho) = \rho\sum_{j_1,\dots,j_m=0}^{1} A_{j_1,\dots,j_m}\prod_{\nu=1}^m \exp((-1)^{j_\nu}\rho\omega_\nu a_\nu)[1],\ A_{j_1,\dots,j_m} \ne 0. \tag{37}$$

Let us now study the behavior of $\Delta(\rho)$ near the real and imaginary axes of the ρ-plane.

Case 1. Let $|Re\ \rho| \le Im\ \rho$. Then it follows from (37) that for $|\rho| \to \infty$

$$\Delta(\rho) = \rho \prod_{\nu \in J^+} \exp(-i\rho a_\nu) \sum_{j_1,\dots,j_p=0}^{1} A'_{j_1,\dots,j_p} \prod_{\nu \in J^-} \exp((-1)^{j_\nu} \rho a_\nu)[1], \qquad A'_{j_1,\dots,j_p} \neq 0. \tag{38}$$

By Theorems 2 and 3, zeros of $\Delta(\rho)$ are pure imaginary: $\rho_k = i\tau_k,\ \tau_k > 0,\ k \ge 1$. On the other hand, using the asymptotic formula (38) by a well-known method (see [24]) we deduce that

(i) The number N_a of zeros of $\Delta(\rho)$ in the segment $[a, a+1]$ is bounded with respect to a.

(ii) Denote

$$G_\delta^- = \{\rho : |\rho| \ge \delta,\ |\rho - \rho_k| \ge \delta,\ k \ge 1\} \cap \{\rho : |Re\ \rho| \le Im\ \rho\}.$$

Then

$$|\Delta(\rho)| \ge C_\delta |\rho| \prod_{\nu \in J^+} \exp(a_\nu Im\ \rho) \prod_{\nu \in J^-} \exp(a_\nu |Re\ \rho|),\ \rho \in G_\delta^-.$$

Case 2. Let $|Re\ \rho| \ge Im\ \rho \ge 0$. Then it follows from (37) that for $|\rho| \to \infty$

$$\Delta(\rho) = \rho \prod_{\nu \in J^-} \exp(\chi \rho a_\nu) \sum_{j_1,\dots,j_q=0}^{1} A^{\chi}_{j_1,\dots,j_q} \prod_{\nu \in J^+} \exp((-1)^{j_\nu} i\rho a_\nu)[1], \tag{39}$$

where $A^{\chi}_{j_1,\dots,j_q} \neq 0$ and $\chi =$ sign $(Re\ \rho)$.
Denote by

$$\Delta_0^{\pm}(\rho) := \rho \prod_{\nu \in J^-} \exp(\pm \rho a_\nu) \sum_{j_1,\dots,j_q=0}^{1} A^{\pm}_{j_1,\dots,j_q} \prod_{\nu \in J^+} \exp((-1)^{j_\nu} i\rho a_\nu)$$

the main part of the asymptotics (39) for $\pm Re\ \rho > 0$.

By Theorems 1-3 the function $\Delta(\rho)$ has no zeros for $|Re\ \rho| \ge Im\ \rho \ge 0$. Notice that the functions $\Delta_0^{\pm}(\rho)$, in general, have countable sets of zeros in this region.

Let $\{\rho_k^0\}_{k \ge 0}$ be the zeros of $\Delta_0^+(\rho)$ for $Re\ \rho > 0$, and let $\{\rho_k^0\}_{k<0}$ be the zeros of $\Delta_0^-(\rho)$ for $Re\ \rho < 0$. Then (see [24])

(i) The number $\{\rho_k^0\}$ lie in the strip $|Im\ \rho| < H$.

(ii) The number N_a^1 of zeros $\{\rho_k^0\}$ in the rectangle $B_a = \{\delta : |Im\ \rho| < H,$ $Re\ \rho \in [a, a+1]\}$ is bounded with respect to a.

(iii) Denote

$$G_\delta^+ = \{\rho : |\rho| \ge \delta,\ |\rho - \rho_{k0}^+| \ge \delta,\ k \in \mathbb{Z}\} \cap \{\rho : |Re\ \rho| \ge Im\ \rho \ge 0\},$$
$$G_\delta = G_\delta^- \cup G_\delta^+.$$

Then

$$|\Delta(\rho) \ge C_\delta |\rho| \prod_{\nu \in J^+} \exp(a_\nu Im\ \rho) \prod_{\nu \in J^-} \exp(a_\nu |Re\ \rho|),\ \rho \in G_\delta. \tag{40}$$

(iiii) There exist numbers $R_n \to \infty$ such that for sufficiently small $\delta > 0$ the semicircles $|\rho| = R_n$, $Im\ \rho \ge 0$ lie in G_δ for all n.
Now let us define the matrix

$$P(x,\lambda) = \begin{bmatrix} P_{11}(x,\lambda) & P_{12}(x,\lambda) \\ P_{21}(x,\lambda) & P_{22}(x,\lambda) \end{bmatrix}$$

by the formula

$$P(x,\lambda) \begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{41}$$

Using (14) we calculate

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi(x,\lambda)\tilde{\varphi}'(x,\lambda), \tag{42}$$

$$P_{22}(x,\lambda) = \Phi(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{\Phi}(x,\lambda). \tag{43}$$

According to (13), $\Phi(x,\lambda) = (\Delta(\rho))^{-1} e(x,\rho)$. Using (34) and (36) we obtain for $|\rho| \to \infty$, $\rho \in S_k^\delta$, $x \in D_\varepsilon$; $j,k = 0,1$,

$$\begin{aligned} \Phi^{(j)}(x,\lambda) = (i\rho)^{j-1} |R(0)|^{-\frac{1}{2}} |R(x)|^{j-\frac{1}{2}} (\exp(i\tfrac{\pi}{2}\theta(x))^j \\ \exp(-\rho \int_0^x |R_-(t)|dt) \exp(i\rho \int_0^x |R_+(t)|dt) K^*(x)[1]. \end{aligned} \tag{44}$$

In view of (14), we transform (42) into

$$P_{11}(x,\lambda) = 1 + (\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda))\tilde{\Phi}'(x,\lambda) - (\Phi(x,\lambda) - \tilde{\Phi}(x,\lambda))\tilde{\varphi}'(x,\lambda). \tag{45}$$

Substituting (31) and (44) into (43) and (45) we get for $|\rho| \to \infty$, $\rho \in S^\delta$,
$x \in D_\varepsilon$

$$P_{11}(x,\lambda) - 1 = O(\rho^{-\Theta_0}),\ P_{12}(x,\lambda) = O(\rho^{-\Theta_0}). \tag{46}$$

Note that according to (40) these estimates are also valid for $\rho \in G\delta$.
Further, using (17) we can rewrite (42) and (43) in the form

$$\left.\begin{aligned} P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{\varphi}'(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda), \\ P_{22}(x,\lambda) = S(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{S}(x,\lambda) + (M(\lambda) - \tilde{M}(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda). \end{aligned}\right\}. \tag{47}$$

By the the hypothesis of Theorem 5, $M(\lambda) = \tilde{M}(\lambda)$, and consequently the functions $P_{11}(x, \lambda)$ and $P_{12}(x, \lambda)$ are entire in λ of order $\frac{1}{2}$. Taking (46) into account we conclude that $P_{11}(x, \lambda) \equiv 1, P_{22}(x, \lambda) \equiv 0$. Substituting into (41) we obtain $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$, $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$ for all x and λ. Consequently, $q(x) = \tilde{q}(x)$ for $x \in I$ and $h = \tilde{h}$. Theorem 5 is proved.

4. Inverse Problem 2. In this section we consider the inverse problem of recovering $q(x)$ and h from the given spectral data $S = (V(\lambda), \{\lambda_k, \alpha_k\})$.

Theorem 6. If $S = \tilde{S}$, then $q(x) = \tilde{q}(x)$ for $x \in I$ and $h = \tilde{h}$.

Proof. In the λ-plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where $\gamma' = \{\lambda : |\rho| = \delta\}$ with a sufficiently small $\delta > 0$, and γ'' is a two-sided cut along the arc $\{\lambda : \lambda \geq \delta\}$. Let $\gamma_n = \gamma \cap \{\lambda : |\lambda| \leq r_n\}$, where $r_n = R_n^2$.

Consider the function

$$I_n(z) := \frac{1}{2\pi i} \oint_{|\lambda|=r_n} \frac{M(\lambda)}{z - \lambda} d\lambda, \; z \notin Sp(\mathcal{L}).$$

It follows from (35) and (40) that for $Im\, \rho \geq 0$, $\rho \in G_\delta$,

$$|e(0, \rho)| \leq C \exp(|Re\, \rho| \int_0^{x_m} |R_-(t)|dt) \exp(Im\, \rho \int_0^{x_m} |R_+(t)|dt), \tag{48}$$

$$|\Delta(\rho)| \geq C_\delta |\rho| \exp(|Re\, \rho| \int_0^{x_m} |R_-(t)|dt) \exp(Im\, \rho \int_0^{x_m} |R_+(t)|dt). \tag{49}$$

Using (16), (48) and (49) we have

$$|M(\lambda)| \leq \frac{C_\delta}{|\rho|}, \; \rho \in G_\delta. \tag{50}$$

It follows from (50) that

$$\lim_{n \to \infty} I_n(z) = 0$$

uniformly on compacts in $\Pi \setminus \Lambda$.
On the other hand, deforming the contour $|\lambda| = r_n$ to γ_n and using Theorem 4 and the residue theorem we get

$$I_n(z) = -M(z) + \sum_{|\lambda_k| < r_n} \frac{1}{\alpha_k (z - \lambda_k)} + \frac{1}{2\pi i} \int_{\gamma_n} \frac{M(\lambda)}{z - \lambda} d\lambda,$$

and consequently uniformly on compacts in $\Pi \setminus \Lambda$

$$M(\lambda) = \sum_{k \geq 1} \frac{1}{\alpha_k (\lambda - \lambda_k)} + \int_0^\infty \frac{V(z)}{\lambda - z} dz, \tag{51}$$

where
$$\sum_{k\geq 1} + \int_0^\infty := \lim_{n\to\infty} (\sum_{|\lambda_k|<r_n} + \int_0^{r_n}), \ r_n = R_n^2.$$
Formula (51) is the expansion the Weyl function into its singularities. We can rewrite (51) with the help of the spectral function $\sigma(\lambda)$ defined by (25):
$$M(\lambda) = \int_{-\infty}^{\infty} \frac{d\sigma(z)}{\lambda - z}.$$
By hypothesis of Theorem 6, $S = \tilde{S}$, and consequently $V(\lambda) = \tilde{V}(\lambda)$, $\lambda > 0;\ \lambda_k = \tilde{\lambda_k},\ \alpha_k = \tilde{\alpha_k},\ k \geq 1$. From this and (51) we get $M(\lambda) = \tilde{M}(\lambda)$. Using Theorem 5 we conclude that $q(x) = \tilde{q}(x)$ for $x \in I$, and $h = \tilde{h}$; hence Theorem 6 is proved. □

Remark 3. By virtue of (25), the specification of the spectral data $S = (V(\lambda), \{\lambda_k, \alpha_k\})$ is equivalent to the specification of the spectral function $\sigma(\lambda)$. Thus, the inverse problem from the spectral data considered in this section is equivalent to the inverse problem from the spectral function.

5. Expansion theorem. The purpose of this section is to prove the following theorem.

Theorem 7. Let $f(x)$ be an absolutely continuous and finite. Then uniformly for $x \geq 0$
$$f(x) = \sum_{k\geq 1} \frac{1}{\alpha_k} F(\lambda_k)\varphi(x, \lambda_k) + \int_0^\infty F(\lambda)\varphi(x, \lambda)V(\lambda)d\lambda, \tag{52}$$
where
$$F(\lambda) = \int_0^\infty R^2(t)\varphi(t, \lambda)f(t)dt,$$
$$\sum_{k\geq 1} + \int_0^\infty := \lim_{n\to\infty} (\sum_{|\lambda_k|<r_n} + \int_0^{r_n}, \ r_n = R_n^2.$$
Proof. Consider the function
$$J_n(x) = \frac{1}{2\pi i} \oint_{|\lambda|=r_n} Y(x, \lambda)d\lambda,$$
where $Y(x, \lambda)$ is defined via (23) and (24), i.e.
$$Y(x, \lambda) = \Phi(x, \lambda) \int_0^x R^2(t)\varphi(t, \lambda)f(t)dt + \varphi(x, \lambda) \int_x^\infty R^2(t)\Phi(t, \lambda)f(t)dt. \tag{53}$$
Since $\varphi(x, \lambda)$ and $\Phi(x, \lambda)$ are solutions of (1) we rewrite (53) as follows
$$Y(x, \lambda) = \frac{\Phi(x, \lambda)}{\lambda} \int_0^x (-\varphi''(t, \lambda) + q(t)\varphi(t, \lambda))f(t)dt +$$
$$+ \frac{\varphi(x, \lambda)}{\lambda} \int_x^\infty (-\Phi''(t, \lambda) + q(t)\Phi(t, \lambda))f(t)dt.$$

We carry out integration by parts in the terms with the second derivatives. Using (13), (14), (31), (34), (35) and (40), we obtain

$$Y(x,\lambda)=\frac{f(x)}{\lambda}+\frac{Z(x,\lambda)}{\lambda}$$

where

$$\lim_{n\to\infty}\max_{|\lambda|=r_n}|Z(x,\lambda)|=0$$

uniformly in x. Hence, uniformly in x

$$\lim_{n\to\infty}J_n(x)=f(x). \tag{54}$$

On the other hand, deforming the contour $|\lambda|=r_n$ to γ_n and using the residue theorem we get

$$J_n(x)=\sum_{|\lambda_k|<r_n}\text{Res}_{\,\lambda=\lambda_k}Y(x,\lambda)+\frac{1}{2\pi i}\int_{\gamma_n}Y(x,\lambda)d\lambda. \tag{55}$$

By virtue of (53), (17) and Theorem 4, we calculate

$$\text{Res}_{\,\lambda=\lambda_k}Y(x,\lambda)=\frac{1}{\alpha_k}F(\lambda_k)\varphi(x,\lambda_k),$$

$$\frac{1}{2\pi i}\int_{\gamma_n}Y(x,\lambda)d\lambda=\int_0^{r_n}F(\lambda)\varphi(x,\lambda)V(\lambda)d\lambda.$$

Substituting into (55) and taking (54) into account we arrive at (52). Hence Theorem 7 is proved.

Remark 4. The expansion (52) can be rewritten with the help of the spectral function $\sigma(\lambda)$:

$$f(x)=\int_{-\infty}^{\infty}F(\lambda)\varphi(x,\lambda)d\sigma(\lambda).$$

6. Solution of the inverse problem. Let us now go on to constructing the solution of the inverse problem. The central role for solving the inverse problem is played by the so-called *main equation* of the inverse problem which connects the spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation which is a linear equation in a suitable Banach space. Moreover, we prove the unique solvability of the main equation. For simplicity, we confine ourselves to the most important particular case when $m=1$ and $T_1=IV$, i.e. the weight-function changes sign exactly once. The general case can be treated analogously. For deriving the main equation of the inverse problem we need more precise asymptotics for the solutions of equation (1). For definiteness, everywhere below $\rho\in S_0\cup S_1$ (the other sectors are considered in the same way). It has been shown in [11] that for $|\rho|\to\infty$, $j=0,1$ the following asymptotic formulas are valid

$$\left.\begin{array}{l} z_1^{(j)}(x,\rho)=\rho^j|R(x)|^{j-\frac{1}{2}}\exp(\rho\int_0^x|R(t)|dt)[1],\ x<x_1,\\ z_1^{(j)}(x,\rho)=(-i\rho)^j|R(x)|^{j-\frac{1}{2}}\ \tfrac{1}{2}\csc\dfrac{\pi\mu_1}{2}\exp(i\dfrac{\pi}{4})\exp(\rho\int_0^{x_1}|R(t)|dt)\\ \times(\exp(-i\rho\int_{x_1}^x|R(t)|dt)[1]+(-1)^{j+1}\,i\exp(i\rho\int_{x_1}^x|R(t)|dt)[1]),\ x>x_1 \end{array}\right\}\tag{56}$$

$$\left.\begin{array}{l} z_2^{(j)}(x,\rho)=(-\rho)^j|R(x)|^{j-\frac{1}{2}}(-i\exp(-\rho\int_0^x|R(t)|dt)[1]+\\ +(-1)^j\exp(\rho\int_0^x|R(t)|dt)\exp(-2\rho\int_0^{x_1}|R(t)|dt)[1]),\ x<x_1,\\ z_2^{(j)}(x,\rho)=(i\rho)^j|R(x)|^{j-\frac{1}{2}}\cdot 2\sin\dfrac{\pi\mu_1}{2}\exp(-i\dfrac{\pi}{4})\exp(-\rho\int_0^{x_1}|R(t)|dt)\\ \times\exp(i\rho\int_{x_1}^x|R(t)|dt)[1],\ x>x_1. \end{array}\right\}\tag{57}$$

By virtue of (32) and (57), we have

$$\left.\begin{array}{l} e^{(j)}(x,\rho)=\rho^j|R(x)|^{j-\frac{1}{2}}\exp(-i\dfrac{\pi}{4})\cdot\dfrac{1}{2}\csc\dfrac{\pi\mu_1}{2}((-1)^j\exp(\rho\int_x^{x_1}|R(t)|dt)[1]+\\ +i\exp(-\rho\int_x^{x_1}|R(t)|dt)[1]),\ x<x_1,\\ e^{(j)}(x,\rho)=(i\rho)^j|R(x)|^{j-\frac{1}{2}}\exp(i\rho\int_{x_i}^x|R(t)|dt[1],\ x>x_1. \end{array}\right\}\tag{58}$$

Further, from (29), in view of (28), (56) and (57), it follows that

$$\left.\begin{array}{l} \varphi^{(j)}(x,\lambda)=\dfrac{1}{2}\rho^j|R(0)|^{\frac{1}{2}}|R(x)|^{j-\frac{1}{2}}(\exp(\rho\int_0^x|R(t)|dt)[1]+\\ (-1)^j\exp(-\rho\int_0^x|R(t)|dt)[1]),\ x<x_1,\\ \varphi^{(j)}(x,\lambda)=\dfrac{1}{2}(i\rho)^j|R(0)|^{\frac{1}{2}}|R(x)|^{j-\frac{1}{2}}(A_1(\rho)\exp(i\rho\int_{x_1}^x|R(t)|dt[1]+\\ (-1)^jA_2(\rho)\exp(-i\rho\int_{x_1}^x|R(t)|dt)[1]),\ x>x_1 \end{array}\right\}\tag{59}$$

where

$$\left.\begin{array}{l} A_2(\rho)=\dfrac{1}{2}\csc\dfrac{\pi\mu_1}{2}\exp(i\dfrac{\pi}{4})(\exp(\rho\int_0^{x_1}|R(t)|dt[1]-\\ \quad i\exp(-\rho\int_0^{x_1}|R(t)|dt)[1]),\\ A_1(\rho)=-iA_2(\rho)+2\sin\dfrac{\pi\mu_1}{2}\exp(i\dfrac{\pi}{4})\exp(-\rho\int_0^{x_1}|R(t)|dt)[1] \end{array}\right\}\tag{60}$$

Remark 5. It was shown in [17] that $A_2(\rho)\equiv C\,\Delta(\rho)$.

Remark 6. Let $\xi = \int_{x_1}^{x} |R(t)|dt$. It follows from the results of [11] that (56)-(59) are also valid uniformly for $|\rho\xi| \geq 1$ with $[1] = 1 + O((\rho\xi)^{-\Theta_0}$, moreover for $|\rho\xi| \leq 1$ we have the estimates

$$\left.\begin{aligned} &|\varphi^{(j)}(x,\lambda)| \leq C|\rho|^j |R(x)|^{j-\frac{1}{2}} |\exp(\rho \int_0^x |R(t)|dt)|, \ x < x_1 \\ &|\varphi^{(j)}(x,\lambda)| \leq C|\rho|^j |R(x)|^{j-\frac{1}{2}} |\exp(\rho \int_0^{x_1} |R(t)|dt)| \ x > x_1 \end{aligned}\right\} \tag{61}$$

By (58),

$$\begin{aligned} &e(0,\rho) = |R(0)|^{-\frac{1}{2}} \exp(-i\frac{\pi}{4}) \cdot \frac{1}{2} \csc\frac{\pi\mu_1}{2} (\exp(\rho \int_0^{x_1} |R(t)|dt)[1] \\ &+ i \exp(-\rho \int_0^{x_1} |R(t)|dt)[1]), \end{aligned} \tag{62}$$

$$\begin{aligned} &\Delta(\rho) = \rho|R(0)|^{\frac{1}{2}} \exp(-i\frac{\pi}{4}) \cdot \frac{1}{2} \csc\frac{\pi\mu_1}{2} (-\exp(\rho \int_0^{x_1} |R(t)|dt) + \\ &i \exp(-\rho \int_0^{x_1} |R(t)|dt)[1]). \end{aligned} \tag{63}$$

It follows from (63) that there is a countable set of eigenvalues $\lambda_k = \rho_k^2$ such that

$$\rho_k = i(k\pi a^{-1} + \frac{\pi}{4} + O(k^{-\Theta_0})), \ k \to \infty \tag{64}$$

where

$$a = \int_0^{x_1} |R(t)|dt.$$

Substituting (64) into (18) we obtain

$$\beta_k = e(0,\rho_k) = 2|R(0)|^{-\frac{1}{2}} \cdot \frac{1}{2} \csc\frac{\pi\mu_1}{2} (1 + O(k^{-\Theta_0})). \tag{65}$$

Denote $G_\delta = \{\rho : |\rho| \geq \delta, \ |\rho - \rho_k| \geq \delta, \ k \geq 1\}$. Then (40) becomes

$$|\Delta(\rho)| \geq C_\delta |\rho| \exp(a|Re\,\rho|), \ \rho \in G_\delta. \tag{66}$$

Lemma 2. For $k \to \infty$

$$\alpha_k = -\frac{1}{2}|R(0)|a(1 + O(k^{-\Theta_0})) \tag{67}$$

Proof. First we consider

$$\alpha_{k1} := \int_0^{x_1} R^2(x)\varphi^2(x,\lambda_k)dx.$$

Denote $I_{k1} = \{x \in [0,x_1] : |\rho_k\xi| \geq 1\}$, $I_{k2} = \{x \in [0,x_1] : |\rho_k\xi| \leq 1\}$, where $\xi = \int_{x_1}^{x} |R(t)|dt$. According to (59).

$$\begin{aligned} &\int_{I_{k1}} R^2(x)\varphi^2(x,\lambda_k)dx = -\frac{1}{4}|R(0)| \int_{I_{k1}} |R(x)| \cdot (\exp(\rho_k \int_0^x |R(t)|dt)[1] + \\ &\exp(-\rho_k \int_0^x |R(t)|dt)[1])^2 dx. \end{aligned}$$

A change of variables gives

$$\int_{I_{k1}} R^2(x)\varphi^2(x,\lambda_k)dx = -\frac{1}{4}|R(0)|\int_{\frac{1}{|\rho_k|}} (\exp(\rho_k(a-\xi))(1+O((\rho_k\xi)^{-\Theta_0}))$$
$$+\exp(-\rho_k(a-\xi))(1+O((\rho_k\xi)^{-\Theta_0})))^2 d\xi.$$

Hence

$$\int_{I_{k1}} R^2(x)\varphi^2(x,\lambda_k)dx = -\frac{1}{2}|R(0)|a(1+O(k^{-\Theta_0})).$$

Further, it follows from (61) that

$$|\int_{I_{k2}} R^2(x)\varphi^2(x,\lambda_k)dx| \le \int_{I_{k2}} |R(x)\exp(2\rho_k\int_0^x |R(t)|dt)|dx.$$

In view of (64), the exponential is bounded here and consequently

$$|\int_{I_{k2}} R^2(x)\varphi^2(x,\lambda_k)dx| \le \int_{I_{k2}} |R(x)|dx = O(\frac{1}{k}).$$

Thus, we arrive at

$$\alpha_{k1} = -\frac{1}{2}|R(0)|a(1+O(k^{-\Theta_0})). \tag{68}$$

Let us now estimate

$$\alpha_{k2} := \int_{x_1}^{\infty} R^2(x)\varphi^2(x,\lambda_k)dx.$$

Denote $J_{k1} = \{x > x_1 : |\xi\rho_k| \ge 1\},\ J_{k2} = \{x > x_1 : |\xi\rho_k| \le 1\}$. In the same way as above one can show that

$$|\int_{J_{k2}} R^2(x)\varphi^2(x,\lambda_k)dx| = O(\frac{1}{k}).$$

This and (19) yields

$$\alpha_{k2} = \frac{1}{\beta_k^2}\int_{J_{k1}} R^2(x)e^2(x,\rho_k)dx + O(\frac{1}{k}).$$

Using (58) we calculate

$$\alpha_{k2} = \frac{1}{\beta_k^2}\int_{J_{k1}} |R(x)|\exp(2i\rho_k\int_{x_1}^x |R(t)|dt)dx + O(\frac{1}{k}).$$

A change of variables leads to

$$\alpha_{k2} = \frac{1}{\beta_k^2}\int_{\frac{1}{|\rho_k|}}^{\infty} \exp(2i\rho_k\xi)d\xi + O(\frac{1}{k}).$$

Taking (64) and (65) into account we get

$$\alpha_{k2} = O(\frac{1}{k}).$$

Combining this with (68) we arrive at (67). Lemma 2 is proved.

Now we go on to the derivation of the main equation of the inverse problem. We assume that the spectral data $S = (V(\lambda), \{\lambda_k, \alpha_k\})$ of $\mathcal{L}$ are given. Let $\tilde{\mathcal{L}} = \mathcal{L}(R^2(x),\ \tilde{q}(x), \tilde{h})$ be a certain known model boundary value problem with the same weight-function $R^2(x)$ and with the spectral data $\tilde{S} = (\tilde{V}(\lambda), \{\tilde{\lambda}_k, \tilde{\alpha}_k\})$.

Lemma 3. For each fixed $x \in I,\ x \neq x_1$, the following relations hold

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int_0^\infty \langle \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\mu) \rangle}{\lambda - \mu} \hat{V}(\mu) \varphi(x,\mu) d\mu +$$
$$\sum_{k \geq 1} \Big(\frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\lambda_k) \rangle}{\alpha_k(\lambda - \lambda_k)} \varphi(x,\lambda_k) - \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\tilde{\lambda}_k) \rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)} \varphi(x,\tilde{\lambda}_k) \Big), \tag{69}$$

$$\frac{\langle \varphi(x,\lambda), \varphi(x,\mu) \rangle}{\lambda - \mu} - \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\mu) \rangle}{\lambda - \mu} +$$
$$\int_0^\infty \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\xi) \rangle}{\lambda - \xi} \cdot \frac{\langle \varphi(x,\xi), \varphi(x,\mu) \rangle}{\xi - \mu} \hat{V}(\xi) d\xi +$$
$$+ \sum_{k \geq 1} \Big(\frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\lambda_k) \rangle}{\alpha_k(\lambda - \lambda_k)} \cdot \frac{\langle \varphi(x,\lambda_k), \varphi(x,\mu) \rangle}{\lambda_k - \mu} - \tag{70}$$
$$- \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\tilde{\lambda}_k) \rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)} \cdot \frac{\langle \varphi(x,\tilde{\lambda}_k), \varphi(x,\mu) \rangle}{\tilde{\lambda}_k - \mu} \Big),$$

where $\hat{V}(\lambda) = V(\lambda) - \tilde{V}(\lambda)$, and

$$\sum_{k \geq 1} + \int_0^\infty := \lim_{n \to \infty} \Big(\sum_{|\lambda_k| < r_n} + \int_0^{r_n} \Big).$$

Proof. Let $P(x,\lambda)$ be the matrix introduced above in (41). In the λ-plane we consider a oriented contour $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^\pm = \{\lambda : \pm Im\,\lambda = \delta,\ -\infty < \mp Re\,\lambda < \infty\}$ with a sufficiently small fixed $\delta > 0$. Put $J_\gamma = \{\lambda : |Im\,\lambda| > \delta\}$, and the contour $\gamma_n = (\gamma \cap \{\lambda : |\lambda| \leq r_n\}) \cup \{\lambda : |\lambda| = r_n,\ \lambda \notin J_\gamma\}$ with clockwise orientation.

By Cauchy's theorem, for $\lambda, \mu \in J_\gamma \cap \{\xi : |\xi| < r_n\}$

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_n} \frac{P_{1k}(x,\xi) - \delta_{1k}}{\lambda - \xi} d\xi,$$
$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_{\gamma_n} \frac{P_{jk}(x,\xi) d\xi}{(\lambda - \xi)(\xi - \mu)},$$

where δ_{jk} is the Kronecker delta. It follows from (13), (41), (43), (45), (58), (59), (66), that

$$P_{jk}(x,\lambda) = O(1),\ P_{1k}(x,\lambda) - \delta_{1k} = O(\rho^{-\Theta_0}),\ \rho \in G_\delta \cap \tilde{G}_\delta.$$

Therefore

$$\lim_{n\to\infty}\int_{|\xi|=r_n}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\lambda-\xi}d\xi=0,\ \lim_{n\to\infty}\int_{|\xi|=r_n}\frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}d\xi=0,$$

and consequently

$$P_{1k}(x,\lambda)=\delta_{1k}+\frac{1}{2\pi i}\int_\gamma\frac{P_{1k}(x,\xi)}{(\lambda-\xi)}\,d\xi,\ \lambda\in J_\gamma, \tag{71}$$

$$\frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu}=\frac{1}{2\pi i}\int_\gamma\frac{P_{jk}(x,\xi)d\xi}{(\lambda-\xi)(\xi-\mu)},\ \lambda,\mu\in J_\gamma, \tag{72}$$

where $\displaystyle\int_\gamma:=\lim_{n\to\infty}\int_{\gamma_n}$.

It follows from (41) that

$$\varphi(x,\lambda)=P_{11}(x,\lambda)\tilde\varphi(x,\lambda)+P_{12}(x,\lambda)\tilde\varphi'(x,\lambda).$$

Then, in view of (71), we get

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma(\tilde\varphi(x,\lambda)P_{11}(x,\xi)+\tilde\varphi'(x,\lambda)P_{12}(x,\xi))\frac{d\xi}{\lambda-\xi}.$$

From this, by virtue of (42) and (43), we infer

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma(\tilde\varphi(x,\lambda)(\varphi(x,\xi)\tilde\Phi'(x,\xi)-\Phi(x,\xi)\tilde\varphi'(x,\xi))+$$

$$\tilde\varphi'(x,\lambda)(\Phi(x,\xi)\tilde\varphi(x,\xi)-\varphi(x,\xi)\tilde\Phi(x,\xi)))\frac{d\xi}{\lambda-\xi}.$$

Using (17), Theorem 4 and the residue theorem we arrive at (69).
From (41) and (14) it follows that

$$\left.\begin{array}{l}P_{11}(x,\lambda)\varphi'(x,\lambda)+P_{21}(x,\lambda)\varphi(x,\lambda)=\tilde\varphi'(x,\lambda),\\ P_{22}(x,\lambda)\varphi(x,\lambda)-P_{12}(x,\lambda)\varphi'(x,\lambda)=\tilde\varphi(x,\lambda),\end{array}\right\} \tag{73}$$

$$P(x,\lambda)\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}=\langle y(x),\,\tilde\Phi(x,\lambda)\rangle\begin{bmatrix}\varphi(x,\lambda)\\ \varphi'(x,\lambda)\end{bmatrix}-\langle y(x),\,\tilde\varphi(x,\lambda)\rangle\begin{bmatrix}\Phi(x,\lambda)\\ \Phi'(x,\lambda)\end{bmatrix} \tag{74}$$

for any smooth $y(x)$. Taking (72) and (74) into account, we calculate

$$\frac{P(x,\lambda)-P(x,\mu)}{\lambda-\mu}\begin{bmatrix}y(x)\\ y'(x)\end{bmatrix}=\frac{1}{2\pi i}\int_\gamma\Big(\langle y(x),\,\tilde\Phi(x,\xi)\rangle\begin{bmatrix}\varphi(x,\xi)\\ \varphi'(x,\xi)\end{bmatrix}-$$
$$\langle y(x),\,\tilde\varphi(x,\xi)\rangle\begin{bmatrix}\Phi(x,\xi)\\ \Phi'(x,\xi)\end{bmatrix}\Big)\frac{d\xi}{(\lambda-\xi)(\xi-\mu)}. \tag{75}$$

Using (73) we get

$$\det\left((P(x,\lambda)-P(x,\mu))\begin{bmatrix}\tilde\varphi(x,\lambda)\\ \tilde\varphi'(x,\lambda)\end{bmatrix},\begin{bmatrix}\varphi(x,\lambda)\\ \varphi'(x,\lambda)\end{bmatrix}\right)=\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle-\langle\tilde\varphi(x,\lambda),\,\tilde\varphi(x,\mu)\rangle,$$

and consequently, in view of (75), we derive

$$\frac{\langle\varphi(x,\lambda),\,\varphi(x,\mu)\rangle}{\lambda-\mu}-\frac{\langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\mu)\rangle}{\lambda-\mu}=$$

$$\frac{1}{2\pi i}\int_\gamma\Big(\frac{\langle\tilde\varphi(x,\lambda),\tilde\Phi(x,\xi)\rangle\langle\varphi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}-\frac{\langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\xi)\rangle\langle\Phi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}\Big)d\xi.$$

By virtue of (17), Theorem 4 and the residue theorem, we arrive at (70), hence Lemma 3 is proved. □

Denote $\lambda_{k0}=\lambda_k,\ \lambda_{k1}=\tilde{\lambda_k},\ \alpha_{k0}=\alpha_k,\ \alpha_{k1}=\tilde{\alpha_k},$

$$\varphi_{kj}(x)=\varphi(x,\lambda_{kj}),\ \tilde\varphi_{kj}(x)=\tilde\varphi(x,\lambda_{kj}),\ \varphi_\lambda(x)=\varphi(x,\lambda),\ \tilde\varphi_\lambda(x)=\tilde\varphi(x,\lambda),$$

$$\begin{aligned}
P_{\lambda,\mu}(x)&=\frac{\langle\varphi_\lambda(x),\varphi_\mu(x)\rangle}{\lambda-\mu}\hat V(\mu) &&= \int_0^x R^2(t)\varphi_\lambda(t)\varphi_\mu(t)dt\cdot\hat V(\mu),\ \lambda,\mu>0\\
P_{\lambda,kj}(x)&=\frac{\langle\varphi_\lambda(x),\varphi_{kj}(x)\rangle}{\alpha_{kj}(\lambda-\lambda_{kj})} &&= \frac{1}{\alpha_{kj}}\int_0^x R^2(t)\varphi_\lambda(t)\varphi_{kj}(t)dt,\ \lambda>0\\
P_{ni,\mu}(x)&=\frac{\langle\varphi_{ni}(x),\varphi_\mu(x)\rangle}{\lambda_{ni}-\mu}\hat V(\mu) &&= \int_0^x R^2(t)\varphi_{ni}(t)\varphi_\mu(t)dt\hat V(\mu),\ \mu>0\\
P_{ni,kj}(x)&=\frac{\langle\varphi_{ni}(x),\varphi_{kj}(x)\rangle}{\alpha_{kj}(\lambda_{ni}-\lambda_{kj})} &&= \frac{1}{\alpha_{kj}}\int_0^x R^2(t)\varphi_{ni}(t)\varphi_{kj}(t)dt.
\end{aligned}$$

We define $\tilde P_{\lambda,\mu}(x)$, $\tilde P_{\lambda,kj}(x)$, $\tilde P_{ni,\mu}(x)$, $\tilde P_{ni,kj}(x)$ by the same formulas but with $\tilde\varphi$ instead of φ.
It follows from (69) and (70) that

$$\begin{aligned}&\tilde\varphi_\lambda(x)=\varphi_\lambda(x)+\int_0^\infty\tilde P_{\lambda,\mu}(x)\varphi_\mu(x)d\mu\\ &+\sum_{k\ge1}(\tilde P_{\lambda,k0}(x)\varphi_{k0}(x)-\tilde P_{\lambda,ki}(x)\varphi_{k1}(x)),\ \lambda>0\end{aligned}\tag{76}$$

$$\begin{aligned}&\tilde\varphi_{ni}(x)=\varphi_{ni}(x)+\int_0^\infty\tilde P_{ni,\mu}(x)\varphi_\mu(x)d\mu\\ &+\sum_{k\ge1}(\tilde P_{ni,k0}(x)\varphi_{k0}(x)-\tilde P_{ni,k1}(x)\varphi_{k1}(x))\end{aligned}\tag{77}$$

$$\begin{aligned}&P_{\lambda,\mu}(x)-\tilde P_{\lambda,\mu}(x)+\int_0^\infty\tilde P_{\lambda,\xi}(x)P_{\xi,\mu}(x)d\xi\\ &+\sum_{s\ge1}(\tilde P_{\lambda,s0}(x)P_{s0,\mu}(x)-\tilde P_{\lambda,s1}(x)P_{s1,\mu}(x))=0\end{aligned}\tag{78}$$

$$\begin{aligned}&P_{\lambda,kj}(x)-\tilde P_{\lambda,kj}(x)+\int_0^\infty\tilde P_{\lambda,\xi}(x)P_{\xi,kj}(x)d\xi\\ &+\sum_{s\ge1}(\tilde P_{\lambda,s0}(x)P_{s0,kj}(x)-\tilde P_{\lambda,s1}(x)P_{s1,kj}(x))=0\end{aligned}\tag{79}$$

$$\begin{aligned}&P_{ni,\mu}(x)-\tilde{P}_{ni,\mu}(x)+\int_0^\infty \tilde{P}_{ni,\xi}(x)P_{\xi,\mu}(x)d\xi\\&+\sum_{s\geq 1}(\tilde{P}_{ni,s0}(x)P_{s0,\mu}(x)-\tilde{P}_{ni,s1}(x)P_{s1,\mu}(x))=0\end{aligned}\tag{80}$$

$$\begin{aligned}&P_{ni,kj}(x)-\tilde{P}_{ni,kj}(x)+\int_0^\infty \tilde{P}_{ni,\xi}(x)P_{\xi,kj}(x)d\xi\\&+\sum_{s\geq 1}(\tilde{P}_{ni,s0}P_{s0,kj}(x)-\tilde{P}_{ni,s1}(x)P_{s1,kj}(x))=0.\end{aligned}\tag{81}$$

Denote

$$\xi_k := |\rho_k-\tilde{\rho}_k|+|\alpha_k-\tilde{\alpha}_k|.$$

Clearly, $\xi_k = O(k^{-\Theta_0})$.

Let for definiteness $x < x_1$. It follows from (59), (64) and Schwarz's lemma that

$$\varphi_\lambda(x) = O(\exp(\rho\int_0^x |R(x)|dt)),\ \rho\geq 0 \tag{82}$$

$$\varphi_{kj}(x)=O(1),\ \varphi_{k0}(x)-\varphi_{k1}(x)=O(\rho_k-\tilde{\rho}_k),\ \ k\geq 1, j=0,1. \tag{83}$$

Using (22) and (63) we calculate

$$V(\lambda)=O(\frac{1}{\rho}\exp(-2\rho\int_0^{x_1}|R(t)|dt)),\ \rho\geq 0. \tag{84}$$

It follows from (59), (61), (64), (67) and (84) that for $n,k\geq 1;\ i,j=0,1;$ $\rho,\theta\geq 0$ $(\lambda=\rho^2,\mu=\theta^2)$,

$$\left.\begin{aligned}&P_{\lambda,\mu}(x)=O(\frac{1}{(|\rho-\theta|+1)(\theta+1)}\exp(\rho\int_0^x|R(t)|dt)\exp(\theta\int_0^x|R(t)|dt)\\&\exp(-2\theta\int_0^{x_1}|R(t)|dt)),\\&P_{\lambda,kj}(x)=O(\frac{1}{\rho+k}\exp(\rho\int_0^x|R(t)|dt)),\\&P_{ni,\mu}(x)=O(\frac{1}{(\theta+n)(\theta+1)}\exp(\theta\int_0^x|R(t)|dt)\exp(-2\theta\int_0^{x_1}|R(t)|dt)),\\&P_{ni,kj}(x)=O(\frac{1}{|n-k|+1}).\end{aligned}\right\}\tag{85}$$

Denote

$$\Omega_\lambda(x):=\exp(\rho\int_0^x|R(t)|dt),\ \rho\geq 0$$

and consider the functions

$$\psi_\lambda(x) = \Omega_\lambda^{-1}(x)\varphi_\lambda(x),$$

$$\psi_{k0}(x) = (\varphi_{ko}(x) - \varphi_{k1}(x))\xi_k^{-1},\ \psi_{k1}(x) = \varphi_{k1}(x),$$

$$H_{\lambda,\mu}(x) = \Omega_\lambda^{-1}(x)P_{\lambda,\mu}(x)\Omega_\mu(x),$$

$$H_{\lambda,k0}(x) = \Omega_\lambda^{-1}(x)P_{\lambda,k0}(x)\xi_k,\ H_{\lambda,k1}(x) = \Omega_\lambda^{-1}(x)(P_{\lambda,k0}(x) - P_{\lambda,k1}(x)),$$

$$H_{n0,\mu}(x) = (P_{n0,\mu})(x) - P_{n1,\mu}(x))\Omega_\mu(x)\xi_n^{-1},\ H_{n1,\mu}(x) = P_{n1,\mu}(x)\Omega_\mu(x),$$

$$H_{n0,k0}(x) = (P_{n0,k0}(x) - P_{n1,k0}(x))\xi_n^{-1}\xi_k,$$

$$H_{n1,k0}(x) = P_{n1,k0}(x)\xi_k,$$

$$H_{n0,k1}(x) = (P_{no,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x))\xi_n^{-1},$$

$$H_{n1,k1}(x) = P_{n1,k0}(x) - P_{n1,k1}(x),$$

$$n, k \geq 1;\ i, j = 0, 1;\ \rho, \theta \geq 0\ (\lambda = \rho^2, \mu = \theta^2).$$

Analogously we define $\tilde{\psi}_\lambda(x), \tilde{\psi}_{kj}(x), \tilde{H}_{\lambda,\mu}(x), \tilde{H}_{\lambda,kj}(x), \tilde{H}_{ni,\mu}(x)$ and $\tilde{H}_{ni,kj}(x)$. It follows from (59), (61), (82) - (85) and Schwarz's lemma that

$$|\psi_\lambda(x)| \leq C,\ |\psi_{kj}(x)| \leq C,$$

$$\left.\begin{array}{l} |H_{\lambda,\mu}(x)| \leq \dfrac{C}{(|\rho-\theta|+1)(\theta+1)} \exp(-2\theta \displaystyle\int_x^{x_1} |R(t)|dt),\ |H_{\lambda,k1}(x)| \leq \dfrac{C\xi_k}{\rho+k}, \\ |H_{ni,\mu}(x)| \leq \dfrac{C}{(\theta+n)(\theta+1)} \exp(-2\theta \displaystyle\int_x^{x_1} |R(t)|dt),\ |H_{ni,kj}(x)| \leq \dfrac{C\xi_k}{|n-k|+1}, \\ n, k \geq 1;\ i, j = 0, 1;\ \rho, \theta \geq 0\ (\lambda = \rho^2, \mu = \theta^2), \end{array}\right\} \tag{86}$$

and consequently,

$$\left.\begin{array}{l} \sup\limits_{\lambda \geq 0}\{\displaystyle\int_0^\infty |H_{\lambda,\mu}(x)|d\mu + \sum_{k,j} |H_{\lambda,kj}(x)|\} < C, \\ \sup\limits_{n,i}\{\displaystyle\int_0^\infty |H_{ni,\mu}(x)d\mu + \sum_{k,j} |H_{ni,kj}(x)|\} < C. \end{array}\right\} \tag{87}$$

The same estimates are also valid for $\tilde{\psi}_\lambda(x), \tilde{\psi}_{kj}(x), \tilde{H}_{\lambda,\mu}(x), \tilde{H}_{ni,\mu}(x), \tilde{H}_{\lambda,kj}(x), \tilde{H}_{ni,kj}(x)$.

Let ω be the set of indices $v = (k, j),\ k \in I\!N; j = 0, 1$. Consider the Banach space m of bounded sequences $\alpha = [\alpha_v]_{v\in\omega}$ with the norm $\|\alpha\|_m = \sup_{v\in\omega} |\alpha_v|$.
Define the vectors

$$\psi(x) = [\psi_v(x)]_{v\in\omega} = \begin{bmatrix} \psi_{k0}(x) \\ \psi_{k1}(x) \end{bmatrix}_{k\in I\!N},$$

$$\tilde{\psi}(x) = [\tilde{\psi}_v(x)]_{v\in\omega} = \begin{bmatrix} \tilde{\psi}_{k0}(x) \\ \tilde{\psi}_{k1}(x) \end{bmatrix}_{k\in I\!N}.$$

It follows from (86) that for each fixed $x \in (0, x_1)$,

$$\psi(x), \tilde{\psi}(x) \in m.$$

Let $\mathcal{C} = C[0, \infty)$ be the Banach space of continuous bounded functions $f = f_\lambda$ on the half-line $\lambda \geq 0$ with the norm $\|f\|_c = \sup_{\lambda \geq 0} |f_\lambda|$. It follows from (86) that for each fixed $x \in (0, x_1)$

$$\psi_\lambda(x), \tilde{\psi}_\lambda(x) \in \mathcal{C}.$$

Consider the Banach space B of vectors

$$F = \begin{bmatrix} f \\ \alpha \end{bmatrix}$$

where $f = f_\lambda \in \mathcal{C},\ \alpha = [\alpha_v]_{v \in \omega} \in m$, with the norm $\|F\|_B = \max(\|f\|_c, \|\alpha\|_m)$. Denote

$$\Psi(x) = \begin{bmatrix} \psi_\lambda(x) \\ \psi(x) \end{bmatrix},\ \tilde{\Psi}(x) = \begin{bmatrix} \tilde{\psi}_\lambda(x) \\ \tilde{\psi}(x) \end{bmatrix}.$$

Clearly, $\Psi(x), \tilde{\Psi}(x) \in B$ for each fixed $x \in (0, x_1)$.
For fixed x, let $\tilde{Q} = \tilde{Q}(x) : B \to B$ be the operator acting from B to B by the formulas

$$F^* = \tilde{Q}F,\ F = \begin{bmatrix} f \\ \alpha \end{bmatrix} \in B,\ F^* = \begin{bmatrix} f^* \\ \alpha^* \end{bmatrix} \in B,$$

$$f^*_\lambda = \int_0^\infty \tilde{H}_{\lambda,\mu} f_\mu d\mu + \sum_{v \in \omega} \tilde{H}_{\lambda,v}\alpha_v,$$

$$\alpha^*_u = \int_0^\infty \tilde{H}_{u,\mu} f_\mu d\mu + \sum_{v \in \omega} \tilde{H}_{u,v}\alpha_v,$$

$$\lambda, \mu \geq 0;\ u = (n, i),\ v = (k, j);\ n, k \geq 1;\ i, j = 0, 1.$$

Analogously we define the operator $Q = Q(x)$.

It follows from (87) that for each fixed $x \in (0, x_1)$ the operators $E + \tilde{Q}(x)$ and $E - Q(x)$ (here E is the identity operator), acting from B to B, are linear bounded operators.

Theorem 8. For each fixed $x \in (0, x_1)$, the vector $\Psi(x) \in B$ is the solution of the equation

$$\tilde{\Psi}(x) = (E + \tilde{Q}(x))\Psi(x) \tag{88}$$

in the Banach space B. The operator $E + \tilde{Q}(x)$ has a bounded inverse operator, i.e. equation (88) is uniquely solvable.

Indeed, taking into account our notations, we can rewrite (76) - (81) in the form

$$\tilde{\Psi}(x) = (E + \tilde{Q}(x)\Psi(x),\ (E + \tilde{Q}(x))(E - Q(x)) = E.$$

Interchanging places for $\mathcal{L}$ and $\tilde{\mathcal{L}}$ we obtain analogously

$$\Psi(x) = (E - Q(x))\tilde{\Psi}(x),\ (E - Q(x))(E + \tilde{Q}(x)) = E.$$

Hence the operator $(E + \tilde{Q}(x))^{-1}$ exists, and it is a linear bounded operator.

Equation (88) is called the main equation of the inverse problem. Solving (88) we find the vector $\Psi(x)$, and consequently, the functions $\varphi_\lambda(x), \varphi_{ni}(x)$. Since $\varphi_\lambda(x)$ and $\varphi_{ni}(x)$ are solutions of (1), we can construct the function $q(x)$ for $x \in (0, x_1)$ and the coefficient h. Thus, the inverse problem has been solved for the interval $x \in (0, x_1)$. For $x > x_1$ the arguments are similar.

Remark 7. To construct $q(x)$ for $x \in (x_1, 1)$ we can act also in another way. Suppose that, using the main equation of the inverse problem, we have constructed $q(x)$ for $x \in (0, x_1)$ and h. Consequently, the solutions $\varphi(x, \lambda)$ and $S(x, \lambda)$ are known for $x \in [0, x_1]$. By virtue of (17), the solution $\Phi(x, \lambda)$ is known for $x \in [0, x_1]$ too. Denote

$$M_1(\lambda) = \frac{\Phi(x_1, \lambda)}{\Phi'(x_1, \lambda)}.$$

The function $M_1(\lambda)$ is the Weyl function for the interval (x_1, ∞). Thus, we can reduce our problem to the inverse problem for (x_1, ∞). In this interval the weight-function $R^2(x)$ does not change sign. We can treat this inverse problem by the same method as above. In this case the main equation will be simpler. We note that the case when the weight-function does not change sign was studied in more general case in [19] and other papers.

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