DISCRETE TIME RICCATI EQUATIONS IN OPEN LOOP NASH AND STACKELBERG GAMES

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Abstract: We study the asymptotic behavior of difference equations appearing in the necessary optimality conditions of noncooperative open loop Nash and Stackelberg games. Since these equations are coupled nonsymmetric Riccati difference equations, their qualitative behavior is essentially different from that of standard (symmetric) Riccati equations. Moreover we study also the properties of the solutions of the corresponding algebraic Riccati equations.

1. Introduction.

Standard Riccati equations are widely studied in the literature due to their prominent role in linear-quadratic optimal control and filtering problems (see for example [6],[10]). In dynamic games where several players or decision makers are acting on the same system, coupled and/or nonsymmetric Riccati equations occur and have to be solved in order to compute each player's strategy. The type of coupling between such equations depends on the information structure in the game (open loop, closed loop, ...) and also on the strategy adopted by the different players. A detailed review of these concepts in dynamic games can be found in [5].

In this paper we consider nonsymmetric coupled Riccati equations associated with open loop discret-time linear-quadratic games when Nash or Stackelberg strategies are applied. For the continuous time case, i.e. for differential games, the corresponding equations were examined in [2], [3] and [4]. Two main problems were tackled: the solution of the coupled Riccati equations [2]-[3], and the asymptotic behavior of the nonsymmetric Riccati equation associated with the Nash strategy to derive necessary conditions for obtaining constant solutions [4]. A recent rigorous exposition of these results can be found in [9]. For the discrete-time case, a closed form solution of the Riccati difference equations associated with the open loop Stackelberg strategy was proposed in [1].

The purpose of this paper is to study the discrete-time coupled nonsymmetric Riccati equations occurring when a Nash or a Stackelberg equilibrium is sought in a linear-quadratic game with an open loop information structure. The main results concern the asymptotic behavior of such equations and the solution of the corresponding algebraic equations. It is shown that, in analogy with the linear-quadratic control problem, the solutions of the algebraic equations can be characterized in terms of invariant subspaces of a matrix associated with the necessary conditions to be satisfied for an equilibrium.
It is worth noting that solvability of the above mentioned set of coupled Riccati equations does not necessarily guarantee the existence of an equilibrium. This problem is examined in [11], [8] and [9] for the continuous time case. The existence of Nash or Stackelberg equilibrium in the discrete-time case is not addressed here.
The paper is structured as follows: discrete-time linear quadratic games are defined in Section 2. Then, Nash and Stackelberg strategies are presented and the associated Riccati equations are derived for an open loop information structure. Sections 3 and 4 are dedicated to the analysis and the solutions of the nonsymmetric Riccati equations for Nash and Stackelberg equilibria respectively. Concluding remarks

make up Section 5.
For further details concerning these topics see [9].

2. Problem formulation.

For the sake of clarity, we restrict the presentation here to two player games. However, the results obtained can be easily extended to the case of n-person dynamic games.
Consider a discrete-time linear system with two decision makers (or players)

$$x(k+1) = Ax(k) + B_1 u_1(k) + B_2 u_2(k), \quad x(0) = x_0, \tag{1}$$

with

$$x(k) \in \mathbf{R}^n, \; u_i(k) \in \mathbf{R}^{r_i}, \; 1 \le i \le 2, \quad 0 \le k \le N-1.$$

Each player is trying to minimize its own cost functional subject to (1) by exploiting the available information to take the correct decision according to the sought strategy. The cost functionals of the players are defined by:

$$J_1 = \frac{1}{2} x^T(N) K_{1N} x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left[x^T(k) Q_1 x(k) + u_1^T(k) R_{11} u_1(k) + u_2^T(k) R_{12} u_2(k) \right],$$

$$J_2 = \frac{1}{2} x^T(N) K_{2N} x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left[x^T(k) Q_2 x(k) + u_1^T(k) R_{21} u_1(k) + u_2^T(k) R_{22} u_2(k) \right],$$

where all matrices are symmetric with

$$R_{ii} > 0 \quad \text{for} \quad i = 1, 2.$$

Notice that we do not assume here that all weighting matrices are positive semidefinite (which would unnecessarily restrict and essentially simplify the problem).
It is assumed here that the information structure of both players is of the open loop type, i.e. no state measurements are available during the optimization period and each player computes its optimal policy at the beginning of the game and is committed to follow that policy during the whole period. Depending on the strategy sought, necessary conditions for an equilibrium can be derived (see [5]). Nash and Stackelberg strategies are now introduced.

2.1. Nash strategy.

The Nash equilibrium strategies (u_1^*, u_2^*) are defined as satisfying the conditions:

$$J_1(u_1^*, u_2^*) \le J_1(u_1, u_2^*),$$

$$J_2(u_1^*, u_2^*) \le J_2(u_1^*, u_2).$$

Thus, the Nash strategy safeguards against a player deviating from the equilibrium strategy. It becomes reasonable when cooperation between decision makers cannot be guarantee. It has been shown by Pindyck [13] (see also [5]) that the necessary conditions for an open loop Nash strategy for the game defined by (1), (2) are given by

$$u_1(k) = -R_{11}^{-1} B_1^T \psi_1(k+1), \;\; u_2(k) = -R_{22}^{-1} B_2^T \psi_2(k+1), \tag{3}$$

where the costate vectors $\psi_i(k)$ must satisfy

$$\psi_i(k) = Q_i x(k) + A^T \psi_i(k+1), \; \psi_i(N) = K_{iN} x(N), \;\; 1 \le i \le 2, \; 0 \le k \le N-1. \tag{4}$$

Due to the linearity of the above equations *we suppose here* that

$$\psi_i(k) = K_i(k) x(k), \;\; 1 \le i \le 2, \;\; 0 \le k \le N, \tag{5}$$

which implies that (1) can be written as

$$0 = Ax(k) - \left[I + S_1 K_1(k+1) + S_2 K_2(k+1) \right] x(k+1) \tag{6}$$

with $x(0) = x_0$ and $S_i = B_i R_{ii}^{-1} B_i^T,\ \ 1 \le i \le 2$. If $[I + S_1 K_1(k+1) + S_2 K_2(k+1)]$ is invertible this means

$$x(k+1) = [I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A x(k),\ \ 0 \le k \le N-1 \tag{7}$$

Here and in the sequel $I \in \mathbf{R}^{n \times n}$ denotes the identity matrix.
From (4),(5) and (6) we infer that K_1, K_2 must be solutions of the discrete -time open loop Nash Riccati difference equations

$$\begin{aligned} K_1(k) &= Q_1 + A^T K_1(k+1)\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A, \quad K_1(N) = K_{1N}, \\ K_2(k) &= Q_2 + A^T K_2(k+1)\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A, \quad K_2(N) = K_{2N}, \end{aligned} \tag{8}$$

provided the inverses in (8) exist.
The asymptotic behavior of such difference equations is discussed in Section 2 of this paper where necessary conditions are established for the existence of constant solutions of (8), i.e. solutions of the coupled algebraic Riccati equations

$$\begin{aligned} K_1 &= Q_1 + A^T K_1\,[I + S_1 K_1 + S_2 K_2]^{-1} A, \\ K_2 &= Q_2 + A^T K_2\,[I + S_1 K_1 + S_2 K_2]^{-1} A. \end{aligned} \tag{9}$$

In the sequel we write the (pairs of) solutions of (9) in the form $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$.

2.2. Stackelberg strategy.

The concept of Stackelberg equilibrium has been introduced in dynamic differential games by Chen and Cruz [7] and Simaan and Cruz [14]. In such a strategy, one of the players, the leader, has the ability to enforce his strategy on the second player, the follower. A hierarchical equilibrium solution concept as proposed by H. von Stackelberg in 1934 (see [5] for details) must be defined. In fact, the follower is restricted to those strategies minimizing its own cost for a given strategy of the leader.
When player 2 acts as a leader and player 1 as follower, necessary conditions for an open loop Stackelberg strategy (3) are given by (see [5], Theorem 7.1 and Corollary 7.1)

$$\begin{aligned} \psi_1(k) &= Q_1 x(k) + A^T \psi_1(k+1), \\ \psi_2(k) &= Q_2 x(k) + A^T \psi_2(k+1) + Q_1 \gamma(k), \\ \gamma(k+1) &= S_{21} \psi_1(k+1) - S_1 \psi_2(k+1) + A \gamma(k), \\ \psi_1(N) = K_{1N} x(N),&\ \ \psi_2(N) = K_{2N} x(N) + K_{1N} \gamma(N),\ \ \gamma(0) = 0, \end{aligned} \tag{10}$$

with $S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T$ and where again $S_i = B_i R_{ii}^{-1} B_i^T,\ \ 1 \le i \le 2$.
If $\psi_i(k) = K_i(k) x(k), 1 \le i \le 2$, and $\gamma(k) = P(k) x(k)$, then again (1) can be rewritten in the form (7), where K_1, K_2 are obtained from the coupled system of difference equations

$$\begin{aligned} K_1(k) &= Q_1 + A^T K_1(k+1)\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A, \\ K_2(k) &= Q_2 + A^T K_2(k+1)\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A + Q_1 P(k), \\ & P(k+1)\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A = \\ & [S_{21} K_1(k+1) - S_1 K_2(k+1)]\,[I + S_1 K_1(k+1) + S_2 K_2(k+1)]^{-1} A + A P(k), \end{aligned} \tag{11}$$

with

$$K_1(N) = K_{1N}, K_2(N) = K_{2N} + K_{1N} P(N), P(0) = 0,$$

provided the inverses in (11) exist.
Notice that - in contrast to (8) - the equations (11) cannot be integrated backwards since they are not decoupled at the terminal condition; i.e. the coupled system (11) constitutes a two-point boundary value problem, since the starting conditions are specified at both endpoints.

3. Nash Games.

In the sequel we *assume that A is regular*; this implies that (after an index transformation) the difference equations in (4) and (6) can be rewritten as

$$\begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}(m+1) = M_{Na} \begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}(m) \tag{12}$$

where $m := N - k,\ \tilde{x}(m) = x(N+1-m),\ \tilde{\psi}_i(m) = \psi_i(N+1-m),\ 1 \le i \le 2$, and

$$M_{Na} = \begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1A^{-1} & A^T + Q_1A^{-1}S_1 & Q_1A^{-1}S_2 \\ Q_2A^{-1} & Q_2A^{-1}S_1 & A^T + Q_2A^{-1}S_2 \end{pmatrix}.$$

Obviously the sequence $\begin{pmatrix} \tilde{x} \\ \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}(m),\ m \ge 1$, is uniquely defined if its initial value is known.
It should be noted that if A is not invertible, a more general matrix pencil approach, in analogy with the standard linear-quadratic case (see [12]), has to be used; we do not discuss this here.

Since it turns out that the asymptotic behavior of this sequence as $m \to \infty$ is related to the behavior of the solutions of the algebraic Riccati equation (9), we present next some results concerning the interconnection of (9) and the corresponding linear difference equation (12).

Theorem 1. *(i) If $S(K_1, K_2) :=$ span $\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} \subset \mathbb{C}^{3n\times n}$ is an invariant subspace of M_{Na} with $\det(I + S_1K_1 + S_2K_2) \ne 0$ then $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is a solution of (9).*
(ii) If $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \in \mathbb{C}^{2n\times n}$ is a solution of the algebraic Riccati equation (9), then $S(K_1, K_2) \subset \mathbb{C}^{3n\times n}$ is an invariant subspace of M_{Na}. Moreover $F_{c\ell}^{-1} = A^{-1}(I + S_1K_1 + S_2K_2)$, which is the inverse of the corresponding closed loop matrix *$F_{c\ell}$, is the matrix of the restriction of M_{Na} to $S(K_1, K_2)$ with respect to the basis defined by the columns of $\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$.*

Proof. (i) If $S(K_1, K_2)$ is M_{Na}-invariant there exists a matrix $R \in \mathbf{C}^{n\times n}$ with

$$\begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1A^{-1} & A^T + Q_1A^{-1}S_1 & Q_1A^{-1}S_2 \\ Q_2A^{-1} & Q_2A^{-1}S_1 & A^T + Q_2A^{-1}S_2 \end{pmatrix} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$$

$$=: M_{Na} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} R \tag{13}$$

The first *row* of (13) yields $R = A^{-1}(I + S_1K_1 + S_2K_2)$, hence we obtain, using the second and third *row* of (13), that

$$\begin{aligned} Q_1A^{-1}(I + S_1K_1 + S_2K_2) + A^TK_1 &= K_1A^{-1}(I + S_1K_1 + S_2K_2), \\ Q_2A^{-1}(I + S_1K_1 + S_2K_2) + A^TK_2 &= K_2A^{-1}(I + S_1K_1 + S_2K_2). \end{aligned} \tag{14}$$

This means that $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is a solution of (9) since we assumed $\det(I + S_1K_1 + S_2K_2) \ne 0$.
(ii) If vice versa $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is a solution of (9) then (14) is verified and (13) holds with
$R := A^{-1}(I + S_1K_1 + S_2K_2)$. □

Remark 1. (i) Theorem 1 shows that the solutions of the algebraic Riccati equation (9) can be determined from the generalized eigenvectors of M_{Na}; more precisely we have:

Let $\mathrm{span}(v_{\nu_1}, \ldots, v_{\nu_n})$ be an M_{Na}-invariant subspace such that $\det X \neq 0$ for $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} := (v_{\nu_1}, \ldots, v_{\nu_n})$,

then $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} := \begin{pmatrix} Y_1 X^{-1} \\ Y_2 X^{-1} \end{pmatrix}$ is a solution of (9) if $\det(I + S_1 K_1 + S_2 K_2) \neq 0$.

If M_{Na} has at least one eigenvalue of geometric multiplicity $\mu > 1$ then (9) may have an uncountable number of (real or complex) solutions; if all eigenvalues of M_{Na} have geometric multiplicity 1 then there exist at most $\begin{pmatrix} 3n \\ n \end{pmatrix}$ solutions of (9).

Notice that a solution $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ of (9) corresponding to $S(K_1, K_2) = \mathrm{span}\{v_{i_1}, \ldots, v_{i_n}\}$ is real if the generalized eigenvectors v_{i_k} corresponding to nonreal eigenvalues of M_{Na} are appearing in $\{v_{i_1}, \ldots, v_{i_n}\}$ in conjugate complex pairs.

(ii) From Theorem 1, (ii) we infer that the closed loop matrix $F_{c\ell} = (I + S_1 K_1 + S_2 K_2)^{-1} A$ is stable (in the discrete-time sense, i.e. that all the eigenvalues of $F_{c\ell}$ have modulus less than one) if the generalized eigenvectors spanning $S(K_1, K_2)$ are corresponding to eigenvalues of M_{Na} lying in the exterior of the closed unit circle; in this case $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is called a *stabilizing solution* of (9). Since M_{Na} has (counting multiplicity) $3n$ eigenvalues it is obvious that (9) may have several stabilizing solutions. Notice that on account of the substitution $m = N - k$ we have to use a solution of (9) corresponding to an invariant subspace of M_{Na} belonging to unstable eigenvalues if we want to have a corresponding stable closed loop matrix.

For the formulation of our next results we introduce the following notations:

Notation. For arbitrary matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times k}$ and $Q \in \mathbb{C}^{m\times n}$we define:
(i) λ is called an *unobservable mode (of rank r)* of the pair (Q, A) if there exist vectors $p_j \in \mathbb{C}^n \setminus \{0\}$, $0 \leq j \leq r-1$, and $p_{-1} = 0$ such that

$$(A - \lambda I)p_j = p_{j-1} \ \text{ and } \ Qp_j = 0 \ \text{ for } \ 0 \leq j \leq r-1.$$

Subspaces of $\mathbb{C}^n$ spanned by chains of such generalized eigenvectors of A and unions of such subspaces are called (Q, A) unobservable subspaces corresponding to the eigenvalue λ of A.
(ii) λ is called an *uncontrollable mode (of rank r)* of the pair (A, B) if there exist vectors $y_j \in \mathbb{C}^n \setminus \{0\}$, $0 \leq j \leq r-1$, and $y_{-1} = 0$ such that

$$y_j^T (A - \lambda I) = y_{j-1}^T \ \text{ and } \ B^T y_j = 0 \ \text{ for } \ 0 \leq j \leq r-1.$$

Subspaces of $\mathbb{C}^n$ spanned by chains of such generalized eigenvectors of A and unions of such subspaces are called (A, B) uncontrollable subspaces corresponding to the eigenvalue λ of A.

Using these notations we get:

Lemma 1. *(i) For* $1 \leq \nu \leq 2$ λ *is an uncontrollable mode of* (A, B_ν) *of rank* r *corresponding to the chain* y_j, $0 \leq j \leq r-1$, *if and only if* $\begin{pmatrix} 0 \\ y_j \\ y_j \end{pmatrix}$, $0 \leq j \leq r-1$, *is a chain of generalized eigenvectors of* M_{Na} *corresponding to the eigenvalue* λ.
(Notice that this implies that λ *is also an uncontrollable mode of rank* r *of* (A, B_0) *with the same chain).*
(ii) λ *is an uncontrollable mode of rank* r *of* (A, B_1) *(respectively* (A, B_2)*) corresponding to the chain* y_j, $0 \leq j \leq r-1$, *if and only if* $\begin{pmatrix} 0 \\ y_j \\ 0 \end{pmatrix}$, $0 \leq j \leq r-1$, *(respectively* $\begin{pmatrix} 0 \\ 0 \\ y_j \end{pmatrix}$, $0 \leq j \leq r-1,$*) is a chain of generalized eigenvectors of* M_{Na} *corresponding to the eigenvalue* λ

Subspaces of $\mathbb{C}^{3n}$ spanned by chains of generalized eigenvectors of M_{Na} as given in Lemma 1, (ii) and unions of such subspaces are called *special* (A, B_1) uncontrollable subspaces (respectively *special* (A, B_2) uncontrollable subspaces) of M_{Na}.

Proof. Let $y_{-1} := 0$. From $\det A \neq 0, R_{\nu\nu} > 0$ and $S_\nu = B_\nu R_{\nu\nu}^{-1} B_\nu^T$ it follows that $A^{-1} S_\nu y = 0$ if and only if $B_\nu^T y = 0$. Therefore it follows from

$$M_{Na} \begin{pmatrix} 0 \\ y_j \\ y_j \end{pmatrix} = \begin{pmatrix} A^{-1}S_1 y_j + A^{-1} S_2 y_j \\ Q_1 A^{-1} S_1 y_j + Q_1 A^{-1} S_2 y_j + A^T y_j \\ Q_2 A^{-1} S_1 y_j + Q_2 A^{-1} S_2 y_j + A^T y_j \end{pmatrix}$$

that $y_j^T (A - \lambda I) = y_{j-1}^T$ and
$B_\nu^T y_j = 0$ for $0 \leq j \leq r-1, 1 \leq \nu \leq 2$ holds if and only if

$$M_{Na} \begin{pmatrix} 0 \\ y_j \\ y_j \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ y_j \\ y_j \end{pmatrix} + \begin{pmatrix} 0 \\ y_{j-1} \\ y_{j-1} \end{pmatrix}, \quad 0 \leq j \leq r-1.$$

This proves (i); similarly one can prove (ii). □

Using $M_{Na} \begin{pmatrix} y_j \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^{-1} y_j \\ Q_1 A^{-1} y_j \\ Q_2 A^{-1} y_j \end{pmatrix}$ we obtain analogously to the proof of Lemma 1:

Lemma 2. *$\lambda \neq 0$ is an unobservable mode of rank r of the pairs (Q_1, A^{-1}) and (Q_2, A^{-1}) corresponding to the chain y_j, $0 \leq j \leq r-1$, if and only if $\begin{pmatrix} y_j \\ 0 \\ 0 \end{pmatrix}$, $0 \leq j \leq r-1$, is a chain of generalized eigenvectors of M_{Na} corresponding to the eigenvalue λ.*

Subspaces of $\mathbb{C}^{3n}$ spanned by chains of generalized eigenvectors of M_{Na} as given in Lemma 2 and unions of such subspaces are called *special* (Q_1, A^{-1}), (Q_2, A^{-1}) unobservable subspaces corresponding to the eigenvalue λ of M_{Na}.

The next Theorem is an immediate consequence of Lemma 1 and Lemma 2.

Theorem 2. *Let $S = span \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$ with $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} = (v_{\nu_1}, \ldots, v_{\nu_n})$ be a n-dimensional M_{Na}-invariant subspace of $\mathbb{C}^{3n}$.*
(i) If S contains a nontrivial special (A, B_ν) uncontrollable subspace of M_{Na} for some $\nu \in \{0,\ 1,\ 2\}$ then $\det X = 0$; i.e. S does not correspond to a (finite) solution of (9).
(ii) If S contains a nontrivial special (Q_1, A) and (Q_2, A) unobservable subspace of M_{Na} then $\det Y_1 = \det Y_2 = 0$.

Notice that part (ii) of Theorem 2 provides necessary conditions for the invertibility of the matrices $Y_1 X^{-1}$ and $Y_2 X^{-1}$, defining a solution of the algebraic Riccati equation (9); a similar statement could be made concerning the relevance of Theorem 7, (ii) and (iv). It is well known that (A, B_ν) is controllable if and only if

$$\text{rank } (B_\nu, AB_\nu, \cdots, A^{n-1} B_\nu) = n,$$

which is equivalent to

$$L_\nu := \text{ Kernel of } \begin{pmatrix} B_\nu^T \\ B_\nu^T A^T \\ \vdots \\ B_\nu^T (A^T)^{n-1} \end{pmatrix} = \{0\}.$$

Notice that $L_0 = L_1 \cap L_2$. Now we prove another characterization of the controllability of (A, B_0).

Theorem 3. *(i) The subspace*

$$\mathcal{S}_0 = \{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \in \mathbb{C}^{3n} | y \in L_0 \} \tag{15}$$

is the maximal M_{Na}-invariant subspace having a basis-matrix of the form $\begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix}$.

(ii) (A, B_0) is controllable if and only if M_{Na} does not have a nontrivial invariant subspace of the form (15).

Proof. We prove (i) as follows:

a) Let $y \in L_0$. Using the theorem of Cayley-Hamilton it follows from the definition of L_0 that $B^T(A^T)^j y = 0$ for $j \in \mathbf{N} \cup \{0\}$, therefore $A^T y \in L_0$.
For $y \in L_0$ we get

$$M_{Na} \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = \begin{pmatrix} A^{-1}S_1 y + A^{-1}S_2 y \\ Q_1 A^{-1} S_1 y + Q_1 A^{-1} S_2 y + A^T y \\ Q_2 A^{-1} S_1 y + Q_2 A^{-1} S_2 y + A^T y \end{pmatrix} = \begin{pmatrix} 0 \\ A^T y \\ A^T y \end{pmatrix} \in \mathcal{S}_0$$

which shows that $\mathcal{S}_0$ is M_{Na}-invariant.

b) Let $\mathcal{S} = \text{ span } \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix}$ be M_{Na}-invariant. Then there exists $P \in \mathbb{C}^{k \times k}$ with

$$M_{Na} \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ YP \\ YP \end{pmatrix} = \begin{pmatrix} A^{-1}S_1 Y & + & A^{-1}S_2 Y \\ Q_1 A^{-1} S_1 Y & + & Q_1 A^{-1} S_2 Y + A^T Y \\ Q_2 A^{-1} S_1 Y & + & Q_2 A^{-1} S_2 Y + A^T Y \end{pmatrix}. \tag{16}$$

Since $\det A \neq 0$ we infer from the first *row* of (16) that $(S_1 + S_2)Y = 0$. Consequently it follows with $R_{ii} > 0, i = 1, 2$, that

$$Y^T B_i R_{ii}^{-1} B_i^T Y = 0, \ \ 1 \leq i \leq 2.$$

This implies

$$B_1^T Y = 0 \ \text{ and } \ B_2^T Y = 0 \tag{17}$$

and, on account of (16),

$$M_{Na} \begin{pmatrix} 0_{n,k} \\ Y \\ Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ A^T Y \\ A^T Y \end{pmatrix} = \begin{pmatrix} 0_{n,k} \\ YP \\ YP \end{pmatrix}. \tag{18}$$

From (17) and (18) we infer that

$$B_i^T A^T Y = B_i^T Y P = 0_{n,k} P = 0_{n,k} \text{ for } i = 1, 2.$$

Repeating this step we obtain, using for $j \geq n$ the Cayley-Hamilton theorem, that

$$B_i^T (A^T)^j Y = 0_{n,k} \text{ for } i = 1, 2 \text{ and } j \in \mathbf{N} \cup \{0\}.$$

Hence $\text{span } Y \subset L_0$ and $\mathcal{S} \subset \mathcal{S}_0$. □

This proves assertion (i).
For the proof of (ii) we use the Hautus-criterion, which implies that (A, B_0) is uncontrollable if and only if there exist a vector $y \neq 0$ and a $\lambda \in \mathbb{C}$ with

$$A^T y = \lambda y, \ \ B_0^T y = 0;$$

notice that in this case $M_{Na}\begin{pmatrix}0\\y\\y\end{pmatrix}=\lambda\begin{pmatrix}0\\y\\y\end{pmatrix}$.
Using this fact and the definition of $\mathcal{S}_0$ and L_0 we get assertion (ii).

Remark 2. In analogy to the continuous-time case we can show that the Riccati difference equation (8) and the linear difference equation (see (12))

$$\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(m+1)=M_{Na}\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(m),\ 1\le m\le N,\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(1)=\begin{pmatrix}I\\ K_{1N}\\ K_{2N}\end{pmatrix} \tag{19}$$

are equivalent in the following sense:

(i) Let $K_1(k),K_2(k),N\ge k\ge 0$, be solutions of (8) with $K_i(N)=K_{iN},\ 1\le i\le 2$.
If (with $m:=N-k$) for $1\le m\le N$,

$$\tilde{X}(m+1)=A^{-1}\tilde{X}(m)+A^{-1}S_1\tilde{\Psi}_1(m)+A^{-1}S_2\tilde{\Psi}_2(m),\quad \tilde{X}(1)=I, \tag{20}$$

$$\tilde{\Psi}_i(m+1):=K_i(N-m)\tilde{X}(m+1),\quad \tilde{\Psi}_i(1)=K_{iN},\ 1\le i\le 2, \tag{21}$$

then $\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(m), 1\le m\le N+1$, is a solution of (19).

(ii) If vice versa $\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(m), 1\le m\le N+1$, is a solution of (19) with invertible matrices $\tilde{X}(m)$ and $I+S_1\tilde{\Psi}(m)^{-1}+S_2\tilde{\Psi}(m)^{-1}, 1\le m\le N+1$, then $K_i(N-m):=\tilde{\Psi}_i(m+1)\tilde{X}(m+1)^{-1}$, $1\le i\le 2, 1\le m\le N+1$, defines a solution of (8).

Proof. (i) Let $K_1(k),K_2(k)$ solve (8) and let $\tilde{X}(m)$ and $\tilde{\Psi}_i(m),\ 1\le m\le N+1$, be defined by (20) and (21). Obviously (20) implies the first *row* of (19). Using this identity it follows from (8) and (21) that for $i=1,2$ and $1\le m\le N$

$$\begin{aligned}
&\tilde{\Psi}_i(m+1)=K_i(N-m)\tilde{X}(m+1)\\
&=\{Q_i+A^TK_i(N-m+1)[I+S_1K_1(N-m+1)+S_2K_2(N-m+1)]^{-1}A\}\tilde{X}(m+1)\\
&=Q_iA^{-1}[I+S_1K_1(N-m+1)+S_2K_2(N-m+1)]\tilde{X}(m)\\
&+A^TK_i(N-m+1)[I+S_1K_1(N-m+1)+S_2K_2(N-m+1)]^{-1}AA^{-1}[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]\\
&=Q_iA^{-1}[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]+A^T\tilde{\Psi}_i(m),
\end{aligned}$$

which is equivalent to the second and third *row* of (19).

(ii) (19) implies for $1\le m\le N$

$$\begin{pmatrix}\tilde{X}\\ \tilde{\Psi}_1\\ \tilde{\Psi}_2\end{pmatrix}(m+1)=\begin{pmatrix}A^{-1}[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]\\ A^T\tilde{\Psi}_1(m)+Q_1A^{-1}[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]\\ A^T\tilde{\Psi}_2(m)+Q_2A^{-1}[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]\end{pmatrix}.$$

Replacing $\tilde{X}(m+1)$ by the expression on the right hand side of this identity we get

$$\begin{pmatrix}\tilde{\Psi}_1(m+1)\tilde{X}(m+1)^{-1}\\ \tilde{\Psi}_2(m+1)\tilde{X}(m+1)^{-1}\end{pmatrix}=\begin{pmatrix}Q_1+A^T\tilde{\Psi}_1(m)[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]^{-1}A\\ Q_2+A^T\tilde{\Psi}_2(m)[\tilde{X}(m)+S_1\tilde{\Psi}_1(m)+S_2\tilde{\Psi}_2(m)]^{-1}A\end{pmatrix},$$

which shows that $K_i(N-m)=\tilde{\Psi}_i(m)\tilde{X}(m)^{-1},\ 1\le i\le 2,\ 1\le m\le N$, solve (8). □

Since the dynamics of the difference equation is very simple it can be used to investigate the asymptotic behavior of the sequences $K_i(k)$ for $k \to -\infty$.
Let $V = (v_1, \cdots, v_{3n}) \in \mathbf{C}^{3nx3n}$ be a matrix defined by a Jordan basis of generalized eigenvectors of M_{Na} such that

$$V^{-1}\, M_{Na}\, V = diag\,(J_1, \cdots, J_p) = \begin{pmatrix} \lambda_1 & * & 0 \\ \ddots & \ddots & * \\ 0 & \ddots & \lambda_{3n} \end{pmatrix}$$

with $* \in \{0, 1\}$ and (without loss of generality)

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{3n}|.$$

M_{Na} (and the system (8) of Riccati difference equations) is called *dichotomically separable* if $|\lambda_{2n}| < |\lambda_{2n+1}|$. If in this case in addition the n-dimensional matrix X defined by $\begin{pmatrix} X \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = (v_{2n+1}, \cdots, v_{3n})$ is invertible then it follows from Theorem 1 that $K_1^d = \Psi_1 X^{-1},\ K_2^d = \Psi_2 X^{-1}$ defines a real solution of the algebraic Riccati equation (9), this solution is called the *dichotomic solution* of (9).
We say that $\begin{pmatrix} K_{1N} \\ K_{2N} \end{pmatrix}$ belongs to $\mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$, the *generalized basin of attraction* of $\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$, if the last n rows of the matrix $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{3n} \end{pmatrix} = V^{-1} \begin{pmatrix} I \\ K_{1N} \\ K_{2N} \end{pmatrix}$ are linearly independent.
Notice that $\mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$ is an open and dense subset of $\mathbf{C}^{2n\times n}$, more precisely, its complement with respect to $\mathbb{C}^{n\times n}$ consists of a finite union of sets of dimensions $<\ 2n^2$.

Theorem 4. *Assume that the dichotomic solution* $\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$ *of (9) exists and that* $\begin{pmatrix} K_{1N} \\ K_{2N} \end{pmatrix} \in \mathbf{GB}\begin{pmatrix} K_1^d \\ K_2^d \end{pmatrix}$. *If the sequence* $\begin{pmatrix} K_1(k) \\ K_2(k) \end{pmatrix}$ *corresponding to the solution of (8) with* $K_i(N) = K_{iN}$ *(for i = 1, 2) is defined for all* $k \leq N$ *then*

$$\lim_{k\to-\infty} K_i(k) = K_i^d \ for\ i = 1, 2.$$

Proof. Let the sequence $\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m),\ m \geq 1$, be defined like in (19). Then it follows (under our assumptions) that

$$\begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m+1) = M_{Na} \begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m) = M_{Na}^m \begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(1) = M_{Na}^{m-1} M_{Na} V V^{-1} \begin{pmatrix} I \\ K_{1N} \\ K_{2N} \end{pmatrix} =$$

$$M_{Na}^{m-1} VJC = M_{Na}^{m-2} VJ^2 C = VJ^m C = (0,\ \cdots\ , 0, v_{2n+1},\ \cdots\ , v_{3n}) J^m C + R(m),$$

where the elements of $R(m)$ are (for $m \to \infty$) bounded by a constant times $|\lambda_{2n}^m|$.
Hence we infer as with the proof of the classical power method that

$$\lim_{m\to\infty} span \begin{pmatrix} \tilde{X} \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}(m) = span\ (v_{2n+1}, \cdots, v_{3n}) = span \begin{pmatrix} I_n \\ K_1^d \\ K_2^d \end{pmatrix}.$$

This proves the assertion of Theorem 4. □

Notice that here the order of convergence depends essentially on the ratio $\delta = \frac{|\lambda_{2n}|}{|\lambda_{2n+1}|}$.

Theorem 4 indicates that for large values of N and for problems having a dichotomic solution with a stable closed loop matrix $(I + S_1 K_1^d + S_2 K_2^d)^{-1} A$ it makes sense to replace the controls $u_i(k)$ defined in (3) by the constant (or static) stabilizing controls

$$\tilde{u}_i(k) = -R_{ii}^{-1} B_i^T K_i^d x(k),\ 1 \le i \le 2,\ 0 \le k \le N.$$

4. Stackelberg Games.

In this section we assume like in section 3 that A is regular. It is possible to formulate most of the results of section 3 in the context of Stackelberg games.
Notice that the difference equations in (1), (10) can be rewritten as

$$\begin{pmatrix} x \\ \gamma \\ \psi_2 \\ \psi_1 \end{pmatrix}(k) = M_{St} \begin{pmatrix} x \\ \gamma \\ \psi_2 \\ \psi_1 \end{pmatrix}(k+1), 0 \le k \le N-1, \tag{22}$$

where

$$M_{St} = \begin{pmatrix} A^{-1} & 0 & A^{-1}S_2 & A^{-1}S_1 \\ 0 & A^{-1} & A^{-1}S_1 & -A^{-1}S_{21} \\ Q_2A^{-1} & Q_1A^{-1} & A^T + Q_1A^{-1}S_1 + Q_2A^{-1}S_2 & Q_2A^{-1}S_1 - Q_1A^{-1}S_{21} \\ Q_1A^{-1} & 0 & Q_1A^{-1}S_2 & A^T + Q_1A^{-1}S_1 \end{pmatrix}.$$

The algebraic (Stackelberg-) Riccati equation corresponding to (11) is

$$\begin{aligned} K_1 &= Q_1 + A^T K_1 [I + S_1K_1 + S_2K_2]^{-1} A, \\ K_2 &= Q_2 + A^T K_2 [I + S_1K_1 + S_2K_2]^{-1} A + Q_1 P, \\ P[I + S_1K_1 + S_2K_2]^{-1} A &= [S_{21}K_1 - S_1K_2][I + S_1K_1 + S_2K_2]^{-1} A + AP. \end{aligned} \tag{23}$$

The following theorem shows that the solutions of (23) can be determined from the n-dimensional invariant subspaces of M_{St}.

Theorem 5.
(i) If $\tilde{S}(K_1, K_2, P) := span\ (I, P^T, K_2^T, K_1^T)^T \subset \mathbb{C}^{4n\times n}$ *is an invariant subspace of* M_{St} *with* $\det(I + S_1K_1 + S_2K_2) \neq 0$, *then* $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ *is a solution of (23).*

(ii) If $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ *is a solution of (23) then* $\tilde{S}(K_1, K_2, P)$ *is an invariant subspace of* M_{St} *and* $A^{-1}(I + S_1K_1 + S_2K_2)$ *is the matrix of the restriction of* M_{St} *to* $\tilde{S}(K_1, K_2, P)$ *with respect to the basis defined by the columns of* $(I, K_1^T, K_2^T, P^T)^T$.

Proof. (i) If $\tilde{S}(K_1, K_2, P)$ is M_{St} - invariant then there exists a matrix $R \in \mathbb{C}^{n\times n}$ with

$$M_{St} \begin{pmatrix} I \\ P \\ K_2 \\ K_1 \end{pmatrix} = \begin{pmatrix} I \\ P \\ K_2 \\ K_1 \end{pmatrix} R. \tag{24}$$

The first line of (24) yields that $R = A^{-1}[I + S_1K_1 + S_2K_2]$, therefore we obtain from the last three *rows* of (24) that

$$A^{-1}P + A^{-1}[S_1K_2 - S_{21}K_1] = PA^{-1}[I + S_1K_1 + S_2K_2], \tag{25}$$

$$\begin{aligned} Q_2A^{-1} + Q_1A^{-1}P + A^TK_2 + Q_1A^{-1}S_1K_2 + Q_2A^{-1}S_2K_2 + Q_2A^{-1}S_1K_1 - Q_1A^{-1}S_{21}K_1 \\ = K_2A^{-1}[I + S_1K_1 + S_2K_2], \end{aligned} \tag{26}$$

$$Q_1A^{-1} + Q_1A^{-1}S_2K_2 + A^TK_1 + Q_1A^{-1}S_1K_1 = K_1A^{-1}[I + S_1K_1 + S_2K_2]. \tag{27}$$

Then we multiply the equations (25), (26), (27) from the right by $[I + S_1K_1 + S_2K_2]^{-1}A$ and in addition (25) also from the left by A. This yields

$$[P + S_1K_2 - S_{21}K_1][I + S_1K_1 + S_2K_2]^{-1}A = AP, \tag{28}$$

and, using (28),

$$Q_2 + A^TK_2[I + S_1K_1 + S_2K_2)^{-1}A + Q_1P = K_2, \tag{29}$$

$$Q_1 + A^TK_2[I + S_1K_1 + S_2K_2]^{-1}A = K_1, \tag{30}$$

which shows that $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ is a solution of (23).

(ii) If vice versa $\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}$ solves (23) then (25) - (30) are verified and (24) holds with $R = A^{-1}[I + S_1K_1 + S_2K_2]$. □

From

$$M_{St}\begin{pmatrix} y \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^{-1}y \\ 0 \\ Q_2A^{-1}y \\ Q_1A^{-1}y \end{pmatrix}, \; M_{St}\begin{pmatrix} 0 \\ 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} A^{-1}S_2y \\ A^{-1}S_1y \\ A^Ty + Q_1A^{-1}S_1y + Q_2A^{-1}S_2y \\ Q_1A^{-1}S_2y \end{pmatrix},$$

$$M_{St}\begin{pmatrix} 0 \\ y \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}y \\ Q_1A^{-1}y \\ 0 \end{pmatrix}, \; M_{St}\begin{pmatrix} 0 \\ 0 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} A^{-1}S_1y \\ -A^{-1}S_{21}y \\ Q_2A^{-1}S_1y - Q_1A^{-1}S_{21}y \\ A^Ty + Q_1A^{-1}S_1y \end{pmatrix}$$

and $S_{21} = B_1R_{11}^{-1}R_{12}R_{11}^{-1}B_1^T$ it follows as with the proof of Lemma 1 and Lemma 2:

Lemma 3. *$\lambda \neq 0$ is an unobservable mode of rank r of the pair (Q_1, A^{-1}) (resp. of the pairs (Q_1, A^{-1}) and (Q_2, A^{-1})) corresponding to the chain $y_j, 0 \leq j \leq r-1$, if and only if*

$$\begin{pmatrix} 0 \\ y_j \\ 0 \\ 0 \end{pmatrix}, 0 \leq j \leq r-1, (\textit{ resp. } \begin{pmatrix} 0 \\ y_j \\ 0 \\ 0 \end{pmatrix} \textit{ and } \begin{pmatrix} y_j \\ 0 \\ 0 \\ 0 \end{pmatrix}, 0 \leq j \leq r-1)$$

is a chain of generalized eigenvectors of M_{St} corresponding to the eigenvalue λ.

Lemma 4. *λ is an uncontrollable mode of rank r of the pair (A, B_1) (resp. of the pairs (A, B_1) and (A, B_2)) corresponding to the chain of generalized left eigenvectors $y_j^T, 0 \leq j \leq r-1$, if and only if*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ y_j \end{pmatrix}, 0 \leq j \leq r-1 \; (\textit{resp. } \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_j \end{pmatrix} \textit{ and } \begin{pmatrix} 0 \\ 0 \\ y_j \\ 0 \end{pmatrix}, 0 \leq j \leq r-1)$$

is a chain of generalized eigenvectors of M_{St} corresponding to the eigenvalue λ.

If we define *special* (Q_1, A^{-1}) unobservable and (A, B_1)- uncontrollable subspaces of M_{St} similarly to section 3 (see Lemma 2), we obtain necessary conditions for the existence of certain solutions of (23):

Theorem 6. *Let $\mathcal{S} = span(X^T, Y_1^T, Y_2^T, Y_3^T)^T$ with $(X^T, Y_1^T, Y_2^T, Y_3^T) =: (v_{\nu_1}, \ldots, v_{\nu_n})$ be a n-dimensional M_{St}-invariant subspace of $\mathbb{C}^{4n}$.*
(i) If $\mathcal{S}$ contains a nontrivial special (Q_1, A^{-1}) unobservable or (A, B_1) uncontrollable subspace of M_{St}, then $\det X = 0$, i.e. $\mathcal{S}$ does not correspond to a (finite) solution of the algebraic Riccati equation (23).

(ii) If $\mathcal{S}$ contains a nontrivial special (A, B_1) and (A, B_2) uncontrollable subspace of M_{St}, then $\det X = \det Y_1 = 0$.
(iii) If $\mathcal{S}$ contains a nontrivial special (Q_1, A^{-1}) unobservable subspace of M_{St}, then $\det Y_2 = 0$ *and* $\det Y_3 = 0$.

If we define the subspaces L_ν like in section 3 then it follows from the preceding results analogously to the proof of Theorem 3:

Theorem 7. *(i) The subspace*

$$\mathcal{S}_{1y} = \{ \begin{pmatrix} 0 \\ 0 \\ y \\ y \end{pmatrix} \in \mathbb{C}^{4n} | y \in L_0 \} \tag{31}$$

is the maximal M_{St}-invariant subspace having a basis-matrix of the form $\begin{pmatrix} 0_{2n,k} \\ Y \\ Y \end{pmatrix}$.

(ii) (A, B_0) is controllable if and only if M_{St} does not have a nontrivial invariant subspace of the form (31).
(iii) The subspace

$$\mathcal{S}_{2y} = \{ \begin{pmatrix} 0 \\ 0 \\ y \\ 0 \end{pmatrix} \in \mathbb{C}^{4n} | y \in L_1 \} \tag{32}$$

is the maximal M_{St}-invariant subspace having a basis-matrix of the form $\begin{pmatrix} 0_{2n,k} \\ Y \\ 0_{n,k} \end{pmatrix}$.

(iv) (A, B_1) is controllable if and only if M_{St} does not have a nontrivial invariant subspace of the form (32).

5. Conclusions.

We study discrete time Riccati difference equations and the corresponding algebraic Riccati equations which appear in the necessary conditions for the optimal strategies of open loop Nash and Stackelberg games.It is been shown how the solutions of these algebraic Riccati equations can be determined from the n-dimensional invariant subspaces of the system matrices M_{Na} and M_{St} of the corresponding linear systems.
Moreover we give necessary conditions for the existence of (stabilizing) solutions of the algebraic Riccati equations in terms of controllability and observability conditions; it is worth mentioning here that there may exist a finite or even uncountable number of stabilizing solutions for the algebraic Riccati equations (9) (see Remark 1) and (23). Notice that in contrast to standard symmetric algebraic Riccati equations it is not possible to give elementary sufficient conditions for the existence of stabilizing solutions.
In the case of Nash games we discuss also the asymptotic behavior of the Riccati difference equations under the assumptions that these solutions are defined for $m > 0$ (or $k < 0$). Sufficient conditions for the existence of these solutions will be derived in a subsequent paper.

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