Reconstructing parameters of a medium from incomplete spectral information

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Abstract. The problem of the synthesis of a stratified medium with specified amplitude and phase properties is investigated. The wave propagation in the medium is described by a system of differential equations. The synthesis problem considered in the paper relates to inverse problems of spectral analysis with incomplete spectral information. Using the contour integral method we study properties of spectral characteristics and obtain algorithms for the solution of the synthesis problem for differential equations with singularities.

AMS classification: 34A55, 34B05, 34E20, 34L05, 47E05

1. Introduction.

The paper deals with the following system of differential equations

$$\frac{dy_1}{dx} = i\rho\, R(x)\, y_2,\ \frac{dy_2}{dx} = i\rho\, \frac{1}{R(x)}\, y_1,\ x \in [0, T] \tag{1}$$

with the initial conditions $y_1(0,\rho) = 1,\ y_2(0,\rho) = -1$. Here $\rho = \sigma + i\tau$ is the spectral parameter, and $R(x)$ is a real function which is called the wave resistance.

System (1) describes the wave propagation in a stratified medium and often appears in optics, spectroscopy, in electrodynamic and acoustic problems. Radio engineering problems of the design of directional couplers for non-uniform electronics lines and synthesizing transitions between acoustic wave guides can also be reduced to studying system (1) (see [1 - 3] and references therein).
Some of these classes of synthesis problems relate to inverse problems of spectral analysis with incomplete spectral information. Several aspects of synthesis problems for system (1) were studied in [3 - 5] with the help of the transformation operator method. In this paper in order to construct the inverse problem theory for system (1) *with singularities and turning points* we use the contour integral method. It gives us an opportunity to obtain effective algorithms for the solution of such classes of inverse problems.

We shall consider three classes of coefficients $R(x)$. We shall say that $R(x) \in B_0$ if $R(x) \in W_2^2(0,T)$, $R(x) > 0$, $R(0) = 1$ and $R'(0) = 0$. We also consider the more general case when $R(x)$ has poles or zeros inside the interval $(0,T)$. We shall say that $R(x) \in B_0^-$ if $R(x)$ has the form

$$R(x) = \sum_{j=1}^{p} \frac{R_j}{(x-x_j)^2} + R_0(x),\ 0 < x_1 < \ldots < x_p < T,\ R_j > 0,$$

and $R_0(x) \in W_2^2(0,T)$, $R(x) > 0\ (x \neq x_j)$, $R(0) = 1$, $R'(0) = 0$. In particular, if $p = 0$, then $R(x) \in B_0$. We shall say that $R(x) \in B_0^+$ if $\dfrac{1}{R(x)} \in B_0^-$.

Let us consider the functions

$$f_1(\rho) = \frac{y_1(T,\rho) - R^0 y_2(T,\rho)}{2\sqrt{R^0}},\ f_2(\rho) = \frac{y_1(T,\rho) + R^0 y_2(t,\rho)}{2\sqrt{R^0}},\ R^0 = R(T),$$

which are called the transmission coefficients. Denote $\alpha_j(\sigma) = |f_j(\sigma)|$, $\sigma = Re\,\rho$. It is shown below (see (69)) that $\alpha_1^2(k) - \alpha_2^2(k) \equiv 1$.

For a wide class of synthesis problems, the phase is difficult or impossible to measure, while the amplitude is easily accessible to measurement. Such cases lead us to inverse spectral problems with incomplete information. In this paper we study the following incomplete inverse problem:

Inverse Problem 1. Given $\alpha_1(\sigma)$, construct $R(x)$.

In order to solve this inverse problem we first should reconstruct the transmission coefficients from their moduli $\alpha_j(\sigma)$ on the real line. Here we face with a lack of information which leads to nonuniqueness of the solution. For the construction of the transmission coefficients we use an additional information about their zeros. Further we calculate the so-called characteristic function: $\Delta(\rho) = f_1(\rho) + f_2(\rho)$, and obtain the solution of the inverse problem of recovering $R(x)$ from the characteristic function $\Delta(\rho)$.

We note that the inverse problem theory for differential operators with integrable coefficients has been described in $[6-10]$ and other works. Inverse problems for Sturm-Liouville equations with turning points habe been studied in [11]. Some aspects of the turning point theory and a number of applications are in [12 - 14].

An investigation of the classes B_0^- and B_0^+ is completely similar because the replacement $R \to 1/R$ is equivalent to the replacement $(y_1, y_2) \to (-y_2, -y_1)$. Below for definiteness we study the case when $R(x) \in R_0^-$.

2. Reduction to the Sturm-Liouville equation.

We introduce the functions

$$u(x,\rho) = y_1(x,\rho)/(R(x))^{1/2}, \quad v(x,\rho) = (R(x))^{1/2}\, y_2(x,\rho), \quad h(x) = R'(x)/(2R(x)). \tag{2}$$

Then the transmission coefficients can be written in the form

$$f_1(\rho) = \frac{u(T,\rho) - v(T,\rho)}{2},\ f_2(\rho) = \frac{u(T,\rho) + v(T,\rho)}{2}. \tag{3}$$

The function $u(x,\rho)$ satisfies the equation

$$-u'' + q(x)u = \lambda u\ ,\ \ \lambda = \rho^2\ , \tag{4}$$

and the initial conditions

$$u(0,\rho) = 1\ ,\ \ u'(0,\rho) = -i\rho\ , \tag{5}$$

where

$$q(x) = h^2(x) - h'(x)\ . \tag{6}$$

By virtue of (1), $y_1(x,0) \equiv 1$. Hence, taking (2) into account, we get

$$(R(x))^{1/2}\, u(x,0) \equiv 1\ . \tag{7}$$

Lemma 1. *$R(x) \in B_0^-$ if and only if $q(x) \in L_2(0,T)$, $u(T,0) \neq 0$.*

Proof. 1) Let $R(x) \in B_0^-$. It follows from (2) that for $x \to x_j$,

$$h(x) \sim -\frac{1}{x - x_j} + h_j^*(x),\ h^*(x) \in W_2^1\ ,\ h_j^*(x_j) = 0\,.$$

Taking (6) into account we conclude that $q(x) \in L_2(0,T)$. By virtue of (7), $u(T,0) \neq 0$.

2) Let now $q(x) \in L_2(0,T)$, $u(T,0) \neq 0$, and let $0 < x_1 < \ldots < x_p < T$ be zeros of $u(x,0)$. It follows from (7) that

$$R(x) = (u(x,0))^{-2}\ .$$

Hence $R(x) > 0 (x \neq x_j)$, $R(0) = 1$, $R'(0) = 0$, and

$$R(x) \sim \frac{R_j}{(x - x_j)^2}\ ,\ x \to x_j\ ,\ R_j > 0\ .$$

Denote

$$R_0(x) := R(x) - \sum_{j=1}^{p} \frac{R_j}{(x - x_j)^2}\ .$$

It is easy to see that $R_0(x) \in W_2^2(0,T)$. □

Remark 1. If $R(x) \in B_0^-$, then the functions $y_1(x,\rho)$ and $v(x,\rho)$ have singularities at $x = x_j$, and there exist finite limits

$$\lim_{x \to x_j} (x - x_j)\, v(x,\rho),\ \lim_{x \to x_j} (x - x_j)\ y_1(x,\rho)\ .$$

It follows from (7) that $\lim_{x \to x_j} (x - x_j)\,(R(x))^{1/2} = R_j$.

3. Properties of the characteristic function.

Denote $\Delta(\rho) = u(T,\rho)$. The function $\Delta(\rho)$ is called the characteristic function. The function

$\Delta(\rho)$ is entire in ρ of exponential type. We interested in zeros of $\Delta(\rho)$ in the upper half-plane $\Pi_+ := \{\rho : Im\,\rho > 0\}$.

Theorem 1. *Let $R(x) \in B_0^-$. Then in $\overline{\Pi}_+$ $\Delta(\rho)$ has at most a finite number of zeros $\{\rho_k\}_{k=\overline{1,m}}$, $m \geq 0$. All zeros of $\Delta(\rho)$ in $\overline{\Pi}_+$ are simple and pure imaginary, i.e. $\rho_k = i\tau_k,\ \tau_k > 0$.*

Proof. It follows from (4), (5) that

$$\overline{u(x,\rho)} = u(x,-\bar{\rho})\ , \tag{8}$$

and consequently

$$\overline{\Delta(\rho)} = \Delta(-\bar{\rho})\ . \tag{9}$$

Denote $\langle y, z\rangle := yz' - y'z$. Using (4), (5) and the Ostrogradskii-Liouville theorem we obtain

$$\langle u(x,\rho),\ u(x,-\rho)\rangle \equiv 2i\rho\ . \tag{10}$$

According to Lemma 1, $\Delta(0) \neq 0$. If $\Delta(\rho_0) = 0$ for a certain real $\rho_0 \neq 0$, then by virtue of (9), $\Delta(-\rho_0) = 0$. But it contradicts to (10) for $x = T$. Thus, $\Delta(\rho)$ has no zeros for real ρ.

Let $\rho_0 = \sigma_0 + i\tau_0 \in \Pi_+$ be zero of $\Delta(\rho)$. Since $u(x,\rho)$ satisfies (4), (5) and $u(T,\rho_0) = \overline{u(T,\rho_0)} = 0$ we calculate

$$(\rho_0^2 - \bar{\rho}_0^2)\int_0^T u(x,\rho_0)\overline{u(x,\rho_0)}dx = \Big|_0^T \langle u(x,\rho_0), \overline{u(x,\rho_0)}\rangle = -i(\rho_0 + \bar{\rho}_0)\ .$$

This yields $\sigma_0 = 0$, i.e. all zeros of $\Delta(\rho)$ in $\overline{\Pi}_+$ lie on the imaginary half-axis.

Let us show that in Π_+ all zeros of $\Delta(\rho)$ are simple. Suppose, on the contrary, that for a certain $\rho_0 = i\tau_0, \tau_0 > 0,\ \Delta(\rho_0) = \dot{\Delta}(\rho_0) = 0$, where $\dot{\Delta}(\rho) = \dfrac{d}{d\rho}\Delta(\rho)$. Denote $u_1(x,\rho) = \dfrac{d}{d\rho}\,u(x,\rho)$. Then, by virtue of (4), (5),

$$-u_1'' + q(x)\,u_1 = \rho^2\,u_1 + 2\rho u,\ u_1(0,\rho) = 0,\ u_1'(0,\rho) = -i\ . \tag{11}$$

In view of (8), the function $u(x,\rho_0)$ is real. Using (4), (5), (11) and the relation $u(T,\rho_0) = u_1(T,\rho_0) = 0$ we get for $\rho = \rho_0$

$$2\rho\int_0^T u^2(x,\rho)dx = \int_0^T u(x,\rho)\cdot(-u_1''(x,\rho) + q(x)\,u_1(x,\rho) - \rho^2\,u_1(x,\rho))dx =$$

$$= -\Big|_0^T \langle u(x,\rho), u_1(x,\rho)\rangle = -i\ ,$$

i.e.

$$2\tau\int_0^T \mid u(x,\rho)\mid^2 dx = -1\ ,$$

which is impossible.

It can be shown by the well-known method (see [15]) that for $\rho \in \overline{\Pi}_+\,,\ |\rho| \to \infty\,,$

$$u^{(j)}(x,\rho) = (-i\rho)^j\,e^{-i\rho x}\,(1 - \frac{1}{2i\rho}\int_0^x q(t)\,dt + \omega(x,\rho))\,,\ j = 0,1 \tag{12}$$

and hence

$$\Delta(\rho) = e^{-i\rho T}\,(1 - \frac{a}{i\rho} + \frac{\omega(\rho)}{\rho}),\ \rho \in \overline{\Pi}_+,\ |\rho| \to \infty\ , \tag{13}$$

where

$$a = \frac{1}{2}\int_0^T q(t)\,dt\ .$$

Here and in the sequel, one and the same symbol $\omega\,(\rho)$ (or $\omega\,(x,\rho)$) denotes various functions such that $\omega = o(1)$ as $|\,\rho\,| \to \infty\ ,\ \rho \in \overline{\Pi}_+$ (uniformly in $x \in [0,T]$) and $\omega \in L_2(-\infty,\infty)$ for real ρ (for each fixed x).

In particular, (13) implies that for large $\rho \in \overline{\Pi}_+$, $\Delta(\rho)$ has no zeros. □

Theorem 2. *The function $\Delta(\rho)$ has no zeros in $\overline{\Pi}_+$ iff $R(x) \in B_0$.*

Proof. Consider the real function $y(x,\tau) := u(x,i\tau),\ x \in [0,T],\ \tau \geq 0$. According to Theorem 1, for each fixed x, the function $y(x,\tau)$ has at most a finite number of simple zeros, and in view of (12) there exists $\tau^* > 0$ such that $y(x,\tau) > 0$ for all $\tau \geq \tau^*, x \in [0,T]$.

If $R(x) \in B_0$, then by virtue of (7), $y(x,0) > 0,\ x \in [0,T]$. Since $y(0,T) = 1, \tau \geq 0$ and $y(x,\tau^*) > 0,\ x \in [0,T]$, we get $y(x,\tau) > 0$ for all $\tau \geq 0,\ x \in [0,T]$. In particular, $\Delta(\tau) = y(T,\tau) > 0$ for all $\tau \geq 0$, i.e. $\Delta(\rho)$ has no zeros in $\overline{\Pi}_+$. The inverse assertion is proved similarly. □

4. Reconstruction of the wave resistance from the characteristic function.

In this section, using the contour integral method, we study the inverse problem of recovering $R(x)$ from the given characteristic function $\Delta(\rho)$. We provide necessary and sufficient conditions of its solvability along with algorithms for the solution.

4.1. Denote

$$p(x) = \begin{cases} q(T-x) & , \quad 0 \leq x \leq T \\ 0 & , \quad x > T\ , \end{cases}$$

and consider the equation

$$-y'' + p(x)y = \lambda y,\ x > 0 \tag{14}$$

on the half-line. Let $S(x,\lambda)$ and $C(x,\lambda)$ be solutions of (14) under the initial conditions $C(0,\lambda) = S'(0,\lambda) = 1, S(0,\lambda) = C'(0,\lambda) = 0$. Denote

$$\Phi(x,\lambda) = \frac{e(x,\rho)}{\delta(\rho)} \tag{15}$$

where $e(x,\rho)$ is the solution of (14) such that $e(x,\rho) \equiv e^{i\rho x}$ for $x \geq T$, and $\delta(\rho) := e(0,\rho)$. Clearly,

$$e(x,\rho) = e^{i\rho T}\,u(T-x,\rho),\ 0 \leq x \leq T\ ;\ \ \delta(\rho) = e^{i\rho T}\,\Delta(\rho)\ . \tag{16}$$

We introduce the function

$$M(\lambda) := \frac{e'(0\,,\,\rho)}{\delta(\rho)}\ , \tag{17}$$

which is called the Weyl function. Since $\Phi(0,\lambda) = 1, \Phi'(0,\lambda) = M(\lambda)$, we have

$$\Phi(x,\lambda) = C(x,\lambda) + M(\lambda)\, S(x,\lambda)\ , \tag{18}$$

$$\langle \Phi(x,\lambda),\, S(x,\lambda)\rangle \equiv 1\ . \tag{19}$$

According to Theorem 1 and the relation $\delta(\rho) = e^{i\rho T}\,\Delta(\rho)$, the function $\delta(\rho)$ has at most a finite number of simple zeros in $\overline{\Pi}_+$ of the form $\rho_k = i\tau_k,\ \tau_k > 0,\ k = \overline{1,m},\ m \geq 0$. Moreover, by virtue of (13) and (16),

$$\delta(\rho) = 1 - \frac{a}{i\rho} + \frac{\omega(\rho)}{\rho}, \quad \rho \in \overline{\Pi}_+,\ |\rho| \to \infty\ . \tag{20}$$

Since $\langle e(x,\rho),\, S(x,\lambda)\rangle \equiv \delta(\rho)$ and $\delta(\rho_k) = 0$, we get

$$e(x,\rho_k) = e'(0,\rho_k)\, S(x,\lambda_k), \quad \lambda_k = \rho_k^2 < 0\ . \tag{21}$$

Denote

$$\beta_k := \left(\int_0^\infty S^2(x,\lambda_k)\, dx\right)^{-1}\ .$$

Lemma 2.

$$\beta_k = -\frac{4i\,\rho_k^2}{\Delta(-\rho_k)\,\dot{\Delta}(\rho_k)} > 0\ . \tag{22}$$

Proof. Since $e(x,\rho)$ satisfies (14) and $e(0,\rho_k) = 0$, we have

$$(\rho^2 - \rho_k^2)\int_0^\infty e(x,\rho)\, e(x,\rho_k)dx = \Big|_0^\infty \langle e(x,\rho),\, e(x,\rho_k)\rangle = -e'(0,\rho_k)\delta(\rho).$$

As $\rho \to \rho_k$ it gives us

$$2\rho_k \int_0^\infty e^2(x,\rho_k)dx = -e'(0,\rho_k)\,\dot{\delta}\,(\rho_k)\ .$$

Using (21) we deduce

$$\beta_k = -\frac{2\rho_k\, e'(0,\rho_k)}{\dot{\delta}(\rho_k)}\ . \tag{23}$$

It follows from (10) and (16) that

$$\langle e(x,\rho),\, e(x,-\rho)\rangle \equiv -2i\rho\ , \tag{24}$$

and consequently

$$\delta(-\rho_k)\, e'(0,\rho_k) = 2i\rho_k\ . \tag{25}$$

Substituting (25) into (23) and taking into account the relations $\dot{\delta}(\rho_k) = e^{i\rho_k T}\dot{\Delta}(\rho_k)$, $\delta(-\rho_k) = e^{-i\rho_k T}\,\Delta(-\rho_k)$, we arrive at (22). □

Denote by Π the λ-plane with the cut $\lambda \geq 0$, and $\Lambda = \{\lambda_k\}_{k=\overline{1,m}},\ \lambda_k = \rho_k^2 < 0$.

Theorem 3. *The Weyl function $M(\lambda)$ is holomorphic in $\Pi \setminus \Lambda$. In the points $\lambda = \lambda_k$ the Weyl function $M(\lambda)$ has simple poles, and*

$$\operatorname*{Res}_{\lambda=\lambda_k} M(\lambda) = -\beta_k \,. \tag{26}$$

For $\lambda > 0$, there exist the finite limits $M^{\pm}(\lambda) = \lim M(\lambda \pm iz)$, $z \to 0, Re\, z > 0$, and

$$V(\lambda) := \frac{1}{2\pi i}(M^{+}(\lambda) - M^{-}(\lambda)) = \frac{\rho}{\pi|\Delta(\rho)|^2} \;, \quad \rho > 0\,. \tag{27}$$

For $|\lambda| \to \infty$,

$$M(\lambda) = i\rho + \omega(\rho) \;, \quad \rho \in \overline{\Pi}_{+} \;, \tag{28}$$

$$V(\lambda) = \frac{\rho}{\pi} + \omega(\rho) \;, \quad \rho > 0 \,. \tag{29}$$

Proof. The domain Π in the λ-plane corresponds to Π_+ in the ρ-plane. By virtue of (17), $M(\lambda)$ is holomorphic in $\Pi \setminus \Lambda$, continuous in $\overline{\Pi} \setminus \Lambda$, and we have

$$\operatorname*{Res}_{\lambda=\lambda_k} M(\lambda) = e'(0,\rho_k) \cdot \left(\left(\frac{d}{d\lambda}\, \delta(\rho) \right)_{/\, \rho=\rho_k} \right)^{-1} .$$

Together with (23) it gives (26).

Further, using (17) and (24), we calculate for $\rho > 0$:

$$M^{+}(\lambda) - M^{-}(\lambda) = \frac{e'(0,\rho)}{\delta(\rho)} - \frac{e'(0,-\rho)}{\delta(-\rho)} = \frac{2i\rho}{|\delta(\rho)|^2} = \frac{2i\rho}{|\Delta(\rho)|^2} \,,$$

i.e. (27) is valid.

According to (12) and (16) we get for $|\rho| \to \infty$, $\rho \in \overline{\Pi}_+$,

$$e^{(j)}\,(x,\rho) = (i\rho)^j\, e^{i\rho x} \left(1 - \frac{1}{2i\rho} \int_x^T p(t)dt + \frac{\omega(x,\rho)}{\rho} \right) \,,\; 0 \le x \le T\,. \tag{30}$$

Hence

$$e'(0,\rho) = (i\rho) \left(1 - \frac{a}{i\rho} + \frac{\omega(\rho)}{\rho} \right) \,. \tag{31}$$

Substituting (20) and (31) into (17) we arrive at (28). At last (29) follows from (13) and (27) . □

Definition. The set $S = (V(\lambda), \{\rho_k, \beta_k\})$ is called the *spectral data.*

4.2. Consider two wave resistances $R(x)$ and $\tilde{R}(x)$ from B_0^-. We agree that, if a certain symbol α denotes an object related to $R(x)$, then $\tilde{\alpha}$ denotes the analogous object related to $\tilde{R}(x)$, and $\hat{\alpha} := \alpha - \tilde{\alpha}$.

Theorem 4.

$$-\hat{M}(\lambda) = \int_0^\infty \frac{\hat{V}(\mu)}{\lambda - \mu}\, d\mu + \sum_{k=1}^{m} \frac{\beta_k}{\lambda - \lambda_k} - \sum_{k=1}^{\tilde{m}} \frac{\tilde{\beta}_k}{\lambda - \tilde{\lambda}_k} \,. \tag{32}$$

Proof. In view of (29), the integral in (32) converges absolutely and uniformly on compacts in Π. In the λ-plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where γ' is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ (i.e. $\Lambda \cup \tilde{\Lambda} \cup \{0\} \subset int\ \gamma'$), and γ'' is the two-sided cut along the arc $\{\lambda : \lambda > 0, \lambda \notin int\ \gamma_0\}$.

Consider the function

$$I_r(z) := \frac{1}{2\pi i} \oint_{|\lambda|=r} \frac{\hat{M}(\lambda) d\lambda}{z-\lambda} .$$

It follows from (28) that

$$\lim_{r\to\infty} I_r(z) = 0 \tag{33}$$

uniformly on compacts in $\Pi \setminus \Lambda$. On the other hand, moving the contour $|\lambda| = r$ to the real axis and using the residue theorem and Theorem 3, we get

$$I_r(z) = -\hat{M}(z) - \int_0^r \frac{\hat{V}(\lambda)}{z-\lambda}\, d\lambda - \sum_{k=1}^{m} \frac{\beta_k}{z-\lambda_k} + \sum_{k=1}^{\tilde{m}} \frac{\tilde{\beta}_k}{z-\tilde{\lambda}_k} .$$

Together with (33) it yields (32) . □

Corollary 1.

$$-M(\lambda) = -i\rho + \int_0^\infty (V(\mu) - \frac{\sqrt{\mu}}{\pi}) \frac{d\mu}{\lambda-\mu} + \sum_{k=1}^{m} \frac{\beta_k}{\lambda-\lambda_k} . \tag{34}$$

Indeed, take $\tilde{R}(x) \equiv 1$. Then $\tilde{M}(\lambda) = i\rho$, $\tilde{V}(\lambda) = \dfrac{\rho}{\pi}$, $\tilde{m} = 0$, and (34) follows from (32).

Corollary 2.

(i) The specification of the spectral data S uniquely determines the Weyl function $M(\lambda)$.

(ii) The specification of the characteristic function $\Delta(\rho)$ uniquely determines the Weyl function $M(\lambda)$.

Denote

$$\tilde{D}(x,\lambda,\mu) := \frac{\langle \tilde{S}(x,\lambda), \tilde{S}(x,\mu)\rangle}{\lambda-\mu} = \int_0^x \tilde{S}(t,\lambda)\, \tilde{S}(t,\mu) dt . \tag{35}$$

The following theorem gives us a tool for the solution of the inverse problem of recovering $R(x)$ from the characteristic function $\Delta(\rho)$.

Theorem 5. *Let $R(x), \tilde{R}(x) \in B_0^-$. Then*

$$\begin{gathered} \tilde{S}(x,\lambda) = S(x,\lambda) + \int_0^\infty \tilde{D}(x,\lambda,\mu)\, \hat{V}(\mu) S(x,\mu)\, d\mu + \\ \sum_{k=1}^{m} \tilde{D}(x,\lambda,\lambda_k)\, \beta_k\, S(x,\lambda_k) - \sum_{k=1}^{\tilde{m}} \tilde{D}(x,\lambda,\tilde{\lambda}_k)\, \tilde{\beta}_k\, S(x,\tilde{\lambda}_k) , \end{gathered} \tag{36}$$

$$p(x) = \tilde{p}(x) - 2\epsilon'(x) \ , \tag{37}$$

where

$$\begin{aligned} \epsilon(x) = \int_0^\infty \ S(x,\lambda)\,\tilde{S}(x,\lambda)\,\hat{V}(\lambda)\,d\lambda + \sum_{k=1}^{m} \ S(x,\lambda_k)\ \tilde{S}(x,\lambda_k)\ \beta_k - \\ - \sum_{k=1}^{\tilde{m}} \ S(x,\tilde{\lambda}_k)\ \tilde{S}(x,\tilde{\lambda}_k)\,\tilde{\beta}_k \ . \end{aligned} \tag{38}$$

Proof. 1) According to (24), the functions $\{e(x,\rho),\, e(x,-\rho)\}$ form a fundamental system of solutions for (14). Using the initial conditions on $S(x,\lambda)$ we calculate

$$S(x,\lambda) = \frac{1}{2i\rho}\,(\delta(-\rho)\ e(x,\rho) - \delta(\rho)\,e(x,-\rho)) \ . \tag{39}$$

By virtue of (15), (20), (30) and (39) one can obtain the following asymptotics for $|\rho| \ \to \infty, \ \rho \in \overline{\Pi}_+, \ j = 0, 1$, uniformly in $x \in [0,T]$:

$$\left.\begin{aligned} \Phi^{(j)}\ (x,\lambda) = (i\rho)^j\ e^{i\rho x}\ (1 + O(\frac{1}{\rho}))\ , \\ S^{(j)}\ (x,\lambda) = \frac{1}{2}\,(i\rho)^{j-1}\ (e^{i\rho x} - (-1)^j\ e^{-i\rho x}) + O(\rho^{j-2}\ e^{-i\rho x})\ . \end{aligned}\right\} \tag{40}$$

It follows from (35) and (40) that

$$|\tilde{D}(x,\lambda,\mu)| \le \frac{C}{(\rho+1)(\theta+1)(|\rho-\theta|\ +1)}\ ,\ \lambda = \rho^2\ ,\ \mu = \theta^2\ ,\ \rho > 0\ ,\ \theta > 0\ . \tag{41}$$

According to (29), (40) and (41), the integrals in (36) and (38) converge absolutely and uniformly in $x \in [0,T], \ \mu > 0$. The function $\epsilon(x)$ is absolutely continuous and $\epsilon'(x) \in L_2(0,T)$.

2) Let us introduce the functions $A_1(x,\lambda)$ and $A_2(x,\lambda)$ by the formulas

$$A_1(x,\lambda) = \Phi(x,\lambda)\,\tilde{S}'(x,\lambda) - S(x,\lambda)\ \tilde{\Phi}'(x,\lambda),\ A_2(x,\lambda) = S(x,\lambda)\ \tilde{\Phi}(x,\lambda) - \Phi(x,\lambda)\,\tilde{S}(x,\lambda)\ . \tag{42}$$

Using (19) one has

$$A_1(x,\lambda) = 1 + (\Phi(x,\lambda) - \tilde{\Phi}(x,\lambda))\ \tilde{S}'(x,\lambda) - (S(x,\lambda) - \tilde{S}(x,\lambda))\ \tilde{\Phi}'(x,\lambda)\ .$$

In view of (40), this yields

$$A_1(x,\lambda) - 1 = O(\frac{1}{\rho}),\ A_2(x,\lambda) = O(\frac{1}{\rho}),\ |\rho|\ \to \infty\,,\ \rho \in \overline{\Pi}_+\ . \tag{43}$$

Let $\gamma_r = \gamma \cap \{\lambda :\ |\lambda|\ \le r\}$, $\gamma_r^0 = \gamma_r \cup \{\lambda :\ |\lambda| = r\}$, $J_\gamma = \{\lambda : \lambda \notin \gamma\ \cap\ int\ \gamma'\}$, where γ was defined above. By Cauchy's theorem

$$A_k(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\gamma_r^0}\ \frac{A_k(x,\mu) - \delta_{1k}}{\lambda - \mu}\,d\mu,$$

where δ_{1k} is the Kronecker delta, and $\lambda \in int\ \gamma_r^0$. It follows from (43) that

$$\lim_{r\to\infty} \frac{1}{2\pi i} \int_{|\mu|=r} \frac{A_k(x,\mu) - \delta_{1k}}{\lambda - \mu}\, d\mu = 0 ,$$

and consequently

$$A_k(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_\gamma \frac{A_k(x,\mu)\, d\mu}{\lambda - \mu} ,\ \lambda \in J_\gamma , \tag{44}$$

where $\int_\gamma := \lim_r \int_{\gamma_r}$. By virtue of (42) and (19),

$$S(x,\lambda) = A_1(x,\lambda)\ \tilde S(x,\lambda) + A_2(x,\lambda)\ \tilde S'(x,\lambda),\ \Phi(x,\lambda) = A_1(x,\lambda)\ \tilde\Phi(x,\lambda) + A_2(x,\lambda)\ \tilde\Phi'(x,\lambda) . \tag{45}$$

Substituting (44) into (45) we get

$$S(x,\lambda) = \tilde S(x,\lambda) + \frac{1}{2\pi i} \int_\gamma (A_1(x,\mu)\ \tilde S(x,\mu) + A_2(x,\mu)\ \tilde S'(x,\lambda)) \frac{d\mu}{\lambda - \mu} .$$

From this and (42) it follows that

$$S(x,\lambda) = \tilde S(x,\lambda) + \frac{1}{2\pi i} \int_\gamma \Big((\Phi(x,\mu)\ \tilde S'(x,\mu) - S(x,\mu)\ \tilde\Phi'(x,\mu))\ \tilde S(x,\lambda) +$$

$$(S(x,\mu)\ \tilde\Phi(x,\mu) - \Phi(x,\mu)\ \tilde S(x,\mu))\ \tilde S'(x,\lambda) \Big)\ \frac{d\mu}{\lambda - \mu} .$$

We replace here $\Phi(x,\mu)$ and $\tilde\Phi(x,\mu)$ by (18). The terms with $C(x,\mu)$ and $\tilde C(x,\mu)$ are equal to zero, and we deduce that

$$\tilde S(x,\lambda) = S(x,\lambda) - \frac{1}{2\pi i} \int_\gamma \tilde D(x,\lambda,\mu)\ \hat M(\mu)\ S(x,\mu)\, d\mu ,\ \lambda \in J_\gamma . \tag{46}$$

From (46) with the help of Theorem 3 we arrive at (36).

We note that similar arguments lead to the formulas

$$\left.\begin{aligned}
&\tilde\Phi(x,\lambda) = \Phi(x,\lambda) - \frac{1}{2\pi i} \int_\gamma \frac{\langle \tilde\Phi(x,\lambda),\ \tilde S(x,\mu)\rangle}{\lambda - \mu}\ \hat M(\mu)\, S(x,\mu)\, d\mu,\ \lambda \in J_\gamma ,\\
&\tilde\Phi(x,\lambda) = \Phi(x,\lambda) + \int_0^\infty \frac{\langle \tilde\Phi(x,\lambda),\ \tilde S(x,\mu)\rangle}{\lambda - \mu} \hat V(\mu)\ S(x,\mu)\, d\mu +\\
&\sum_{k=1}^m \frac{\langle \tilde\Phi((x,\lambda),\ \tilde S(x,\lambda_k)\rangle}{\lambda - \lambda_k}\ \beta_k\ S(x,\lambda_k) - \sum_{k=1}^{\tilde m} \frac{\langle \tilde\Phi(x,\lambda),\ \tilde S(x,\tilde\lambda_k)\rangle}{\lambda - \tilde\lambda_k}\ \tilde\beta_k\ S(x,\tilde\lambda_k) .
\end{aligned}\right\} \tag{47}$$

3) Differentiating (46) twice with respect to x and using (14) and the relation

$$\frac{d}{dx}\tilde D(x,\lambda,\mu) = \tilde S(x,\lambda)\ \tilde S(x,\mu) ,$$

we obtain

$$(\tilde p(x)-\lambda)\ \tilde S(x,\lambda)=(p(x)-\lambda)\ S(x,\lambda)-\frac{1}{2\pi i}\int_\gamma \tilde D(x,\lambda,\mu)\ \hat M(\mu)\,(p(x)-\mu)\ S\,(x,\mu)\,d\mu$$

$$-2\tilde S(x,\lambda)\cdot\frac{1}{2\pi i}\int_\gamma \tilde S(x,\mu)\ S^{'}(x,\mu)\ \hat M(\mu)d\mu-\frac{1}{2\pi i}\int_\gamma \frac{d}{dx}\Big(\tilde S(x,\lambda)\ \tilde S(x,\mu)\Big)\ \hat M(\mu)\ S(x,\mu)\,d\mu\,.$$

We replace here $S(x,\lambda)$ by (46). Then

$$\tilde p(x)\ \tilde S(x,\lambda)=p(x)\ \tilde S(x,\lambda)-\frac{1}{2\pi i}\int_\gamma \langle\tilde S(x,\lambda),\ \tilde S(x,\mu)\rangle\ \hat M(\mu)\ S(x,\mu)\,d\mu-$$

$$2\,\tilde S(x,\lambda)\cdot\frac{1}{2\pi i}\int_\gamma \tilde S(x,\mu)\ S'(x,\mu)\ \hat M(\mu)\,d\mu-\frac{1}{2\pi i}\int_\gamma \Big(\ \tilde S'(x,\lambda)\ \tilde S(x,\mu)+$$
$$\tilde S(x,\lambda)\ \tilde S'(x,\mu)\Big)\ \hat M(\mu)\ S(x,\mu)\,d\mu\ .$$

After cancellation of the terms with $\tilde S'(x,\lambda)$ we get

$$p(x)-\tilde p(x)=\frac{1}{\pi i}\ \frac{d}{dx}\int_\gamma S(x,\lambda)\ \tilde S(x,\lambda)\ \hat M(\lambda)\,d\lambda\ .$$

By virtue of Theorem 3, it gives (37) . □

Denote $\lambda_{k0}=\lambda_k,\ \lambda_{k1}=\tilde\lambda_k,\ \beta_{k0}=\beta_k,\ \beta_{k1}=\tilde\beta_k,\ S_{kj}(x)=S(x,\lambda_{kj}),\ \tilde S_{kj}(x)=\tilde S(x,\lambda_{kj})$. Then (36) yields

$$\left.\begin{aligned}&\tilde S(x,\lambda)=S(x,\lambda)+\int_0^\infty \tilde D(x,\lambda,\mu)\ \hat V(\mu)\ S(x,\mu)\,d\mu+\\&\sum_{k=1}^m \tilde D(x,\lambda,\lambda_{k0})\ \beta_{k0}\ S_{k0}(x)-\sum_{k=1}^{\tilde m}\tilde D(x,\lambda,\lambda_{k1})\ \beta_{k1}\ S_{k1}/x)\ ,\lambda>0\ ,\\&\tilde S_{ni}(x)=S_{ni}(x)+\int_0^\infty \tilde D(x,\lambda_{ni},\mu)\ \hat V(\mu)\ S(x,\mu)\,d\mu+\\&\sum_{k=1}^m \tilde D(x,\lambda_{ni},\lambda_{k0})\ \beta_{k0}\ S_{k0}(x)-\sum_{k=1}^{\tilde m}\tilde D(x,\lambda_{ni},\lambda_{k1})\ \beta_{k1}\ S_{k1}(x)\ .\end{aligned}\right\}\qquad(48)$$

For each fixed x, (49) are linear equations with respect to $S(x,\lambda),\ \lambda>0$ and $S_{ni}(x)$.

Denote

$$N=m+\tilde m;\ \Omega(\lambda)=\min(1,\frac{1}{\rho}),\rho>0;\ \psi_\lambda(x)=(\Omega(\lambda))^{-1}\ S(x,\lambda);$$

$$\psi(x)=[\psi_k(x)]_{k=\overline{1,N}};\ \ \psi_k(x)=S_{k0}(x),k=\overline{1,m};\ \psi_{k+m}(x)=S_{k1}(x),k=\overline{1,\tilde m}\ .$$

Analogously we define $\tilde\psi_\lambda(x),\tilde\psi(x)$. Clearly,

$$|\psi_\lambda(x)|\ \le C,\ |\tilde\psi_\lambda(x)|\ \le C,\ \lambda\ge 0,\ x\in[0,T]\ .\qquad(49)$$

Let $\mathbb{C} = \mathbb{C}[0,\infty)$ be the Banach space of continuous bounded functions $f = f_\lambda$ on the half-line $\lambda \geq 0$ with the norm $||f||_{\mathbb{C}} = \sup\limits_{\lambda\geq 0} |f_\lambda|$. It follows from (49) that for each fixed $x \in [0,T]$, $\psi_\lambda(x), \tilde{\psi}_\lambda(x) \in \mathbb{C}$. Consider the Banach space B of vectors

$$F = \begin{bmatrix} f \\ \alpha \end{bmatrix}$$

where $f = f_\lambda \in \mathbb{C}$, $\alpha = [\alpha_k]_{k=\overline{1,N}} \in R^N$, with the norm $||F||_B = \max(||f||_{\mathbb{C}},\ ||\alpha||_{R^N})$. Denote

$$\Psi(x) = \begin{bmatrix} \psi_\lambda \\ \psi \end{bmatrix},\ \tilde{\Psi}(x) = \begin{bmatrix} \tilde{\psi}_\lambda \\ \tilde{\psi} \end{bmatrix}.$$

Obviously, $\Psi(x), \tilde{\Psi}(x) \in B$ for each fixed $x \in [0,T]$. Let

$$\begin{aligned}
&\tilde{H}_{\lambda,\mu}(x) = (\Omega(\lambda))^{-1}\ \tilde{D}(x,\lambda,\mu)\ \hat{V}(\mu)\ \Omega(\mu),\\
&\tilde{H}_{\lambda,k}(x) = (\Omega(\lambda))^{-1}\ \tilde{D}(x,\lambda,\lambda_{k0})\ \beta_{k0},\ k = \overline{1,m},\\
&\tilde{H}_{\lambda,k+m}(x) = -(\Omega(\lambda))^{-1}\ \tilde{D}(x,\lambda,\lambda_{k1})\ \beta_{k1},\ k = \overline{1,\tilde{m}},\\
&\tilde{H}_{n,\mu}(x) = \tilde{D}(x,\lambda_{n0},\mu)\,\hat{V}(\mu)\ \Omega(\mu), n = \overline{1,m},\\
&\tilde{H}_{n+m,\mu}(x) = \tilde{D}(x,\lambda_{n1},\mu)\ \hat{V}(\mu)\ \Omega(\mu),\ n = \overline{1,\tilde{m}},\\
&\tilde{H}_{nk}(x) = \tilde{D}(x,\lambda_{n0},\lambda_{k0})\ \beta_{k0},\ n,k = \overline{1,m},\\
&\tilde{H}_{n+m,k}(x) = \tilde{D}(x,\lambda_{n1},\lambda_{k0})\,\beta_{k0}\ ,\ n = \overline{1,\tilde{m}},\ k = \overline{1,m},\\
&\tilde{H}_{n,k+m}(x) = -\tilde{D}(x,\lambda_{n0},\lambda_{k1})\ \beta_{k1},\ n = \overline{1,m},\ k = \overline{1,\tilde{m}},\\
&\tilde{H}_{n+m,k+m}(x) = -\tilde{D}(x,\lambda_{n1},\lambda_{k1})\ \beta_{k1}, n,k = \overline{1,\tilde{m}}.
\end{aligned}$$

By virtue of (29) and (41) we get

$$|\tilde{H}_{\lambda,\mu}(x)|\ \leq \frac{\omega(\theta)}{(|\rho-\theta|\ +1)(\theta+1)^2}\ ,\ |\tilde{H}_{n,\mu}(x)|\ \leq \frac{\omega(\theta)}{(\theta+1)^3}\ ,\theta > 0,\ \rho > 0\ . \tag{50}$$

Let $\tilde{Q} : B \to B$ be the operator acting from B to B by the formulas

$$F^* = \tilde{Q}F,\ F = \begin{bmatrix} f \\ \alpha \end{bmatrix} \in B,\ F^* = \begin{bmatrix} f^* \\ \alpha^* \end{bmatrix} \in B\ ,$$

$$\left.\begin{aligned}
f^*_\lambda &= \int_0^\infty\ \tilde{H}_{\lambda,\mu}\ f_\mu\ d\mu + \sum_{k=1}^N\ \tilde{H}_{\lambda,k}\ \alpha_k\ ,\\
\alpha^*_n &= \int_0^\infty\ \tilde{H}_{n,\mu}\ f_\mu\, d\mu + \sum_{k=1}^N\ \tilde{H}_{nk}\,\alpha_k\ .
\end{aligned}\right\}$$

It follows from (50) that for each fixed $x \in [0,T]$, the operator $E + \tilde{Q}(x)$ (here E is the identity operator), acting from B to B, are a linear bounded operator.

Taking into account our notations we can rewrite (49) to the form

$$\tilde{\Psi}(x) = (E + \tilde{Q}(x))\ \Psi(x)\ . \tag{51}$$

Thus, we proved the following assertion.

Theorem 6. *For each fixed $x \in [0,T]$, the vector $\Psi(x) \in B$ is a solution of equation (51). Equation (51) is called the main eqution of the inverse problem.*

4.3. In this subsection we present a theorem on the solvability of the main equation. We also provide necessary and sufficient conditions on $\Delta(\rho)$ along with algorithms for the solution of the inverse problem of recovering $R(x)$ from the characteristic function $\Delta(\rho)$.

Denote by Ω_0 the set of entire functions $\Delta(\rho)$ of exponential type T such that

(i) (9) and (13) are valid;

(ii) In $\overline{\Pi}_+$, $\Delta(\rho)$ has at most a finite number of simple zeros $\rho_k = i\tau_k,\ \tau_k > 0,\ k = \overline{1,m},\ m \geq 0$, (in general, the set $\{\rho_k\}$ is different for each $\Delta(\rho)$);

(iii) $\beta_k > 0$, where β_k are defined via (22).

Clearly, if $\Delta(\rho)$ is the characteristic function for $R(x) \in B_0^-$, then $\Delta(\rho) \in \Omega_0$.

Theorem 7. *Let $\Delta(\rho) \in \Omega_0,\ \tilde{R}(x) \in B_0^-$. Then, for each fixed $x \in [0,T]$, equation (51) has a unique solution in B.*

Proof. It is sufficient to show that the homogeneous equation

$$(E + \tilde{Q}(x))\, G(x) = 0,\ G(x) = \begin{bmatrix} g_\lambda(x) \\ g(x) \end{bmatrix} \in B,\ g(x) = [g_k(x)]_{k=\overline{1,N}} \tag{52}$$

has only the zero solution.

Let $G(x)$ be a solution of (52). Then (52) implies

$$\left.\begin{aligned} g_\lambda + \int_0^\infty \tilde{H}_{\lambda,\mu}\, g_\mu\, d\mu + \sum_{k=1}^N \tilde{H}_{\lambda,k}\, g_k = 0,\ \lambda > 0\ , \\ g_n + \int_0^\infty \tilde{H}_{n,\mu}\, g_\mu\, d\mu + \sum_{k=1}^N \tilde{H}_{n,k}\, g_k = 0,\ n = \overline{1,N}. \end{aligned}\right\} \tag{53}$$

Denote $s(x,\lambda) = \Omega(\lambda)\, g_\lambda(x);\ s_{k0}(x) = g_k(x),\ k = \overline{1,m}\ ,\ s_{k1}(x) = g_{k+m}(x),\ k = \overline{1,\tilde{m}}$. Then (53) takes the form

$$\left.\begin{aligned} s(x,\lambda) + \int_0^\infty \tilde{D}(x,\lambda,\mu)\, \hat{V}(\mu)\, s(x,\mu)\, d\mu + \sum_{k=1}^m \tilde{D}(x,\lambda,\lambda_{k0})\, \beta_{k0}\, s_{k0}(x) - \\ \sum_{k=1}^{\tilde{m}} \tilde{D}(x,\lambda,\lambda_{k1})\, \beta_{k1}\, s_{k1}(x) = 0,\ \lambda > 0\ , \\ s_{ni}(x) + \int_0^\infty \tilde{D}(x,\lambda_{ni},\mu)\, \hat{V}(\mu)\, s(x,\mu)\, d\mu + \sum_{k=1}^m \tilde{D}(x,\lambda_{ni},\lambda_{k0})\, \beta_{k0}\, s_{k0}(x) - \\ \sum_{k=1}^{\tilde{m}} \tilde{D}(x,\lambda_{ni},\lambda_{k1})\, \beta_{k1}\, s_{k1}(x) = 0\ . \end{aligned}\right\} \tag{54}$$

Further, we construct the functions $\gamma(x,\lambda)$ and $\Gamma(x,\lambda)$ by the formulas

$$\left.\begin{aligned} -\gamma(x,\lambda) = \int_0^\infty \tilde{D}(x,\lambda,\mu)\,\hat{V}(\mu)\,s(x,\mu)\,d\mu + \sum_{k=1}^{m} \tilde{D}(x,\lambda,\lambda_{k0})\,\beta_{k0}\,s_{k0}(x) \\ -\sum_{k=1}^{\tilde{m}} \tilde{D}(x,\lambda,\lambda_{k1})\,\beta_{k1}\,s_{k1}(x)\;, \\ -\Gamma(x,\lambda) = \int_0^\infty \frac{\langle\tilde{\Phi}(x,\lambda),\ \tilde{S}(x,\mu)\rangle}{\lambda-\mu}\,\hat{V}(\mu)\,s(x,\mu)\,d\mu \\ +\sum_{k=1}^{m}\frac{\langle\tilde{\Phi}(x,\lambda),\ \tilde{S}(x,\lambda_{k0})\rangle}{\lambda-\lambda_{k0}}\,\beta_{k0}\,s_{k0}(x) - \sum_{k=1}^{\tilde{m}}\frac{\langle\tilde{\Phi}(x,\lambda),\tilde{S}(x,\lambda_{k1})\rangle}{\lambda-\lambda_{k1}}\,\beta_{k1}\,s_{k1}(x)\;. \end{aligned}\right\} \tag{55}$$

The function $\gamma(x,\lambda)$ is entire in λ, $\overline{\gamma(x,\lambda)} = \gamma(x,\overline{\lambda})$, and by virtue of (54), $\gamma(x,\lambda_{kj}) = s_{kj}(x)$, $\gamma(x,\lambda) = s(x,\lambda)$, $\lambda > 0$. The function $\Gamma(x,\lambda)$ is holomorphic in $\Pi\setminus(\Lambda\cup\tilde{\Lambda})$ and continuous in $\overline{\Pi}\setminus(\Lambda\cup\tilde{\Lambda})$. Denote

$$d(x,\lambda) = \frac{1}{2\pi i}(\Gamma^-(x,\lambda)-\Gamma^+(x,\lambda)), \lambda>0;$$

$$\Gamma^\pm(x,\lambda) := \lim\Gamma(x,\lambda\pm iz), z\to 0, Re\, z>0.$$

Since $\tilde{\Phi}(x,\lambda) = \tilde{C}(x,\lambda) + \tilde{M}(\lambda)\,\tilde{S}(x,\lambda)$, we obtain from (55)

$$-\Gamma(x,\lambda) = -\tilde{M}(\lambda)\,\gamma(x,\lambda) + \int_0^\infty \frac{\langle\tilde{C}(x,\lambda),\ \tilde{S}(x,\mu)\rangle}{\lambda-\mu}\,\hat{V}(\mu)\,s(x,\mu)\,d\mu +$$

$$+\sum_{k=1}^{m}\frac{\langle\tilde{C}(x,\lambda),\ \tilde{S}(x,\lambda_{k0})\rangle}{\lambda-\lambda_{k0}}\,\beta_{k0}\,s_{k0}(x) - \sum_{k-1}^{\tilde{m}}\frac{\langle\tilde{C}(x,\lambda),\ \tilde{S}(x,\lambda_{k1})\rangle}{\lambda-\lambda_{k1}}\,\beta_{k1}\,s_{k1}(x)\;.$$

From the relation

$$\frac{d}{dx}\langle\tilde{C}(x,\lambda),\ \tilde{S}(x,\mu)\rangle = (\lambda-\mu)\,\tilde{C}(x,\lambda)\,\tilde{S}(x,\mu)$$

it follows that

$$\frac{\langle\tilde{C}(x,\lambda),\ \tilde{S}(x,\mu)\rangle}{\lambda-\mu} = \frac{1}{\lambda-\mu} + \int_0^x \tilde{C}(t,\lambda)\,\tilde{S}(t,\mu)\,dt\;.$$

It yields

$$\begin{aligned} -\Gamma(x,\lambda) = -\tilde{M}(\lambda)\,\gamma(x,\lambda) + \int_0^\infty \frac{\hat{V}(\mu)\,s(x,\mu)}{\lambda-\mu}\,d\mu \\ +\sum_{k=1}^{m}\frac{\beta_{k0}\,s_{k0}(x)}{\lambda-\lambda_{k0}} - \sum_{k=1}^{\tilde{m}}\frac{\beta_{k1}\,s_{k1}(x)}{\lambda-\lambda_{k1}} + \epsilon_0(x,\lambda), \end{aligned} \tag{56}$$

where $\epsilon_0(x,\lambda)$ is entire in λ. Using (56) and Theorem 3 we calculate

$$\begin{gathered} \operatorname*{Res}_{\lambda=\lambda_{k0}}\ \Gamma(x,\lambda) = -\beta_{k0}\,s_{k0}(x),\ \operatorname*{Res}_{\lambda=\lambda_{k1}}\ \Gamma(x,\lambda) = 0\;, \\ d(x,\lambda) = V(\lambda)\,s(x,\lambda)\;. \end{gathered} \tag{57}$$

Now we introduce the function

$$A(x,\lambda) = \gamma(x,\lambda)\,\Gamma(x,\lambda)\;. \tag{58}$$

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{\gamma_r^0} A(x,\lambda)\,d\lambda = 0\ .$$

Using (55) and (58) one can show that

$$\lim_{r\to\infty}\frac{1}{2\pi i}\int_{|\lambda|=r} A(x,\lambda)\,d\lambda = 0\ ,$$

and consequently

$$\frac{1}{2\pi i}\int_{\gamma} A(x,\lambda)\,d\lambda = 0\ .$$

Taking (57) into account, we calculate

$$\int_0^\infty |s(x,\lambda)|^2\,V(\lambda)\,d\lambda + \sum_{k=1}^{m}\beta_k\,|s_{k0}(x)|^2 = 0\ .$$

Since $\beta_k > 0, V(\lambda) > 0$ we deduce $s_{k0}(x) = 0,\ s(x,\lambda) = 0$. Hence $\gamma(x,\lambda) \equiv 0,\ \lambda > 0$. The function $\gamma(x,\lambda)$ is entire in λ, and consequently $\gamma(x,\lambda) \equiv 0$ for all λ and x. From this it follows that $s_{k1}(x) = \gamma(x,\lambda_{k1}) = 0$. □

Theorem 8. *For a function $\Delta(\rho)$ to be the characteristic function for $R(x) \in B_0^-$ it is necessary and sufficient that $\Delta(\rho) \in \Omega_0$. The specification of the characteristic function $\Delta(\rho)$ uniquely determines $R(x)$. The function $R(x)$ can be constructed by the following algorithm.*

Algorithm 1. *Given $\Delta(\rho)$. Then*

(1) Calculate $\lambda_k, \beta_k, V(\lambda), \lambda > 0$ by (22), (27).

(2) Choose $\tilde{R}(x) \in B_0^-$.

(3) Construct $\tilde{p}(x), \tilde{S}(x,\lambda), \tilde{V}(\lambda)$ and $\tilde{D}(x,\lambda,\mu)$.

(4) Find $S(x,\lambda), \lambda > 0$ and $S_{ki}(x)$ by solving equation (48) which is equivalent to (51).

(5) Calculate

$$\epsilon(x) = \int_0^\infty S(x,\lambda)\,\tilde{S}(x,\lambda)\hat{V}(\lambda)\,d\lambda + \sum_{k=1}^{m} S_{k0}(x)\tilde{S}_{k0}(x)\beta_{k0} - \sum_{k=1}^{\tilde{m}} S_{k1}(x)\,\tilde{S}_{k1}(x)\,\beta_{k1}. \quad (59)$$

(6) Construct

$$R(x) := \frac{1}{y^2(T-x)}$$

where $y(x)$ is the solution of the Cauchy problem

$$y'' = p(x)y,\ y(T) = 1,\ y'(T) = 0,\ p(x) = \tilde{p}(x) - 2\epsilon'(x).$$

Remark 2. We also can construct $R(x)$ using (2), (6), namely:

$$R(x) = \exp(2\int_0^x h(t)dt), \tag{60}$$

where $h(x)$ is the solution of the equation

$$h(x) = \int_0^x h^2(t)dt - \int_0^x \tilde{p}(T-t)dt - 2(\epsilon(T-x) - \epsilon(T)). \tag{61}$$

In this case it is not necessary to calculate the derivative $\epsilon'(x)$.

Proof of Theorem 8. Necessity is obvious, so we will prove sufficiency. Let

$$\Psi(x) = \begin{bmatrix} \psi_\lambda(x) \\ [\psi_k(x)]_{k=\overline{1,N}} \end{bmatrix} \in B$$

be the solution of equation (51). Denote

$$S(x,\lambda) = \Omega(\lambda)\psi_\lambda(x)\,,\ S_{n0}(x) = \psi_n(x), n = \overline{1,m};\ S_{n1}(x) = \psi_{n+m}(x), n = \overline{1,\tilde{m}}.$$

Then the functions $S(x,\lambda), S_{ni}(x)$ satisfy (48). We construct the entire function $S(x,\lambda)$ by (48) for all λ. Then $S(x,\lambda_{ni}) = S_{ni}(x)$. Further, we introduce the function $\Phi(x,\lambda)$ via (47) and $p(x), R(x)$ according to Algorithm 1. Clearly, $p(x) \in L_2(0,T)$, $R(x) \in B_0^-$. Following the method described in [16] one can show that $S(x,\lambda), \Phi(x,\lambda)$ are solutions of (14) under the conditions $S(0,\lambda) = 0, S'(0,\lambda) = \Phi(0,\lambda) = 1\,,\ \Phi(x,\lambda) = O(e^{i\rho x}), x \to \infty$, and $\Delta(\rho)$ is the characteristic function for $R(x)$. Uniqueness is obvious. □

Now let us consider the particular case when $R(x) \in B_0$. In this case the inverse problem becomes simpler.

If $R(x) \in B_0$, then, according to Theorem 2, $\Delta(\rho)$ has no zeros in $\overline{\Pi}_+$, i.e. $m = 0$. Take $\tilde{R}(x) \in B_0$. Then the main equation (48) takes the form

$$\tilde{S}(x,\lambda) = S(x,\lambda) + \int_0^\infty \tilde{D}(x,\lambda,\mu)\,\hat{V}(\mu)\,S(x,\mu)\,d\mu,\ \lambda > 0\,. \tag{62}$$

The function $p(x)$ can be constructed by the formulas

$$p(x) = \tilde{p}(x) - 2\epsilon'(x)\,, \tag{63}$$

$$\epsilon(x) = \int_0^\infty S(x,\lambda)\,\tilde{S}(x,\lambda)\,\hat{V}(\lambda)d\lambda\,. \tag{64}$$

If we choose $\tilde{R}(x) \equiv 1$, then

$$\left.\begin{aligned} &\tilde{p}(x) \equiv 0,\ \tilde{V}(\lambda) = \frac{\rho}{\pi},\ \tilde{S}(x,\lambda) = \frac{\sin\rho x}{\rho}\,, \\ &\tilde{D}(x,\lambda,\mu) = \int_0^x \frac{\sin\rho t}{\rho}\cdot\frac{\sin\theta t}{\theta}\,dt,\quad \lambda = \rho^2, \mu = \theta^2,\ \rho > 0, \theta > 0\,. \end{aligned}\right\} \tag{65}$$

Thus we obtain the following algorithm for the case $R(x) \in B_0$.

Algorithm 2. *Given* $\alpha(\sigma) := |\Delta(\sigma)|\,,\ \sigma > 0$. *Then*

(1) Calculate $V(\lambda) = \dfrac{\sigma}{\pi\alpha^2(\sigma)}\,,\ \lambda = \sigma^2\,,\ \sigma > 0\,.$

(2) Choose $\tilde{R}(x) \in B_0$ *(for example,* $\tilde{R}(x) \equiv 1$*).*

(3) Calculate $\tilde{p}(x),\ \tilde{S}(x,\lambda),\ \tilde{V}(\lambda),\ \tilde{D}(x,\lambda,\mu),\ \lambda,\mu > 0$ *(if* $\tilde{R}(x) \equiv 1$ *use (65)).*

(4) Find $S(x,\lambda), \lambda > 0$ *by solving the main equation (62).*

(5) Construct $\epsilon(x)$ *by (64).*

(6) Construct $R(x)$ *by the same was as in Algorithm 1 or by (60), (61).*

5. Algorithm for the solution of Inverse Problem 1.

It follows from (3) that

$$\Delta(\rho) = f_1(\rho) + f_2(\rho)\ . \tag{66}$$

Therefore in order to solve Inverse Problem 1 we need to reconstruct the transmission coefficients from their moduli and then reduce Inverse Problem 1 to the inverse problem of recovering $R(x)$ from the characteristic function described in Section 4.

By virtue of (1), (2),

$$u'(x,\rho) + h(x)\,u(x,\rho) = i\rho v(x,\rho)\ .$$

Substituting into (3), we get

$$\left.\begin{aligned} f_1(\rho) &= \frac{1}{2}\,u(T,\rho) - \frac{1}{2i\rho}\,\left(u'(T,\rho) + hu(T,\rho)\right)\,,\\ f_2(\rho) &= \frac{1}{2}\,u(T,\rho) + \frac{1}{2i\rho}\,\left(u'(T,\rho) + hu(T,\rho)\right)\,,\end{aligned}\right\} \tag{67}$$

where $h := h(T)$.

The functions $f_j(\rho)$ are entire in ρ of exponential type T. It follows from (10) and (67) that

$$f_1(\rho)f_1(-\rho) - f_2(\rho)f_2(-\rho) \equiv 1\ . \tag{68}$$

Since $\overline{f_j(\rho)} = f_j(-\overline{\rho})$, we have from (68) that

$$\alpha_1^2(\sigma) - \alpha_2^2(\sigma) \equiv 1 \tag{69}$$

where $\alpha_j(\sigma) = |f_j(\sigma)|$ as above.

Using (12) and (67), we obtain that for $|\rho| \to \infty,\ \rho \in \overline{\Pi}_+$,

$$f_1(\rho) = e^{-i\rho T}\left(1 + \frac{b}{i\rho} + \frac{\omega(\rho)}{\rho}\right),\ f_2(\rho) = e^{-i\rho T}\left(\frac{h}{2i\rho} + \frac{\omega(\rho)}{\rho}\right)\,, \tag{70}$$

where

$$b = -\frac{1}{2}\int_0^T h^2(t)dt,\quad h = h(T)\ .$$

It follows from (69) and (70) that for real $\rho = \sigma,\ |\sigma| \to \infty$,

$$\alpha_1^2(\sigma) = 1 + \frac{h^2}{4\sigma^2} + \frac{\omega(\sigma)}{\sigma^2}, \quad \alpha_2^2(\sigma) = \frac{h^2}{4\sigma^2} + \frac{\omega(\sigma)}{\sigma^2} \ . \tag{71}$$

In view of (69), the function $f_1(\rho)$ has no zeros for real ρ. Using similar arguments as in Section 3, one can show that zeros of $f_1(\rho)$ in Π_+ are simple and pure imaginary: $\rho_k^* = i\tau_k^*,\ \tau_k^* > 0,\ k = \overline{1, m^*}$. If $R(x) \in B_0$, then $f_1(\rho)$ has no zeros in $\overline{\Pi}_+$, i.e. $m^* = 0$.

Lemma 3. *Suppose that a function $\gamma(\rho)$ is regular in $\overline{\Pi}_+$, has no zeros in Π_+ and for $|\rho| \to \infty,\ \rho \in \overline{\Pi}_+,\ \gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(\sigma) = |\gamma(\sigma)| e^{-i\beta(\sigma)},\ \sigma = Re\,\rho$. Then*

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \sigma}\, d\xi \ . \tag{72}$$

In (72) (and everywhere below, where necessary) the integral is understood in the principal value sense.

Proof. First we suppose that $\gamma(\sigma) \neq 0$ for real σ. By Cauchy's theorem, taking into account the hypothesis of the lemma, we obtain

$$\frac{1}{2\pi i} \int_{C_{r,\epsilon}} \frac{\ln \gamma(\xi)}{\xi - \sigma}\, d\xi = 0 \ , \tag{73}$$

where $C_{r,\epsilon}$ is the closed contour (with counterclockwise circuit) consisting of the semicircles $C_r = \{\xi : \xi = re^{i\varphi}\ ,\ \varphi \in [0, \pi]\},\ \Gamma_\epsilon = \{\xi : \xi - \sigma = \epsilon e^{i\varphi},\ \varphi \in [0, \pi]\}$ and the intervals $\xi \in [-r, r] \setminus [\sigma - \epsilon, \sigma + \epsilon]$ of the real axis. Since

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{\ln \gamma(\xi)}{\xi - \sigma}\, d\xi = -\frac{1}{2} \ln \gamma(\sigma) \ ,$$

$$\lim_{r \to \infty} \frac{1}{2\pi i} \int_{C_r} \frac{\ln \gamma(\xi)}{\xi - \sigma} d\xi = 0 \ ,$$

we get from (73) that

$$\ln \gamma(\sigma) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - \sigma}\, d\xi \ .$$

Separating here real and imaginary parts, we arrive at (72).

Suppose now that for real ρ, the function $\gamma(\rho)$ has one zero $\rho_0 = 0$ of multiplicity s (the general case is treated in the same way). Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho) \left(\frac{\rho + i\epsilon}{\rho} \right)^s,\ \epsilon > 0\,;\ \ \tilde{\gamma}(\sigma) = |\tilde{\gamma}(\sigma)| \cdot e^{-i\tilde{\beta}(\sigma)} \ .$$

Then

$$\beta(\sigma) = \tilde{\beta}(\sigma) + s \,\mathrm{arctg} \frac{\epsilon}{\sigma} \ . \tag{74}$$

For the function $\tilde{\gamma}(\rho)$, (72) has been proved. Hence, (74) takes the form

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - \sigma} \, d\xi + \frac{s}{2\pi} \int_{-\infty}^{\infty} \frac{\ln (1 + \frac{\epsilon^2}{\xi^2})}{\xi - \sigma} \, d\xi + s \, \mathrm{arctg} \, \frac{\epsilon}{\sigma} \; .$$

As $\epsilon \to 0$, it gives us (72). □

Lemma 4. *Suppose that a function $\gamma(\rho)$ is regular in $\overline{\Pi}_+$, $\gamma(-\overline{\rho}) = \overline{\gamma(\rho)}$, and for $|\rho| \to \infty$, $\rho \in \overline{\Pi}_+$, $\gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(\sigma) = |\gamma(\sigma)| e^{-i\beta(\sigma)}$, $\sigma = Re\, \rho$, and let $\rho_k = \sigma_k + i\tau_k$, $\tau_k > 0$, $k = \overline{1,s}$ be zeros of $\gamma(\rho)$ in Π_+. Then*

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \sigma} \, d\xi + 2 \sum_{k=1}^{s} \; arctg \frac{\tau_k}{\sigma - \sigma_k} \; . \tag{75}$$

Proof. Since $\gamma(-\overline{\rho}) = \overline{\gamma(\rho)}$, the zeros of $\gamma(\rho)$ in Π_+ are symmetrical with respect to the imaginary axis.

If $\rho_k = i\tau_k, \tau_k > 0$, then

$$\arg \frac{\sigma + \rho_k}{\sigma - \rho_k} = 2 \mathrm{arctg} \, \frac{\tau_k}{\sigma} \; . \tag{76}$$

If $\rho_k = \sigma_k + i\tau_k, \sigma_k > 0, \tau_k > 0$, then

$$\arg \frac{(\sigma + \rho_k)(\sigma - \overline{\rho}_k)}{(\sigma - \rho_k)(\sigma + \overline{\rho}_k)} = 2 \, \mathrm{arctg} \frac{\tau_k}{\sigma - \sigma_k} + 2 \, \mathrm{arctg} \, \frac{\tau_k}{\sigma + \sigma_k} \; . \tag{77}$$

Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho) \prod_{k=1}^{s} \frac{\rho + \rho_k}{\rho - \rho_k} \; .$$

The function $\tilde{\gamma}(\rho)$ satisfies the hypothesis of Lemma 3. Then using (72), (76) and (77) we arrive at (75). □

Using Lemma 4 one can construct the transmission coefficients from their moduli and information about their zeros in Π_+. For definiteness we confine ourselves to the case $h \neq 0$.

Theorem 9. *Let*

$$f_1(\sigma) = \alpha_1(\sigma) \, e^{-i\delta_1(\sigma)} \; , \tag{78}$$

and let $\rho_k^* = i\tau_k^*, \tau_k^* > 0, k = \overline{1, m^*}$ be the zeros of $f_1(\rho)$ in Π_+.
Then

$$\delta_1(\sigma) = \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \, \alpha_1(\xi)}{\xi - \sigma} \, d\xi + 2 \sum_{k=1}^{m^*} \; arctg \, \frac{\tau_k^*}{\sigma} \; . \tag{79}$$

In particular, if $R(x) \in B_0$, then

$$\delta_1(\sigma) = \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \, \alpha_1(\xi)}{\xi - \sigma} \, d\xi \; . \tag{80}$$

Proof. Denote $\gamma(\rho) = e^{i\rho T} f_1(\rho)$. It follows from (70) that the function $\gamma(\rho)$ satisfies the hypothesis of Lemma 4. Using (75) and the relations $|\gamma(\sigma)| = \alpha_1(\sigma),\ \delta_1(\sigma) = \sigma T + \beta(\sigma)$, we arrive at (79). $\square$

Similarly one can prove the following theorem.

Theorem 10. *Let*

$$f_2(\sigma) = \alpha_2(\sigma)\, e^{-i\delta_2(\sigma)} \ , \tag{81}$$

and let $\rho_k^0 = \sigma_k^0 + i\tau_k^0,\ \tau_k^0 > 0, k = \overline{1, m^0}$ *be the zeros of* $f_2(\rho)$ *in* Π_+*. Then*

$$\delta_2(\sigma) = \frac{\pi}{2}\, sign\left(\frac{\sigma}{h}\right) + \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\frac{2\xi}{h}\,\alpha_2(\xi)|}{\xi - \sigma}\, d\xi + 2\sum_{k=0}^{m^0} arctg\,\frac{\tau_k^0}{\sigma - \sigma_k^0} \ . \tag{82}$$

In particular, if $f_2(\rho)$ *has no zeros in* Π_+*, then*

$$\delta_2(\sigma) = \frac{\pi}{2}\, sign\left(\frac{\sigma}{h}\right) + \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\frac{2\xi}{h}\,\alpha_2(\xi)|}{\xi - \sigma}\, d\xi \ . \tag{83}$$

Thus, the specification of $\alpha_j(\sigma)$ *uniquely determines the transmission coefficients only when they have no zeros in* Π_+*.*

Using (66), (78) and (81) we calculate $\alpha(\sigma) := |\triangle(\sigma)|$:

$$\alpha^2(\sigma) = \alpha_1^2(\sigma) + \alpha_2^2(\sigma) + 2\alpha_1(\sigma)\,\alpha_2(\sigma)\ \cos(\delta_1(\sigma) - \delta_2(\sigma)) \ . \tag{84}$$

From the obtained results one can construct various algorithms for synthesizing $R(x)$ from spectral characteristics. As example, we provide one of the possible algorithms. For definiteness we confine ourselves to the case $h \neq 0,\ \tilde{R}(x) \equiv 1,\ R(x) \in B_0$.

Algorithm 3. *Given* $\alpha_1(\sigma)$

$$\left(|\,\alpha_1(\sigma)| \geq 1,\ \alpha_1(-\sigma) = \alpha_1(\sigma),\ \alpha_1^2(\sigma) = 1 + \frac{h^2}{4\sigma^2} + \frac{\omega(\sigma)}{\sigma^2}\right) \ .$$

Then

(1) Calculate $\delta_1(\sigma)$ *by (80).*

(2) Construct $\delta_2(\sigma)$ *by (83) where* $\alpha_2(\sigma)$ *is defined by (69).*

(3) Find $\alpha(\sigma)$ *by (84).*

(4) Calculate

$$V(\lambda) = \frac{\sigma}{\pi\,\alpha^2(\sigma)} \ , \ \lambda = \sigma^2 > 0 \ .$$

(5) Construct $\tilde{V}(\lambda),\ \tilde{S}(x,\lambda),\ \tilde{D}(x,\lambda,\mu)$ *by (65).*

(6) Find $S(x,\lambda),\ \lambda > 0$ *from equation (62).*

(7) Calculate $\epsilon(x)$ by (64).

(8) Construct $R(x)$ by

$$R(x) = \exp\,(2\int_0^x h(t)\,dt)$$

where $h(x)$ is the solution of the integral equation

$$h(x) = \int_0^x h^2(t)\,dt - 2(\epsilon(T-x) - \epsilon(T))$$

or by

$$R(x) = \frac{1}{y^2(T-x)}\ ,$$

where $y(x)$ is the solution of the Cauchy's problem

$$y'' = p(x)y, \quad y(T) = 1, \quad y'(T) = 0, \quad p(x) = -2\,\epsilon'(x)\ \ .$$

Acknoledgment. The authors gratefully acknowledge support from Volkswagen Foundation.

References

1. Sveshnikov, A.G. and Il'inskii, A.S., Design problems in electrodynamics, Dokl. Akad. Nauk SSSR, 204 (1972), 5, 1077 - 1080.

2. Meshanov, V.P. and Feldstein, A.L., Automatic Design of Directional Couplers, Moscow: Sviaz', 1980 (in Russian).

3. Litvinenko, O.N. and Soshnikov, V.I., The Theory of Heterogenous Lines and their Applications in Radio Engineering, Moscow: Radio, 1964 (in Russian).

4. Tikhonravov, A.V., The accuracy obtainable in principle when solving synthesis problems, Zh. Vychisl. Mat. mat. Fiz. 22 (1982), 6, 1421 - 1433; English transl. in USSR Comput. Maths. Math. Phys. 22 (1982), 6, 143 - 157.

5. Freiling, G. und Yurko, V.A., On constructing differential equations with singularities from incomplete spectral information, Schriftenreihe des Fachbereichs Mathematik der Universität Duisburg, SM-DU- 405 , Duisburg 1998.

6. Marchenko, V.A., Sturm-Liouville Operators and their Applications, Kiev: Naukova Dumka, 1977, (Engl. transl. 1986 (Basel: Birkhäuser)).

7. Levitan, B.M., Inverse Sturm-Liouville Problems, Moscow: Nauka, 1984, (Engl. transl. 1987 (Utrecht: VNU Science Press)).

8. Pöschel, J. and Trubowitz, E., Inverse Spectral Theory, New York: Academic, 1987.

9. Beals, R., Deiff, P. and Tomei, C., Direct and Inverse Scattering on the Line, Math. Surveys and Monographs, 28, Amer. Math. Soc. Providence: RI, 1988.

10. Yurko, V.A., Inverse Spectral Problems for Differential Operators and their Applications, New York: Gordon & Breach, 1998.

11. Freiling, G. and Yurko, V.A., Inverse problems for differential equations with turning points, Inverse Problems, 13 (1997), 1247-1263.

12. Wasow, W., Linear Turning Point Theory, Berlin: Springer, 1985.

13. Mc Hugh, J.M., An historical survey of ordinary linear differential equations with a large parameter and turning points, Arch. Hist. Exact Sci 7 (1971), 277 - 324.

14. Eberhard, W., Freiling, G. and Schneider, A., Connection formula for second-order differential equations with a complex parameter and having an arbitrary number of turning points, Math. Nachr. 165 (1994), 205 - 229.

15. Levitan, B.M. and Sargsyan, I.S., Sturm-Liouville and Dirac operators. Kluwer Academic Publishers, Dordrecht, 1991.

16. Yurko, V.A., On determination of selfadjoint differential operators on the half-line, Matemat. Zametki 57 (1995), no. 3, 451-462; English transl. in Math. Notes 57 (1995), no. 3-4, 310-318.

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Eingegangen am 14. Mai 1998