Nonsymmetric discrete-time difference and algebraic Riccati equations: Some representation formulae and comments

G. Freiling* and V. Ionescu†

**Universität Duisburg, D-47048 Duisburg, Germany*
† Faculty of Automatic Control and Computers, University Polytechnica Bucharest, Bucharest, Romania

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ABSTRACT

A representation formula for the solution of the Riccati difference equation

$$W(k+1) = -M_{21} - M_{22}\, W(k)\, (I - M_{12}\, W(k))^{-1}\, M_{11}\ ;\ W(0) = W_0,$$

is derived. It reflects precisely the dependence of $W(k)$ on the initial value W_0.
A parametrization of all solutions to the corresponding algebraic Riccati equation and results on the dynamic behavior
of the sequence $(W(k))$ are provided as well.

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1. INTRODUCTION

We shall analyze first the solutions of the linear matrix difference equation

$$Z(k+1) = T\, Z(k), \quad Z(0) = Z_0, \quad k \geq 0. \tag{1.1}$$

Subsequently we use these results for the investigation of the asymptotic behavior of the solution of the associated matrix Riccati difference equation

$$W(k+1) = -M_{21} - M_{22}\, W(k)\, (I - M_{12}\, W(k))^{-1}\, M_{11}\ , \quad W(0) = W_0. \tag{1.2}$$

In Section 2 we recall that (1.2) and the linear system (1.1) are, in a restricted sense, equivalent and that the solutions of the algebraic matrix Riccati equation

$$0 = -M_{21} - M_{22}\, W\, (I - M_{12}\, W)^{-1} M_{11} \tag{1.3}$$

can be determined from certain n-dimensional T-invariant subspaces of $\mathbb{C}^{n+m}$.
A representation formula for the solutions of (1.2) in dependence on the initial matrix W_0 is derived in an elementary way in Section 3. This formula allows us to give a nice and complete description of all solutions of (1.3) and of the behavior

of the sequence $(W(k))$ for $k \to \infty$; see Section 4.
These results extend earlier ones obtained in [4] for the solutions of (1.2) and (1.3) with coefficients M_{12} and M_{21} of rank 1.
It turns out that each solution $(W(k))$ of (1.2), if it is defined for all $k \geq 0$, is either convergent to a finite or infinite limit or it is asymptotically periodic or almost periodic. Similar results have been obtained for nonsymmetric matrix Riccati differential equations in [16] and [8], where further detailed results on the geometry of matrix Riccati equations can be found.
Moreover we use the results of Section 3 for the investigation of the solution of time-varying Riccati difference equations with asymptotically constant coefficients.

2. PRELIMINARY RESULTS AND NOTATIONS

In this section we introduce some notations used in this paper and we state some basic facts for linear matrix difference equations and in particular for general matrix Riccati difference equations. By

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$$

we denote an arbitrary matrix where the submatrices $M_{11}, M_{12}, M_{21}, M_{22}$ are of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. If $\det M_{11} \neq 0$, then we associate with M the matrix

$$(T(M) =)\, T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{21} \end{pmatrix}$$

$$:= \begin{pmatrix} I_n & 0 \\ -M_{21} & I_m \end{pmatrix} \begin{pmatrix} M_{11}^{-1} & 0 \\ 0 & -M_{22} \end{pmatrix} \begin{pmatrix} I_n & -M_{12} \\ 0 & I_m \end{pmatrix} \tag{2.1}$$

$$= \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1} M_{12} \\ -M_{21} M_{11}^{-1} & -M_{22} + M_{21} M_{11}^{-1} M_{12} \end{pmatrix} ;$$

if $\det M_{22} \neq 0$ we consider instead the matrix

$$\tilde{T} = \begin{pmatrix} M_{11} - M_{12} M_{22}^{-1} M_{21} & -M_{12} M_{22}^{-1} \\ -M_{22}^{-1} M_{21} & -M_{22}^{-1} \end{pmatrix} .$$

Here $I_n \in \mathbb{C}^{n\times n}$ stands for the unit matrix. If both $\det M_{11}$ and $\det M_{22}$ are nonzero then we have $\tilde{T} = T^{-1}$. Since the cases $\det M_{11} \neq 0$ and $\det M_{22} \neq 0$ can be treated similarly, we assume henceforth that $\det M_{11} \neq 0$.
Further we associate with T and $\tilde{T}$ the (discrete-time) algebraic Riccati equations

$$W = -M_{21} - M_{22}\, W (I - M_{12} W)^{-1} M_{11} \tag{DARE}$$

and the Riccati difference equation

$$W(k+1) = -M_{21} - M_{22}\, W(k)\, (I - M_{12}\, W(k))^{-1} M_{11}, \quad W(0) = W_0 . \tag{DRDE}$$

Notice that (DARE) and (DRDE) can also be written as

$$W = -M_{21} - M_{22}\,(I - W\,M_{12})^{-1}\,W\,M_{11}, \tag{$DARE$}$$

$$W\,(k+1) = -M_{21} - M_{22}\,(I - W\,M_{12}\,(k))^{-1}\,W\,(k)\,M_{11},\ \ W\,(0) = W_0\,. \tag{$DRDE$}$$

Sometimes one considers instead of (DRDE) the reverse difference Riccati equation

$$W\,(k) = -M_{21} - M_{22}\,W\,(k+1)\,(I - M_{12}\,W\,(k+1))^{-1}\,M_{11}, \tag{$RDRDE$}$$

with $W\,(0) = W_0$ for $-k \in I\!N$.

Nonsymmetric Riccati equations of this form appear in polynomial factorization problems and have been investigated in [4], where the corresponding continuous-time Riccati equations have also been studied. Further results on nonsymmetric matrix Riccati differential equations and the associated algebraic Riccati equations can be found in [15], [8] and [10].

Here we recall and generalize some basic facts for discrete-time problems obtained essentially in [4]; for the convenience of the reader we add simple proofs.

For $W \in \mathbb{C}^{m\times n}$ let us denote by $\mathcal{S}_W = \operatorname{Im}\binom{I_n}{W}$ the graph subspace of W.

Lemma 2.1 *There exists a bijection* φ *between the set* $\mathcal{S}$ *of all solutions of* $(DARE)$ *and the set* $\mathbf{I}^{\mathbf{n}}_{\mathbf{T}}$ *of all* n*-dimensional* T*-invariant graph subspaces* $\mathcal{S}_W$ *of* $\mathbb{C}^{2n}$ *with* $\det\,(I_n - M_{12}\,W) \neq 0$*;* φ *is defined by* $\varphi\,(W) = \mathcal{S}_W$*.*
Moreover the matrix of $T_{|\mathcal{S}_W}$ *with respect to the basis (-matrix)* $\binom{I_n}{W}$ *of* $\mathcal{S}_W$ *is the invertible matrix*

$$K_0 = M_{11}^{-1}\,(I_n - M_{12}\,W)\,. \tag{2.2}$$

Proof. From

$$T\begin{pmatrix} I_n \\ W \end{pmatrix} = \begin{pmatrix} M_{11}^{-1}\,(I_n - M_{12}\,W) \\ -M_{22}\,W - M_{21}\,M_{11}^{-1}\,(I_n - M_{12}\,W) \end{pmatrix} \tag{2.3}$$

we infer that $\mathcal{S}_W$ is T-invariant if and only if

$$T\begin{pmatrix} I_n \\ W \end{pmatrix} = \begin{pmatrix} I_n \\ W \end{pmatrix} K_0\ , \tag{2.4}$$

where K_0 is defined by (2.2). Obviously (2.4) holds if and only if

$$W\,M_{11}^{-1}\,(I_n - M_{12}\,W) = -M_{22}\,W - M_{21}\,M_{11}^{-1}\,(I_n - M_{12}\,W),$$

which is for $\det K_0 \neq 0$ equivalent to $(DARE)$.
Notice that (2.2) results from (2.3) and (2.4). □

In order to analyze $(DRDE)$ we introduce the linear difference equation

$$Z\,(k+1) = T\,Z\,(k)\,,\ Z\,(0) = \begin{pmatrix} I_n \\ W_0 \end{pmatrix},\ k \geq 0\ , \tag{L}$$

where $Z(k) = \binom{X(k)}{Y(k)}$ with $X(k) \in \mathbb{C}^{n\times n}$ and $Y(k) \in \mathbb{C}^{m\times n}$.

Equations (L) and $(DRDE)$ are equivalent in the following (restricted) sense:

Lemma 2.2 (i) *If* $(Z(k))_{0\le k\le\nu}$ *satisfies* (L) *and* $\det X(k) \neq 0$ *for* $0 \le k \le \nu$ *then* $(W(k))_{0\le k\le\nu}$*, with* $W(k) := Y(k)\,X(k)^{-1}$*, satisfies* $(DRDE)$*; in particular* $W(0) = W_0$.

(ii) *If* $(W(k))_{0\le k\le\nu}$ *satisfies* $(DRDE)$ *then the sequence* $\binom{X(k)}{Y(k)}_{0\le k\le\nu}$ *defined by*

$$X(k+1) = M_{11}^{-1}\,X(k) - M_{11}^{-1}\,M_{12}\,W(k)\,X(k), \quad X(0) = I_n$$

and

$$Y(k) = W(k)\,X(k)\,,\ 0 \le k \le \nu\ ,$$

satisfies (L).

Proof. (i) Under the assumptions of statement (i) we have $W(0) = Y(0)\,X(0)^{-1} = W_0$ and

$$\det\,(X(k) - M_{12}\,Y(k)) = \det X(k)\,\det\,(I_n - M_{12}\,W(k)) \neq 0\ .$$

Therefore assertion (i) follows from

$$W((k+1) = Y(k+1)\,X(k+1)^{-1}$$

$$= [-M_{22}\,Y(k) - M_{21}\,M_{11}^{-1}\,(X(k) - M_{12}\,Y(k))]\,(X(k) - M_{12}\,Y(k))^{-1}\,M_{11}$$

$$= -M_{22}\,Y(k)\,X(k)^{-1}\,(I_n - M_{12}\,Y(k)\,X(k)^{-1})^{-1}\,M_{11} - M_{21},\ 0 \le k \le \nu - 1.$$

(ii) Under the assumptions of statement (ii) we have

$$X(0) = I_n,\ Y(0) = W(0) = W_0\ \ ,$$

$$X(k+1) = M_{11}^{-1}\,(I_n - M_{12}\,W(k))\,X(k)$$

and

$$Y(k+1) = W(k+1)\,X(k+1) = -M_{21}\,M_{11}^{-1}\,(I_n - M_{12}\,W(k))\,X(k)$$

$$-M_{22}\,W(k)\,(I_n - M_{12}\,W(k))^{-1}\,M_{11}\,M_{11}^{-1}\,(I_n - M_{12}\,W(k))\,X(k)$$

$$= -M_{21}\,M_{11}^{-1}\,X(k) - [\,-M_{22} + M_{21}\,M_{11}^{-1}\,M_{12}]Y(k)$$

for $0 \le k \le \nu - 1$. Hence the assertion of (ii) is proved. □

3. A REPRESENTATION FORMULA

In order to analyze the asymptotic behavior
of the sequences $(Z(k))_{k\geq 0}$ and $(W(k))_{k\geq 0}$ (if the last is defined) for $k \to \infty$ we represent appropriately these sequences as functions of the initial data.
Let $V = (v_1, \ldots, v_{n+m}) \in \mathbb{C}^{(n+m)\times(n+m)}$ be a Jordan basis (matrix) of T, i.e.

$$V^{-1}\,T\,V = J_T = \operatorname{diag}(J_1, \ldots, J_p) = \begin{pmatrix} \lambda_1 & * & & 0 \\ & \lambda_2 & \ddots & \\ & & \ddots & * \\ 0 & & & \lambda_{n+m} \end{pmatrix} \tag{3.1}$$

with $* \in \{0, 1\}$ and with Jordan blocks

$$J_\nu = \begin{pmatrix} \tilde{\lambda}_\nu & 1 & \ldots & 0 \\ 0 & \tilde{\lambda}_\nu & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & 0 & \tilde{\lambda}_\nu \end{pmatrix} \in \mathbb{C}^{j_\nu \times j_\nu},\ \tilde{\lambda}_\nu \in \{\lambda_1, \ldots, \lambda_{n+m}\},\ 1 \leq \nu \leq p\,.$$

We assume here and in the sequel that

$$|\,\lambda_1\,| \leq \ldots \leq \,|\,\lambda_{n+m}\,|\,.$$

The matrix

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix} = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{n+m,1} & \ldots & c_{n+m,n} \end{pmatrix} := V^{-1} \begin{pmatrix} I_n \\ W_0 \end{pmatrix} \tag{3.2}$$

is uniquely determined by the initial data of the sequences $(Z(k))$ and/or $W(k)$.

For a parametrization of these sequences it turns out that it is more convenient to use C instead of W_0.
Using (3.1) we get for $k \geq 0$

$$Z(k+1) = T^k \begin{pmatrix} I_n \\ W_0 \end{pmatrix} (V\,J_T\,V^{-1})^k \begin{pmatrix} I_n \\ W_0 \end{pmatrix} = V\,J_T^k\,C =: \begin{pmatrix} Q(k) \\ P(k) \end{pmatrix}. \tag{3.3}$$

If T is semisimple then

$$V(k) := V\,J_T^k = (\,\lambda_1^k\,v_1,\,\lambda_2^k\,v_2, \cdots,\,\lambda_{n+m}^k\,v_{n+m})\,, \tag{3.4}$$

otherwise

$$V(k) = V\,J_T^k = (x_1(k),\ x_2(k), \cdots,\ x_{n+m}(k))\,, \tag{3.5}$$

where to every Jordan block $J_\nu, 1 \leq \nu \leq p$, there correspond the vectors

$$x_{j_1+\cdots+j_{\nu-1}+s}(k) = \sum_{i=1}^{s} \tilde{\lambda}_\nu^{k+i-s} \binom{k}{s-i} v_{j_1+\cdots+j_{\nu-1}+i},\ 1 \leq i \leq j_\nu\,,\ j_0 = 0\,. \tag{3.6}$$

Introduce now for arbitrary vectors $x = (x_1, \cdots, x_{n+m})^T \in \mathbb{C}^{n+m}$ the abbreviations

$$\tilde{x} = (x_1, \cdots, x_n)^T \,, \ \hat{x} = (x_{n+1}, \cdots, x_{n+m})^T$$

and

$$x^{(\ell,j)} = (x_1, \cdots, x_{j-1}, x_{n+\ell}, x_{j+1}, \cdots, x_n)^T \,.$$

Consequently (3.3), (3.4), (3.5) yield $Q(k) = (\tilde{x}_1\,(k), \cdots, \tilde{x}_{n+m}(k))C$ and $P\,(k) = (\hat{x}_1(k), \cdots, \hat{x}_{n+m}(k))C$, and the Binet-Cauchy formula (see [11], Section 1.2) gives

$$\det Q\,(k) = \sum_{1 \le \nu_1 < \nu_2 < \cdots < \nu_n \le n+m} |\; \tilde{x}_{\nu_1}(k), \cdots, \tilde{x}_{\nu_n}(k) \;| \left| \begin{matrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right| \tag{3.7}$$

for $k \ge 0$. For the entries $w\,(k)_{\ell\,j}$ of the matrix $W\,(k) = P\,(k)\;Q\,(k)^{-1}$ we get analogously to (3.7) by an elementary calculation (similar to [8], p. 263)

$$w\,(k)_{\ell\,j} \cdot \det Q\,(k) = (P\,(k)\;Adj\;Q\,(k))_{\ell\,j}$$

$$= \sum_{1 \le \nu_1 < \cdots < \nu_n \le n+m} |\; x_{\nu_1}(k)^{(\ell,j)}, \cdots, x_{\nu_n}(k)^{(\ell,j)} \;| \left| \begin{matrix} c_{\mu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right| . \tag{3.8}$$

Summarizing we get from (3.4) - (3.8) the following representation formula for the elements of the sequence $W\,(k)$.

Theorem 3.1 (Representation Formula)
As long as $W\,(k)$ *is defined,*

$$w\,(k)_{\ell\,j} = \frac{\displaystyle\sum_{1 \le \nu_1 \cdots < \nu_n \le n+m} |\; x^{(\ell,j)}_{\nu_1}(k)\,, \cdots, x^{(\ell,j)}_{\nu_n}(k) \;| \left| \begin{matrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right|}{\displaystyle\sum_{1 \le \nu_1 < \cdots < \nu_n \le n+m} |\; \tilde{x}_{\nu_1}(k), \cdots, \tilde{x}_{\nu_n}(k) \;| \left| \begin{matrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right|} \tag{3.10}$$

for $1 \le \ell \le m,\ 1 \le j \le n.$
If in addition T *is semisimple then*

$$w\,(k)_{\ell j} = \frac{\displaystyle\sum_{1 \le \nu_1 < \cdots < \nu_n \le n+m} \lambda^k_{\nu_1} \cdots \lambda^k_{\nu_n} \;|\; v^{(\ell,j)}_{\nu_1}, \cdots, v^{(\ell,j)}_{\nu_n} \;| \left| \begin{matrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right|}{\displaystyle\sum_{1 \le \nu_1 < \cdots < \nu_n \le n+m} \lambda^k_{\nu_1} \cdots \lambda^k_{\nu_n} \;|\; \tilde{v}_{\nu_1}, \cdots, \tilde{v}_{\nu_n} \;| \left| \begin{matrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{matrix} \right|} \,. \tag{3.11}$$

4. INTERPRETATION OF THE REPRESENTATION FORMULA

In spite of their quasi-elementary derivations formulae (3.10) and (3.11) are of utmost relevance for obtaining the whole information on the dynamics of the solution $(W(k))$ of $(DRDE)$ and also on the structure of the set of *all* solutions of the algebraic Riccati equations $(DARE)$ this will be discussed subsequently.

For sake of simplicity we assume henceforth that T *is semisimple*; the investigation of the general case runs analogously but is much more technical, moreover, T is generically semisimple.

4.1 Properties of the solutions to $(DARE)$

(**a**) Notice first that the determinants $\begin{vmatrix} c_{\nu_1} \\ \vdots \\ c_{\nu_n} \end{vmatrix}$, appearing in (3.10), (3.11), are the so-called Plückerian coordinates of the subspace Im $V^{-1}\binom{I^n}{W_0}$. Hence, for a fixed Jordan basis V, the initial value is uniquely determined by these Plückerian coordinates (and vice versa).

(**b**) If T has only simple eigenvalues and if the *complementarity* (*disconjugacy*) *condition*

$$|\tilde{v}_{\nu_1}, \cdots, \tilde{v}_{\nu_n}| \neq 0 \ , \tag{4.1}$$

is satisfied, then the matrix $W = (w_{\ell j})_{\substack{1 \leq \ell \leq m \\ 1 \leq j \leq n}}$ with

$$w_{\ell j} = \frac{|\ v_{\nu_1}^{(\ell,j)}, \cdots, v_{\nu_n}^{(\ell j)}\ |}{|\ \tilde{v}_{\nu_1}, \cdots, \tilde{v}_{\nu_n}\ |} \tag{4.2}$$

is a solution of $(DARE)$; moreover, each solution of $(DARE)$ has a representation of the form (4.2), and in this case there are obviously at most $\binom{n+m}{n}$ solutions of $(DARE)$. For the product $\lambda_{\nu_1} \cdots \lambda_{\nu_n} \neq 0$ these solutions are obtained by choosing in (3.11) $\begin{vmatrix} c_{\mu_1} \\ \vdots \\ c_{\nu_n} \end{vmatrix} \neq 0$ for one n-tuple $(\nu_1, \cdots, \nu_n)$ (with property (4.1)) while all other Plückerian coordinates are zero.
Notice also that for $\lambda_{\nu_1} \cdots \lambda_{\nu_n} = 0$ the matrix W, defined by (4.2), (under assumption (4.1)) is not a solution of $(DARE)$ since in this case, according to (2.2), (2.4), $I_n - M_{12} W$ is not invertible.

(**c**) For $\nu = (\nu_1, \cdots, \nu_n)$ with $1 \leq \nu_1 < \cdots < \nu_n \leq n+m$ we define

$$I(\nu) = \left\{ \mu = \begin{pmatrix} \mu_{\nu_1} \\ \vdots \\ \mu_{\nu_n} \end{pmatrix} \Bigg| 1 \leq \mu_1 < \cdots < \mu_n \leq n+m \text{ and } \begin{pmatrix} \lambda_{\nu_1} \\ \vdots \\ \lambda_{\nu_n} \end{pmatrix} = \begin{pmatrix} \mu_{\nu_1} \\ \vdots \\ \mu_{\nu_n} \end{pmatrix} \right\} .$$

In this case each matrix $W = (w_{\ell j})_{\substack{1\le \ell\le m\\ 1\le j\le n}}$ with

$$w_{\ell j} = \frac{\displaystyle\sum_{\mu\in I(\nu)} \mid v^{(\ell,j)}_{\mu_1},\cdots,v^{(\ell,j)}_{\mu_n} \mid \begin{vmatrix} c_{\mu_1}\\ \vdots \\ c_{\mu_n}\end{vmatrix}}{\displaystyle\sum_{\mu\in I(\nu)} \mid \tilde{v}_{\mu_1},\cdots,\tilde{v}_{\mu_n} \mid \begin{vmatrix} c_{\mu_1}\\ \vdots \\ c_{\mu_n}\end{vmatrix}} =: \frac{N}{D}, \tag{4.3}$$

satisfying the complementarity condition $D \neq 0$ and the feasability conditions

$$\lambda_{\nu_1}\cdots\lambda_{\nu_n} \neq 0 \text{ and} \begin{vmatrix} c_{\mu_1}\\ \vdots \\ c_{\mu_n}\end{vmatrix} = 0 \text{ for } \mu \notin I(\nu)$$

is a solution of $(DARE)$. Notice that $\lambda_{\nu_1}\cdots\lambda_{\nu_n} = \mu_{\nu_1}\cdots\mu_{\nu_n}$ does not imply that $(\lambda_{\nu_1},\cdots,\lambda_{\nu_n}) = (\mu_{\nu_1},\cdots,\mu_{\nu_n})$, moreover, if one of these feasability conditions is not fulfilled then (4.3) does not define a solution of $(DARE)$ since otherwise either (as before) $\det(I_n - M_{12}(k)) = 0$ or $\mathcal{S}_W$ is not T-invariant. This last conclusion is derived by using (3.3) and (3.4) in conjunction with the main idea of the proof of Lemma 3 in [8].

Moreover, according to (3.10), each solution of $(DARE)$ can be represented in the form (4.3); however in this case there may be solutions of $(DARE)$ which cannot be represented in the form (4.2) (for a fixed Jordan basis V).
If $I(\nu)$ contains $p_0 > 1$ elements then the family $\mathcal{F}(\nu)$ of solutions of $(DARE)$, represented by (4.3), contains (after normalizing one nonzero Plückerian coordinate $\begin{vmatrix} c_{\mu_1}\\ \vdots \\ c_{\mu_n}\end{vmatrix}$ to 1) a subset with $q \le p_0 - 1$ free parameters. If for each $\mu \in I(\nu)$ at least one of the determinants

$$\mid \tilde{v}_{\mu_1},\cdots,\tilde{v}_{\mu_n} \mid, \ \mid v^{(\ell,j)}_{\mu_1},\cdots,v^{(\ell,j)}_{\mu_n} \mid$$

in nonzero, then it is easy to see that $\mathcal{F}(\nu) = \cup_{j=0}^{p_0-1}\mathcal{F}_j(\nu)$, where $\mathcal{F}_j(\nu)$, $0 \le j \le p_0 - 1$, is a j-parametric set; in particular $\mathcal{F}(\nu)$ is uncountable for $p_0 > 1$. Consequently the set $\mathcal{F}(\nu)$ is either finite or a uncountable set.
If $\mathcal{F}(\nu)$ is uncountable then formula (4.3) shows that no solution $W \in \mathcal{F}(\nu)$ is isolated - this corresponds to the main result obtained for continuous-time algebraic Riccati equations in [5].

We have to point out that the condition $\lambda_{\nu_1}\cdots\lambda_{\nu_n} \neq 0$ is always fulfilled if M_{11} and M_{12} are both invertible since in this case T is invertible as well. In fact for nonsymmetric difference Riccati equations only this special case has been treated up to now in the literature (see [4]).
If T is invertible then (L) generates the sequence $(\operatorname{Im} Z(k))_{k\ge 0}$ of points from $G(n, m+n)$, the Grassmannian manifold of n-dimensional subspaces of $\mathbb{C}^{m+n}$; this allows us to interpret $(\operatorname{Im} Z(k))_{k\ge 0}$ as the solution of an *extended Riccati equation* on $G(n, n+m)$ which

is defined for any initial value $\mathcal{S}_{W_0} \in G(n, n+m)$.
The extended Riccati equation has in the continuous-time case been studied in detail in [16]. In the discrete-time case the extended Riccati equation is equivalent to the (trivial) linear difference equation (L).

(**d**) T is called *dichotomic (of type (m,n))* if

$$| \lambda_1 | \leq \cdots \leq | \lambda_m | < | \lambda_{m+1} | \leq \cdots \leq | \lambda_{m+n} | .$$

In this case $\mathcal{S}_d := \operatorname{Im}(v_{m+1}, \cdots, v_{m+n})$ is called the *dichotomic (invariant) subspace* of T; moreover, if in addition the complementarity (disconjugacy) condition
$| \tilde{v}_{m+1}, \cdots, \tilde{v}_{m+n} | \neq 0$ is fulfilled, then

$$W_d = (\hat{v}_{m+1}, \cdots, \hat{v}_{m+n}) \, (\tilde{v}_{m+1}, \cdots, \tilde{v}_{m+n})^{-1} \tag{4.4}$$

is called the *dichotomic solution* of $(DARE)$. According to Lemma 2.1 the matrix $M_{11}^{-1}(I - M_{12} W_d)$ has the eigenvalues $\lambda_{m+1}, \cdots, \lambda_{m+n}$ (counted with multiplicity).

(**e**) T is called *reverse dichotomic* if $| \lambda_n | < | \lambda_{n+1} |$; if in this case $| \tilde{v}_1, \cdots, \tilde{v}_n | \neq 0$ and $\lambda_1 \neq 0$ then

$$W_{rd} = (\hat{v}_1, \cdots, \hat{v}_n) \, (\tilde{v}_1, \cdots, \tilde{v}_n)^{-1} \tag{4.5}$$

is called the *reverse dichotomic solution* and $M_{11}^{-1}(I - M_{12} W_{rd})$ has the eigenvalues $\lambda_1, \cdots, \lambda_n$.

4.2 Properties of the solutions to $(DRDE)$

Let $W_0 \in \mathbb{C}^{m \times n}$ be given and let us define as before C by $V^{-1} \binom{I}{W_0} = C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix}$ and denote by $(W(k))$ the solution of $(DRDE)$.

(**a**) If T is dichotomic we shall say that W_0 is contained in the *basin of attraction* of $\mathcal{S}_d$ (in symbols: $W_0 \in BA(\mathcal{S}_d)$) if $| c_{m+1}, \cdots, c_{m+n} | \neq 0$. If in addition W_d exists we shall say that W_0 is contained in the *generalized basin of attraction* of W_d (in symbols: $W_0 \in GBA(W_d)$); in this case $BA(\mathcal{S}_d) = GBA(W_d)$.

It can be easily seen that $BA(\mathcal{S}_d)$ is of dimension $m \cdot n$ if $\mathcal{S}_d$ is defined; moreover $\mathbb{C}^{m \times n} \setminus BA(\mathcal{S}_d)$ can be represented as the union of a finite number of sets of dimensions less than $m \cdot n$. Consequently $BA(\mathcal{S}_d)$ is in particular open and dense in $\mathbb{C}^{m \times n}$.

$$BA(W_d) = \{ W_0 \in GBA(W_d) \mid W(k) \text{ exists for } k \geq 0 \}$$

is called the *basin of attraction* of W_d.

(**b**) Assume now that W_d exists and that $W_0 \in BA(W_d)$. Then it follows from (3.6) that

$$| \tilde{x}_{m+1}(k), \cdots, \tilde{x}_{n+m}(k) | = \lambda_{m+1}^k \cdots \lambda_{n+m}^k \, | \tilde{v}_{m+1}, \cdots, \tilde{v}_{m+n} | \neq 0 \tag{4.6}$$

and

$$| x_{m+1}^{(\ell,\alpha)}(k), \cdots, x_{n+m}^{(\ell,\alpha)}(k) | = \lambda_{m+1}^k \cdots \lambda_{m+n}^k \, | v_{m+1}^{(\ell,\alpha)}, \cdots, v_{m+n}^{(\ell,\alpha)} | . \tag{4.7}$$

Since all remaining summands in the denominator and in the nominator of (3.10) are dominated by the terms (4.6) and (4.7) (if the last determinant is nonzero), it follows that (under these assumptions)

$$\lim_{k\to\infty} W(k) = W_d \,. \tag{4.8}$$

From Theorem 3.1 combined with (3.6) it follows in addition that there exist constants $c \geq 0$ and $\kappa \in \mathbb{N}_0$ with

$$|\,(W(k) - W_d)_{i,j}\,| \leq c\,k^{\kappa}\,|\,\frac{\lambda_m}{\lambda_{m+1}}\,|^k \,,\; 1 \leq i \leq m \,,\; 1 \leq j \leq n \,, \tag{4.9}$$

where κ depends on the size of the Jordan blocks J_ν, in particular $\kappa = 0$ if T is semisimple. Notice that $W_0 \in GBA(W_d)$ does in general not guarantee that $(W(k))$ is defined for all $k \geq 0$. Nevertheless it follows like before that

$$\lim_{k\to\infty} \operatorname{Im} Z(k) = \mathcal{S}_d \,. \tag{4.10}$$

Notice that (4.10) also holds true if W_d does not exist but $W_0 \in BA(\mathcal{S}_d)$.
Formulae (4.8) and (4.10) justify the notation introduced in (**a**).

(**c**) If T^{-1} exists, analogous results hold for the reverse Riccati difference equation $(RDRDE)$. Notice that $(RDRDE)$ corresponds to the linear difference equation

$$Z(k+1) = T^{-1}\,Z(k)\,,\; Z(0) = \begin{pmatrix} I \\ W_0 \end{pmatrix},\; k \geq 0 \,.$$

(**d**) Assume now that T is not dichotomic. In this case there exist no n-dimensional T-invariant subspace and no solution of $(DARE)$ having an $m \cdot n$-dimensional basin of attraction. Define

$$J_{dic}\,\{\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \Big|\, 1 \leq \mu_1 < \cdots < \mu_n \leq n+m \;\text{ and }\; |\,\lambda_{\mu_1}\cdots\lambda_{\mu_n}\,| = |\,\lambda_{m+1}\cdots\lambda_{n+m}\,|\}.$$

Then (using the notation of (3.2)) it follows that

$$D_{dic} = \{W_0 \in \mathbb{C}^{m\times n} \,\Big|\, \begin{vmatrix} c_{\mu_1} \\ \vdots \\ c_{\mu_n} \end{vmatrix} \neq 0 \;\text{ for at least one }\; \mu \in J_{dic}\}$$

is open and dense in $\mathbb{C}^{m\times n}$.
For $W_0 \in D_{dic}$ the sequence $(W(k))$ behaves - as long as it is defined - asymptotically like its leading term $R(k)$, which is defined by

$$(R(k))_{\ell j} = \frac{\displaystyle\sum_{\mu\in J_{dic}} \lambda_{\mu_1}^k \cdots \lambda_{\mu_n}^k \,|\, v_{\mu_1}^{(\ell,j)}, \cdots, v_{\mu_n}^{(\ell,j)} \,|\, \begin{vmatrix} c_{\mu_1} \\ \vdots \\ c_{\mu_n} \end{vmatrix}}{\displaystyle\sum_{\mu\in J_{dic}} \lambda_{\mu_1}^k \cdots \lambda_{\mu_n}^k \,|\, \tilde{v}_{\mu_1}, \cdots, \tilde{v}_{\mu_n} \,|\, \begin{vmatrix} c_{\mu_1} \\ \vdots \\ c_{\mu_n} \end{vmatrix}} \,. \tag{4.11}$$

If J_{dic} contains τ elements then there are numbers $\varphi_\mu \in [0, 2\pi)$, $\mu \in J_{dic}$, such that (4.11) can be written as

$$(R(k))_{\ell j} = \frac{\displaystyle\sum_{\mu \in J_{dic}} e^{i\,k\,\varphi_\mu} \left| v_{\mu_1}^{(\ell,j)}, \cdots, v_{\mu_n}^{(\ell,j)} \;\middle|\; \begin{matrix} c_{\mu_1} \\ \vdots \\ c_{\mu_n} \end{matrix} \right|}{\displaystyle\sum_{\mu \in J_{dic}} e^{i\,k\varphi_\mu} \left| \tilde{v}_{\mu_1}, \cdots, \tilde{v}_{\mu_n} \;\middle|\; \begin{matrix} c_{\mu_1} \\ \vdots \\ c_{\mu_n} \end{matrix} \right|} . \tag{4.12}$$

This clearly shows that R is (as a function of k) either constant, periodic or almost periodic. Consequently the sequences $(W(k))$ behave for each initial value $W_0 \in D$ (if they are defined for $k \geq 0$) asymptotically like (4.12).
Notice that the constant sequences of the form (4.12) are solutions of $(DARE)$. Moreover (4.12) is periodic if and only if all differences $\varphi_\nu - \varphi_\alpha$, $1 \leq \nu \leq \tau$, corresponding to nontrivial summands in (4.12), are rational multiples of π; here $\alpha \in \{1, \cdots, \tau\}$ is fixed and belongs to one nontrivial summand in (4.12). If T is dichotomic then $\tau = 1$ and $D_{dic} = BA(\mathcal{S}_d)$.

(**e**) In linear-quadratic discrete-time control problems one has to study special cases of $(DARE)$ and $(DRDE)$ where

$$M_{11} = -M_{22}^* = A, \quad M_{12} = -B\,R^{-1}\,B^* =: -S, \quad M_{21} = -Q,$$

with $\det A \neq 0$, $Q = Q^*$ and invertible $R = R^*$. In this case one is interested mainly in *hermitian* or *symmetric* solutions of the resulting *hermitian* Riccati difference equations

$$W(k+1) = Q + A^*W(k)(I - SW(k))^{-1}A, \tag{$HDRDE$}$$

$$W(k) = Q + A^*W(k+1)(I - SW(k+1))^{-1}A, \tag{$RHDRDE$}$$

and the corresponding hermitian algebraic Riccati equation

$$W = Q + A^*W(I - SW)^{-1}A (= Q + A^*(I - WS)^{-1}WA). \tag{$HDARE$}$$

There exist well known sufficient conditions for the existence of solutions of $(HDARE)$ or $(RHDARE)$ (see [13], [14], Part III, and [18]). The existence and convergence properties of the solutions of *symmetric* Riccati difference equations have been investigated in [1], [2] and [3].

(**f**) Under the additional assumption that M_{12} and M_{21} are matrices of rank 1, sufficient conditions for the existence of solutions of nonsymmetric algebraic Riccati equations $(DARE)$ have been derived in [4], Section 3.2. This result is in general not valid for matrices M_{12}, M_{12} of rank greater than 1 (see [17] and also Section 4.3).

The next theorem shows how the precedent results can be used to obtain easily information on the dynamics of (HDRDE) and (RHDRDE).
For convenience we assume here in addition to the preceding assumptions that

$$Q = C^*C, \; R > 0, \; (A, B) \text{ is controllable and } (C, A) \text{ is detectable.} \tag{4.13}$$

Under these assumptions it is well known ([14], [7]) that

(i) $\sigma(T) \cap i\mathbb{R} = \emptyset$, i.e. T and T^{-1} are dichotomic of type (n, n).

(ii) The dichotomic solution W_d and the reverse dichotomic solution W_{rd} of (HDARE) exist and $W_{rd} \leq \tilde{W} \leq W_d$ for any solution $\tilde{W}$ of (HDARE). Moreover, in the notation of discrete-time control theory, W_d (and W_{rd}) is the unique stabilizing (and antistabilizing, respectively) solution of (HDARE).

(iii) For any $W_0 \geq W_{rd}$ (or $W_0 \leq W_d$) the solution $W(k)$ of (HDRDE) with $W(0) = W_0$ exists for $k \geq 0$ (or $k \leq 0$, respectively); analogous statements hold for (RHDRDE).

Theorem 4.1 Let (4.13) be fulfilled. Then the following holds for the solution W of (HDRDE) and the solution $\tilde{W}$ of (RHDRDE) with $W(0) = \tilde{W}(0) = W_0$:

(i) For $W_0 > W_{rd}$ and $k \geq 0$ $W(k)$ exists and $\lim\limits_{k \to \infty} W(k) = W_d$,

(ii) for $W_0 < W_{rd}$ $W(k)$ blows up after a finite number of steps,

(iii) for $W_0 < W_d$ and $k \geq 0$ $\tilde{W}(k)$ exists and $\lim\limits_{k \to \infty} \tilde{W}(k) = W_{rd}$,

(iv) for $W_0 > W_d$ $\tilde{W}(k)$ blows up after a finite number of steps. Similar results are valid for $k \leq 0$ and $k \to -\infty$.

Proof. We prove here only (i) and (ii) since (iii) and (iv) can be proved analogously.

(i): Let $W_0 > W_{rd}$. Since $BA(\mathcal{S}_d)$ is open and dense in $\mathbb{C}^{n \times n}$ (see Section 4.2) there exists a matrix $W_1 \in BA(\mathcal{S}_d)$ with $W_{rd} < W_1 < W_0$. On account of the remarks precedent to this theorem we also have $W_1 \in BA(W_d)$. If $(W_1(k))$ denotes the solution of (HDRDE) with $W_1(0) = W_1$, then it follows from the comparison theory of (HDRDE) (see [7]) that $W_1 \leq W_1(k) \leq W(k) \leq W_d$ for $k \geq 0$. Hence $\lim\limits_{k \to \infty} W_1(k) = W_d$ implies $\lim\limits_{k \to \infty} W(k) = W_d$.
(ii): For the proof of (ii) we use the fact that there exists a matrix $W_2 \in BA(\mathcal{S}_d)$ with $W_0 < W_2 < W_{rd}$. Using the above mentioned comparison results it is obvious that the solution $(W_2(k))$ of (DHRDE) with $W(0) = W_2$ (and hence also the solution with $W(0) = W_0$) must blow up after a finite number of steps, since otherwise $W_2(k)$ exists for $k \geq 0$ and $\lim\limits_{k \to \infty} W_2(k) = W_d$, which contradicts $W_2 < W_{rd} \leq W_d$. □

With some more effort it is possible to prove (under weaker assumptions) a refined version of Theorem 4.1 and also an analoguous result for hermitian Riccati difference equations with periodic coefficients - these results are the discrete-time versions of the convergence results in [6].

4.3 Necessary complementarity conditions

(**a**) $\lambda \in \mathbb{C}$ is called an *unobservable mode (of rank r)* of the pair $(M_{12}, -M_{22})$ if there exist vectors $q_j \in \mathbb{C}^m \setminus \{0\}$, $0 \leq j \leq r-1$, such that

$$(-M_{22} - \lambda\, I_{n+m})\, q_j = q_{j-1} \text{ and } M_{12}\, q_j = 0 \text{ for } 0 \leq j \leq r-1;$$

here $q_{-1} = 0$.
Subspaces spanned by chains of such generalized eigenvectors are called $(M_{12}, -M_{22})$ - *unobservable subspaces* corresponding to the eigenvalue λ of $-M_{22}$.
Notice that a subspace $\mathcal{S}$ is $(M_{12}, -M_{22})$-unobservable if and only if it is $-M_{22}$-invariant with $\mathcal{S} \subset \text{Ker } M_{12}$.

(**b**) The subsequent assertion follows, using (2.1), by straightforward calculation:
λ is an unobservable mode (of rank r) of the pair $(M_{12}, -M_{22})$ corresponding to the chain q_j, $0 \leq j \leq r = 1$, if and only if (with the notations of (2.1))

$$(T - \lambda\, I_{n+m}) \begin{pmatrix} 0 \\ q_j \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ q_{j-1} \end{pmatrix} , \quad 0 \leq j \leq r - 1 \, .$$

(**c**) As an immediate consequence of (b) we get:

Let $\mathcal{S} = \text{Im } (v_{\nu_1}, \cdots, v_{\nu_n})$ *be an* n*-dimensional* T*-invariant subspace with* $(v_{\nu_1}, \cdots, v_{\nu_n}) = \binom{X}{Y}$ *and* $X \in \mathbb{C}^{n \times n}$*. Then the complementarity condition* $\det X \neq 0$ *can only be fulfilled if* $\mathcal{S}$ *does not contain an* $(M_{12}, -M_{22})$ *- unobservable subspace.*

This shows that the observability of $(M_{12}, -M_{22})$ is a necessary condition for all possible eigenvector selections $(v_{\nu_1}, \cdots, v_{\nu_n})$ to give nonsingular X. From the results in Section 3 of [17], where similar problems have been studied , we infer that these necessary conditions are not sufficient for the invertibility of X.

Necessary conditions for the existence of solutions of two special classes of nonsymmetric algebraic Riccati equations, appearing in discrete-time Nash and Stackelberg games, can be found in [9].

(**d**) Analogously to (a), (b), (c) we can derive by direct calculations that the observability of (M_{11}^{-1}, M_{21}) is a necessary (but non-sufficient) condition for all eigenvector selections $(v_{\nu_1}, \cdots, v_{\nu_n}) = \binom{X}{Y}$ to give rank $Y = n$.
Notice that

$$(T - \lambda\, I_{n+m}) \begin{pmatrix} p_j \\ 0 \end{pmatrix} = \begin{pmatrix} p_{j-1} \\ 0 \end{pmatrix}, \quad 0 \leq j \leq r - 1,$$

holds (with $p_{-1} = 0$) if and only if

$$(M_{11}^{-1} - \lambda\, I_n)\, p_j = \, p_{j-1}, \;\; M_{21}^{T}\, p_j = 0, \;\; 0 \leq j \leq r - 1,$$

i. e. if and only if λ is an unobservable mode of rank r of the pair (M_{11}^{-1}, M_{21}).
(**e**) If W is a solution of the $(DARE)$

$$W = -M_{21} - M_{22}\, W\, (I - M_{12}\, W)^{-1}\; M_{11},$$

then $V = W^*$ is a solution of the modified algebraic Riccati equation

$$V = -M_{21}^* \; - M_{11}^*\, (I - V \; M_{12}^*)^{-1}\; V\, M_{22}^* \, ;$$

hence each solution V corresponds to an n-dimensional $\hat{T}$-invariant subspace, where

$$\hat{T} = \begin{pmatrix} -M_{22}^{-*} & -M_{22}^{-*}\; M_{12}^* \\ -M_{21}^*\; M_{22}^{-*} & -M_{11}^* + M_{21}^*\; M_{22}^{-*}\; M_{12}^* \end{pmatrix} \, .$$

As before it follows that λ is an $(M_{12}^*,\ -M_{11}^*\ )$ - unobservable mode of rank r corresponding to the chain $q_j, 0 \leq j \leq r-1$, if and only if

$$(\hat{T} - \lambda\, I_{n+m}) \begin{pmatrix} 0 \\ q_j \end{pmatrix} = \begin{pmatrix} 0 \\ q_{j-1} \end{pmatrix}, \quad 0 \leq j \leq r-1\ .$$

This shows that the observability of $(M_{12}^*, -M_{11}^*)$ (or the controllability of $(-M_{11},\ M_{12})$) is a (second) necessary condition for all eigenvector selections to give (as before) a nonsingular X.

Summarizing the preceding results we obtain:

Theorem 4.2 *If each n-dimensional T-invariant subspace* Im $\binom{X}{Y}$ *has the property that X is nonsingular, then (*$W = X^{-1}Y$ *is a solution of (DARE) and)*

(i) det $T \neq 0$ *(i. e. det* $M_{11} \neq 0$ *and* $M_{22} \neq 0$*).*

(ii) $(M_{12},\ -M_{22})$ *is observable;*

(iii) $(M_{11},\ M_{12})$ *is controllable.*

Notice that assertion (i) of Theorem 4.2 follows from the last sentence of Lemma 2.1; moreover (M_{11}, M_{12}) is controllable if and only if $(-M_{11},\ M_{12})$ is controllable.

4.4 Nonautonomous Riccati difference equations

In this subsection we assume that $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is as in Section 2 and that $(M(k))_{k\geq 0}$ with $\begin{pmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{pmatrix}$ is a sequence of invertible $(n+m)\times(n+m)$ matrices with $\lim\limits_{k\to\infty} M(k) = M$, where M is dichotomic of type (m,n).
We associate with $(M(k))_{k\geq 0}$ the nonautonomous Riccati difference equation

$$W(k+1) = -M_{21}(k) - M_{22}(k)W(k)(I - M_{12}(k))^{-1}M_{11}(k),\ W(0) = W(0). \tag{4.14}$$

Let $T(k) := T(M(k)) =: \begin{pmatrix} T_{11}(k) & T_{12}(k) \\ T_{21}(k) & T_{22}(k) \end{pmatrix}$, $k \in I\!N$ (see (2.1)), then (4.14) corresponds to the nonautonomous linear difference equation

$$Z(k+1) = T(k)Z(k),\ Z(0) = \begin{pmatrix} I_n \\ W_0 \end{pmatrix},\ k \geq 0, \tag{4.15}$$

and we can use the results of Kooman, [12], on (4.15) for the study of (4.14).

Let $V = (v_1, \ldots, v_{n+m})$ be a Jordan-basis of M such that

$$V^{-1}T(M)V = \begin{pmatrix} \Delta_m & 0 \\ 0 & \Delta_n \end{pmatrix},$$

where all eigenvalues of Δ_m have smaller modulus than each eigenvalue of Δ_n.

From [12], Theorem 3.2, it follows that there exists a sequence $(L(k))_{k\geq 0}$ of invertible matrices with

$$\lim_{k\to\infty} L(k) = I_{n+m} \tag{4.16}$$

and

$$L(k+1)V^{-1}T(M(k))VL(k) = \begin{pmatrix} \Delta_m(k) & 0 \\ 0 & \Delta_n(k) \end{pmatrix}, \tag{4.17}$$

where $\Delta_m(k) \in \mathbb{C}^{m\times m}$, $\Delta_n(k) \in \mathbb{C}^{n\times n}$ and $\lim\limits_{k\to\infty} \begin{pmatrix} \Delta_m(k) & 0 \\ 0 & \Delta_n(k) \end{pmatrix}$.
Denote

$$D_+ = \{W_0 \in \mathbb{C}^{m\times n} | \det Z_2 \neq 0 \text{ for } \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = L(0)V^{-1} \begin{pmatrix} I_n \\ W_0 \end{pmatrix}\}, \tag{4.18}$$

here we used the preceding notations and $Z_1 \in \mathbb{C}^{m\times m}$, $Z_2 \in \mathbb{C}^{n\times n}$.
As in Section 4.2, **a**, we infer that D_+ has dimension $m\cdot n$ and $\mathbb{C}^{m\times n} \setminus D_+$ is the finite union of sets of dimensions less than $m\cdot n$; hence D_+ is open and dense in $\mathbb{C}^{m\times n}$.

Lemma 4.3 *Let $\mathcal{S}_d = Im(v_{m+1},\dots,v_{m+n})$. Then*

$$\lim_{k\to\infty} Im\ Z(k+1) = \mathcal{S}_d \textit{ for } Z(0) = \begin{pmatrix} I_n \\ W_0 \end{pmatrix} \textit{ with } W_0 \in D_+. \tag{4.19}$$

Proof. From (4.15) - (4.17) we infer that

$$\begin{aligned} Z(k+1) &= \prod_{j=0}^{k} T(k) \begin{pmatrix} I \\ W_0 \end{pmatrix} \\ &= VL(k+1)^{-1} \prod_{j=0}^{k} \begin{pmatrix} \Delta_m(j) & 0 \\ 0 & \Delta_n(j) \end{pmatrix} L(1)V^{-1} \begin{pmatrix} I \\ W_0 \end{pmatrix} \\ &= VL(k+1)^{-1} \begin{pmatrix} \prod\limits_{j=0}^{k} \Delta_m(j) Z_1 \\ \prod\limits_{j=0}^{k} \Delta_n(j) Z_2 \end{pmatrix} = V \begin{pmatrix} \tilde{Z}_1(k) \\ \tilde{Z}_2(k) \end{pmatrix}. \end{aligned}$$

Here $\det Z_2 \neq 0$ (since $W_0 \in D_+$) and therefore, on account of $\det \Delta_n(k) \neq 0$ and (4.16) also $\det \tilde{Z}_2(k) \neq 0$ for $k \geq k_0$. This implies (4.19). □

In the remaining part of this section we assume in addition to the preceding assumptions that

$$M_{11}(k) = -M_{22}^*(k) = A(k),\ M_{12}(k) = -B(k)R^{-1}(k)B^*(k),\ M_{21}(k) = -C^*(k)C(k), k \geq 0,$$

where $\det A(k) \neq 0$ and $R(k) > 0$ for $k \geq 0$ and

$$\lim_{k\to\infty} (A(k), B(k), C(k), R(k)) = (A, B, C, R)$$

with $\det A \neq 0, R > 0$ such that

$$(A, B) \text{ is stabilizable and } (C, A) \text{ is dectectable.}$$

Then it follows easily from the comparison results in [7] that for $W_0 \geq 0$ the solution $(W(k))_{k \geq 0}$ of (RHDRDE) with $W(0) = W_0$ is a well defined sequence of positive semidefinite matrices. Moreover the stabilizing solution $W_+ = W_d$ of (HDARE) exists and is positive semidefinite.

Theorem 4.4 *For* $W_0 > 0$ *the solution* $W(k)$ *of (RHDRDE) with* $W(0) = 0$ *exists and* $\lim_{k \to \infty} W(k) = W_+$.

Proof. For $W_0 > 0$ there exists, according to Lemma 4.3, W_1 with $0 < W_1 < W_0$ and $W_1 \in GBA(W_+)$.
Since the solution $W_1(k)$ of (RHDRDE) with $W_1(0) = W_1$ exists for $k \geq 0$ (which follows for example from the comparison theorem), it follows that $\lim_{k \to \infty} W_1(k) = W_+$ and consequently, using again the comparison theorem, $\lim_{k \to \infty} W(k) = W_+$. □

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