Non - Blow - Up Conditions for Riccati - type Matrix Differential and Difference Equations *

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Abstract: We present several methods to obtain global existence results for solutions of non-symmetric Riccati matrix differential equations and for generalized or perturbed symmetric Riccati differential equations. One approach is to derive sufficient conditions ensuring that the spectral norm of the solutions remain uniformly bounded in an interval $(-\infty, t_0]$ or, weaker, that the minimal an the maximal eigenvalue of a hermitian solution remains bounded in $(-\infty, t_0]$. If however, there exists a linearizing transformation, as in the case of a non-symmetric Riccati differential equation, then with the aid of an appropriate Lyapunov-type function we obtain sufficient conditions guaranteeing that no escape finite time can occur. This method also applies to non-symmetric matrix Riccati difference equations.

These results, among others, can then be applied to control problems like H_∞-control, Markovian Jump Linear Quadratic control, Minimal Cost Variance control and to open loop and memoryless feedback Nash games as well.

1. Introduction.

We consider non-symmetric non-autonomous matrix Riccati differential equations of the form

$$\dot{W} = B_{21}(t) + B_{22}(t)W - WB_{11}(t) - WB_{12}(t)W, \quad W(t_0) = W_0 \qquad (RDE),$$

where the coefficients $B_{11}, B_{12}, B_{21}, B_{22}$ are piecewise continuous, locally integrable matrix functions of dimensions $n \times n, n \times m, m \times n$ and $m \times m$, defined for $t \in \mathbf{R}$ (or on some subinterval $I \subset \mathbf{R}$).

Motivations for studying this class of differential equations come from various branches of applied mathematics such as differential games theory [1], [2], [6], optimal control [43], spectral factorization [7] and invariant embedding and scattering processes [26], [22].

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The main problem arising in connection with (RDE) is the finite escape time phenomenon which is typical for Riccati equations - this means that the solution of (RDE) may not exist on the whole interval $[t_0, \infty)$ (or $(-\infty, t_0]$). This phenomenon has been shown to be intimately related to the properties of the solutions of the problems we have just mentioned

For symmetric or hermitian matrix Riccati differential equations

$$\dot{K} = -A^*(t)K - KA(t) - Q(t) + KS(t)K, \qquad K(t_0) = K_0, \qquad (SRDE)$$

with

$$Q(t) = Q^*(t), S(t) = S^*(t) \text{ and } K_0 = K_0^*,$$

there exist many results ensuring the existence of the solution $K(t)$ for all $t \geq t_0$ (or $t \leq t_0$), see e.g. [36], [38], [13]; these results are mainly based on controllability/observability assumptions or on geometric properties of symmetric Riccati equations.

In contrast to this rather *exceptionally nice* behavior of the solutions to (SRDE) there exist only few results concerning the behavior of solutions of non-symmetric Riccati equations. In Section 2 we summarize for convenience of the reader some of the most important results concerning non blow up conditions for the solutions of (RDE). Further information concerning structural and geometric properties of non-symmetric Riccati equations can be found in [39], [13], [9] [29], [28], [9] and [41].
The main results of this paper are presented in Sections 2, 3 and 4. In Sections 2 and 3 we use a Lyapunov-type approach for the proof of the global existence of the solution of (RDE) and also for the global existence of the solutions to the non-symmetric Riccati difference equation

$$W(k+1) = -B_{21}(k) + B_{22}(k)W(k)(I - B_{12}(k)W(k))^{-1}W(k)B_{11}(k)$$
$$(= -B_{21}(k) + B_{22}(k)(I - W(k)B_{12}(k))^{-1}W(k)B_{11}(k)).$$

Here $B_{11}(k), B_{12}(k)$ and $B_{22}(k)$ are matrices of dimensions $n \times n, n \times m, m \times n$ and $m \times m$ respectively. For a general treatment of difference equations see for instance [5], [27]

In Section 4 we derive non-blow up conditions for two classes of perturbed Riccati equations appearing in stochastic control theory [10] and in closed loop Nash differential games [1], these results complement the existence results proved in [3] and [10].
In the second part of Section 2 and in Section 5 we indicate how our theoretical existence results can be applied in applications to control theory and differential games.
For convenience of the reader we include a detailed list of references to papers dealing with existence results for non-symmetric Riccati differential equations.

2. Existence results for (RDE).

If Y is the solution of the initial value problem

$$\dot{Y} = B(t)Y, \qquad Y(t_0) = \begin{pmatrix} I_n \\ W_0 \end{pmatrix}, \tag{2.1}$$

with

$$B(t) = \begin{pmatrix} B_{11}(t) B_{12}(t) \\ B_{21}(t) B_{22}(t) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix},$$

then (see [36], p. 11) $W(t) := Y_2(t)Y_1(t)^{-1}$ is a solution of (2.1) as long as $Y_1(t)$ is invertible.

Using this fact it can be shown that in the time-invariant case $B(t) \equiv B$ there exists at most one anti-stable (and also at most one stable) equilibrium of (RDE) which is called dichotomic (anti-dichotomic) solution of (RDE). It is known from [28], [13] that the dichotomic (anti-dichotomic) solution of (RDE) - if it exists - is the only equilibrium solution of time-invariant (RDE) having a neighborhood being negative (positive) invariant under (RDE).
First results on the global existence for the solutions of non-symmetric (RDE) for $t \geq t_0$ (or $t \leq t_0$) were obtained by Redheffer [30], [31] and Reid [35]. Redheffer and Volkmann [32], [33] obtained - as far as we know for the first time - conditions for the existence of an invariant ball for operator differential equations, which include (RDE) as a special case (see also [34]). Kuiper [26] gave another proof for the existence of an invariant ball for (RDE), which has been slightly extended in [15], where the following version of the invariance result was presented:

2.1 Theorem. *If for some constants a, $\gamma > 0$, a positive definite matrix $C \in \mathbf{C}^{m\times m}$ and every $t \leq t_0$ there holds*

$$\eta^* \begin{pmatrix} -a(B_{11}(t) + B_{11}^*(t)) & B_{21}^*(t)C - aB_{12}(t) \\ CB_{21}(t) - aB_{12}^*(t) & CB_{22}(t) + B_{22}^*(t)C \end{pmatrix} \eta \geq \gamma \|\eta\|^2$$

for all $\eta \in \mathbf{C}^{n+m}$, then the set $\{W \in \mathbf{C}^{m\times n} |$ spectral radius of $W^*CW < \sqrt{a}\}$ is negative invariant under (RDE).

In [14] it has been discussed how this theorem should be applied - in particular it has been proposed to apply Theorem 2.1 in the autonomous case after a transformation $W \to W - W_d$ if the dichotomic solution W_d of (RDE) exists. Notice that a, γ, and C can be used here as parameters in order to achieve a maximal negative invariant ellipsoid; moreover, in the same way we can check the existence of a positive invariant ellipsoid.

In the special case of square matrices W (i.e. $n = m$) Knobloch and Pohl [25] have derived recently the following existence result which is based on a special maximum principle for second order linear differential equations ([23]).

2.2 Theorem. *Given the RDE*

$$\dot{X} + XR(t)X = -A^T(t)X - XA(t) - Q(t), \tag{2.2}$$

where all (matrix-) coefficients A, R, Q *are continuous and locally integrable functions of* t *and* $R(t) = R^T(t)$ *is positive definite. Let* $A_1 = A_1^T = \sqrt{R}$,

$$B = \dot{A}_1(A^{-1})^T - A_1 A^T A_1^{-1}$$

and assume that

$$[-\sqrt{R}\,Q\sqrt{R} - \frac{1}{2}(B + B^T) + \frac{1}{4}(B - B^T)^2 + BB^T](t) \geq 0 \text{ for } t \geq 0$$

and that there exists some initial value $X = X(0)$ *such that the inequality*

$$[-\frac{1}{2}(B + B^T) + \sqrt{R}\,X\sqrt{R}](t) \geq 0 \tag{2.3}$$

holds for $t = 0$.
Then the solution $X(t)$ *of (2.2) with this initial value exists and is bounded for* $t \geq 0$ *and satisfies (2.3) for* $t \geq 0$.

Notice that the coefficient Q in (2.2) may be non-symmetric. The inequality (2.3) provides a *lower bound* for $X(t)$ and the term $XR(t)X$ is essential for obtaining an *upper bound* for $t \geq 0$.

It is worthwhile to mention that Reid used in §9 of his book [36] an *elementwise* comparison method in order to prove an existence theorem for non-symmetric Riccati differential equations. This method has been modified and extended by Gewert [17] and also by Jonq [21], [22].
Moreover, rough bounds for the interval of existence of the solution of (RDE) have been derived in [19] (see also [20]).

In what follows we suggest to use a Lyapunov-type function V, which is applied to (2.1), in order to prove global existence for the solution of (RDE) for a big class of initial values. Notice that here V is not a Lyapunov function of (RDE).

2.3 Theorem. *Let* $B_{11}, B_{12}, B_{21}, B_{22}$ *be piecewise continuous and locally integrable on* $(-\infty, T]$. *If for some matrices* $C \in \mathbf{C}^{n\times n}$, *with* $C^* = C$, $D \in \mathbf{C}^{n\times m}$, *with*

$$L = \begin{pmatrix} CB_{11} + DB_{21} & CB_{12} + B_{11}^* D + DB_{22} \\ 0 & B_{12}^* D \end{pmatrix} \tag{2.4}$$

the condition

$$L(t) + L^*(t) \leq 0, \tag{2.5}$$

holds for all $t \leq t_0 (\leq T)$ *and if*

$$C + DW_0 + W_0^* D^* > 0, \tag{2.6}$$

for some $W_0 \in \mathbf{C}^{m \times n}$*, then the solution* $W(t, W_0)$ *of (RDE) with* $W(t_0, W_0) = W_0$ *exists for all* $t \leq t_0$*.*

Proof. Here we make use of the reduction of Riccati differential equations to a linear system as described at the beginning of this section in (2.1). From the remarks made there it is clear that a *blow up* of the solution $W = W(\cdot, W_0)$ of the Riccati differential equation occurs at the moments where $\det Y_1(t)$ vanishes.
To formulate non-blow-up conditions we introduce the quadratic Lyapunov-type function

$$V(t) := x^*(Y_1^*(t) C Y_1(t) + Y_1^*(t) D Y_2(t) + Y_2^*(t) D^* Y_1(t))x. \tag{2.7}$$

Here $0 \neq x \in bfC^n, C \in \mathbf{C}^{n \times n}, C^* = C, D \in \mathbf{C}^{n \times m}$ are some parameters. Evidently

$$V_0 := V(t_0) = x^*(C + DW_0 + W_0^* D^*)x. \tag{2.8}$$

Calculating the derivative of $V(t)$ in (2.7), along a solution of (2.1), we get (suppressing t)

$$\begin{aligned} \dot{V} &= 2x^*[Y_1^* C(B_{11} Y_1 + B_{12} Y_2) + \\ &+ (Y_1^* B_{11}^* + Y_2^* B_{12}^*) D Y_2 + Y_1^* D(B_{21} Y_1 + B_{22} Y_2)]x \end{aligned}$$

and therefore

$$\begin{aligned} \dot{V} &= 2x^T Y_1^T (CB_{11} + DB_{21}) Y_1 x + \\ &+ 2x^T Y_1^T (CB_{12} + B_{11}^T D + DB_{22})) Y_2 x + +2x^T Y_2^T B_{12}^T D Y_2 x. \end{aligned}$$

This can be represented as

$$\begin{aligned} \frac{1}{2}\dot{V} &= (x^* Y_1^*, x^* Y_2^*) \times \\ &\times \begin{pmatrix} CB_{11} + DB_{21} & CB_{12} + B_{11}^* D + DB_{22} \\ 0 & B_{12}^* D \end{pmatrix} \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \\ &= (x^* Y_1^*, x^* Y_2^*) L \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \end{aligned}$$

or after a symmetrization

$$\dot{V} = (x^* Y_1^*, x^* Y_2^*)(L + L^*) \begin{pmatrix} Y_1 x \\ Y_2 x \end{pmatrix} \tag{2.9}$$

where L was defined in (2.4). ¿From (2.1), (2.7), (2.9), (2.4) and (2.5) we conclude that $\dot{V}(t)$ is piecewise continuous and $\dot{V}(t) \leq 0$ in $(-\infty, t_0]$. Hence, $V(t)$ is monotonically decreasing in $(-\infty, t_0]$.

Assume now that $C + DW_0 + W_0^*D^* > 0$ holds; then from (2.1), (2.7), (2.9) and (2.6) we infer

$$V(t_0) = x^*(C + DW_0 + W_0^*D^*)x > 0,$$

which together with the monotonicity of $V(t)$ implies that $V(t) > 0$ for all $t \leq t_0$. Hence, $Y_1(t)$ must be regular in $(-\infty, t_0]$ which implies the existence of $W(t)$ in $(-\infty, t_0]$. □

The condition (2.6) is rather restrictive, even in particular applications as for example in the standard control theoretic case. Further we suggest a refined approach.

2.4 Corollary. *If with the notations of Theorem 2.3 condition (2.5) holds and if instead of (2.6) it also holds*

$$C + DW_0 + W_0^*D^* \geq 0 \tag{2.10}$$

and

$$\operatorname{rank}\begin{pmatrix} C + DW_0 + W_0^*D^* \\ \begin{pmatrix} I & W_0^* \end{pmatrix} (L(t_0) + L^*(t_0)) \begin{pmatrix} I \\ W_0 \end{pmatrix} \end{pmatrix} = n \tag{2.11}$$

then the solution $W(t, W_0)$ of (RDE) with $W(t_0, W_0) = W_0$ exists for all $t \leq t_0$.

Proof. Integrating the equation (2.9) from $t \leq t_0$ to t and observing that $V(t_0) = x^*(C + DW_0 + W_0^*D^*)x$, we obtain

$$V(t) = V(t, x) =$$

$$x^*(C + DW_0 + W_0^*D^*)x - x^*(\int_t^{t_0} \begin{pmatrix} Y_1^*(s) & Y_2^*(s) \end{pmatrix} (L(s) + L^*(s)) \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} ds)x. \tag{2.12}$$

¿From the assumptions (2.5) and (2.10) it follows

$$x^*(C + DW_0 + W_0^*D^*)x \geq 0$$

and

$$-x^*(\int_t^{t_0} \begin{pmatrix} Y_1^*(s) & Y_2^*(s) \end{pmatrix} (L(s) + L^*(s)) \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} ds)x \geq 0.$$

If $0 \neq x \notin \ker(C + DW_0 + W_0^*D^*)$ then from (2.12) it follows

$$V(t, x) > 0 \quad \text{for} \quad \text{all} \quad t \in (-\infty, t_0] \tag{2.13}$$

If $0 \neq x \in \ker(C + DW_0 + W_0^*D^*)$ then for $0 < \delta < t_0 - t$ we get from (2.12)

$$V(t, x) = -x^*(\int_t^{t_0} \begin{pmatrix} Y_1^*(s) & Y_2^*(s) \end{pmatrix} (L(s) + L^*(s)) \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} ds)x \geq$$

$$-x^*(\int_{t_0-\delta}^{t_0} \begin{pmatrix} Y_1^*(s) & Y_2^*(s) \end{pmatrix} (L(s) + L^*(s)) \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} ds)x \tag{2.14}$$

Since $Y_1(t_0) = I$, by continuity $Y_1(t)^{-1}$ exists in $(t_0 - \delta, t_0]$ for sufficiently small δ, hence, together with (2.1), i.e. with $W(t) = Y_2(t)Y_1(t)^{-1}$, we infer from (2.14)

$$V(t, x) \geq -\delta x^* Y_1^*(s_0) \begin{pmatrix} I & W^*(s_0) \end{pmatrix} (L(s_0) + L^*(s_0)) \begin{pmatrix} I \\ W(s_0) \end{pmatrix} Y_1(s_0)x \tag{2.15}$$

for some intermediate $s_0 \in (t_0 - \delta, t_0]$. With $W(t_0) = W_0$ and the rank condition (2.11) we infer

$$-\delta x^* \begin{pmatrix} I & W_0^* \end{pmatrix} (L(t_0) + L^*(t_0)) \begin{pmatrix} I \\ W_0 \end{pmatrix} x > 0.$$

Again by continuity and for sufficiently small δ we infer from (2.15)

$$V(t,x) > 0 \quad \text{for} \quad \text{all} \quad t \in (-\infty, t_0]$$

This, together with (2.13) completes the proof since we now conclude as in the proof of the previous theorem that $\det Y_1(t) \neq 0$ for all $t \in (-\infty, t_0]$. □

In [15] global existence has been concluded under the weaker assumption $C + DW_0 + W_0^* D^* \geq 0$ using an argument from [11] but it seems that there is some gap in that argumentation. However, from the proof to Corollary 2.4 it also becomes clear that the sufficient conditions for global existence in (2.10) and (2.11) are not the weakest possible.

On the other hand, by choosing $C = 0$ and $D = 0$ in (2.4) and (2.10), respectively, conditions (2.5) and (2.10) are fulfilled for any Riccati differential equation. The rank condition (2.11) is not fulfilled and, indeed, for instance the real solutions of the scalar Riccati differential equation $\dot{x} = 1 + x^2$ all have a finite escape time, since they are of the type $x(t) = \tan(t + c)$ for some real constant c depending on the initial value. This shows that (2.4) and (2.10) cannot be sufficient without any further assumption.

We finish this section presenting possible applications of these results to optimal control and game theoretic problems. Since applicability of Theorem 2.1 to such problems was already discussed in [14], [15] we emphasize here the application of Theorem 2.3 and Corollary 2.4. The most important examples are symmetric (or hermitian) matrix Riccati differential equations and their generalizations. We consider here Riccati matrix differential equations

$$\dot{K} = -A^*(t)K - KA(t) - Q(t) + KS(t)K, \qquad K(t_0) = K_0. \tag{2.16}$$

This equation is denoted ($SRDE$), i.e. symmetric Riccati differential equation, if $Q(t) = Q^*(t), S(t) = S^*(t)$ and $K_0 = K_0^*$. In classical *Optimal Control Theory* there holds moreover $Q(t), S(t) \geq 0$. In [25] one dealt with the case where $S(t) \leq 0$ and Q may be non-symmetric, while in *H_∞-control problems* $S(t)$ is in general indefinite and $Q(t) \geq 0$. Another generalization is investigated in [43] where in general $Q^*(t) \neq Q(t)$ and therefore the solutions under consideration are usually non-symmetric.

Complex valued matrix solutions have been considered in [44]. In order to apply Theorem 2.3 we can choose the parameters C and D in an adequate way; let us choose for instance $C = 0_n$ and $D = I_n$. Then the matrix L in (2.4) has a simple form and we obtain

$$L + L^* = \begin{pmatrix} -(Q + Q^*) & 0_n \\ 0_n & -(S + S^*) \end{pmatrix}. \tag{2.17}$$

¿From (2.5) and (2.6) we then infer global existence of solutions for $t \leq t_0$ if $Q(t) + Q^*(t) \geq 0$ and $S(t) + S^*(t) \geq 0$ for $t \leq t_0$ and if the terminal value fulfills

$$K_0 + K_0^* > 0.$$

Applying Corollary 2.4, this latter condition could be weakened to

$$K_0 + K_0^* \geq 0$$

and

$$\text{rank}\begin{pmatrix} K_0 + K_0^* \\ -(Q(t_0) + Q^*(t_0)) - K_0^*(S(t_0) + S^*(t_0))K_0 \end{pmatrix} = \text{n}.$$

This yields appropriate global existence results in all cases mentioned above. Choosing other parameter matrices C, D one would obtain different conditions for global solvability.
Non-symmetric matrix Riccati differential equations appear for instance in the theory of differential games. We shall discuss here two relevant examples, namely, open loop Nash and open loop Stackelberg differential games.
In order to obtain an *open loop Nash* equilibrium in feedback synthesis, in the two player case, one has to solve the coupled system of Riccati matrix differential equations which can be written as a single non-symmetric Riccati matrix differential equation (see for instance [1]):

$$\begin{pmatrix} \dot{K_1} \\ \dot{K_2} \end{pmatrix} = -\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (S_1 S_2) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \qquad (2.18)$$

$$A,\ K_i,\ Q_i,\ S_i \in \mathbf{R}^{n\times n}, Q_i = Q_i^T,\ S_i = S_i^T,\ i = 1, 2,$$

where all coefficient matrices are supposed to be constant.
In this case, using again $C = 0_n$ and $D = (I_n, I_n)$, the matrix L in Theorem 2.3 becomes

$$L = \begin{pmatrix} -(Q_1 + Q_2) & 0_n & 0_n \\ 0_n & -S_1^* & -S_1^* \\ 0_n & -S_2^* & -S_2^* \end{pmatrix}.$$

¿From Theorem 2.3 we then infer that (2.18) admits a solution for $t \leq t_0$ if $L + L^* \leq 0$, i.e. if

$$Q_1 + Q_2 + Q_1^* + Q_2^* \geq 0$$

and

$$\begin{pmatrix} S_1 + S_1^* & S_1^* + S_2 \\ S_1 + S_2^* & S_2 + S_2^* \end{pmatrix} \geq 0,$$

and if moreover for the terminal values $K_1(t_0) = K_{10}, K_2(t_0) = K_{20}$ (2.6) is fulfilled, i.e.

$$K_{10} + K_{20} + K_{10}^* + K_{20}^* > 0$$

or, in the weaker form of Corollary 2.4,

$$K_{10} + K_{20} + K_{10}^* + K_{20}^* \geq 0$$

and

$$\text{rank}\begin{pmatrix} K_{10} + K_{20} + K_{10}^* + K_{20}^* \\ \tilde{K} \end{pmatrix} = \text{n},$$

where

$$\tilde{K} = -(Q_1 + Q_2) - (Q_1^* + Q_2^*) - K_{10}^*(S_1 + S_1^*)K_{10} - K_{20}^*(S_1 + S_2^*)K_{10} - K_{10}^*(S_1^* + S_2)K_{20}$$

$$-K_{20}^*(S_2 + S_2^*)K_{20}.$$

Choosing other parameter matrices C, D, this again would lead to different types of global existence theorems.

In the case of an *open loop Stackelberg game* for two players it is also of great importance to assure global solvability of a coupled system of Riccati matrix differential equations (see [2]). In order to calculate an equilibrium control in feedback representation one has to solve the following system (here already written as a single non-symmetric Riccati matrix differential equation)

$$\frac{d}{dt}\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} = -\begin{pmatrix} Q_1 \\ Q_2 \\ 0 \end{pmatrix} - \begin{pmatrix} A^T & 0 & 0 \\ 0 & A^T & -Q_1 \\ S_{21} & -S_1 & -A \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} A + \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}(S_1, S_2, 0)\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix},$$

$$\begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}(t_0) = \begin{pmatrix} K_{10} \\ K_{20} \\ 0 \end{pmatrix}.$$

$$A\ , P,\ K_i,\ K_{i0},\ Q_i,\ S_i \in \mathbf{R}^{n\times n}, Q_i = Q_i^T,\ S_i = S_i^T,\ K_{i0} = K_{i0}^T,\ \ i = 1, 2,$$

Choosing appropriate parameter matrices C, D as before one also can obtain global existence of solutions to this type of equations. We omit details.

3. Existence results for matrix Riccati difference equations.

In this section we consider *non-symmetric matrix Riccati difference equations* of the form

$$W(k+1) = -B_{21} + B_{22}W(k)(I - B_{12}W(k))^{-1}B_{11}, \quad W(0) = W_0. \tag{3.1}$$

Here

$$B_{11} \in \mathbf{C}^{n\times n},\ B_{12} \in \mathbf{C}^{n\times m},\ B_{21} \in \mathbf{C}^{m\times n},\ B_{22} \in \mathbf{C}^{m\times m},\ W \in \mathbf{C}^{m\times n}.$$

In the sequel we use the *operator norms* in the spaces of rectangular matrices, i.e. for $C \in \mathbf{C}^{m\times n}$:

$$\|C\| = \max\{\|Cx\|_{\mathbf{C}^m} : \|x\|_{\mathbf{C}^n} = 1\}.$$

Moreover we shall assume that at least one of the matrices B_{11} or B_{22} is invertible. These two cases are treated similarly; therefore we assume for definiteness that $\det B_{11} \neq 0$.

In order to analyze the equation (3.1) we shall introduce the following (see [12]) *linear difference equation*

$$Z(k+1) = TZ(k), \quad k \geq 0, \quad Z(0) = \begin{pmatrix} I_n \\ W_0 \end{pmatrix}, \tag{3.2}$$

where

$$T = T(B) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$
$$= \begin{pmatrix} I_n & 0 \\ -B_{21} & I_m \end{pmatrix} \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} I_n & -B_{12} \\ 0 & I_m \end{pmatrix}$$

The following lemma is proved in [12, Lemma 2.2])

3.1 Lemma. *i) If $(Z(k))_{0\le k\le\nu}$ is a trajectory of the linear difference equation (3.2), $Z_k = \binom{X_k}{Y_k}$ and $\det X_k \neq 0$ for $0 \le k \le \nu$, then $(W(k))_{0\le k\le\nu}$ with $W(k) = Y(k)X^{-1}(k)$ is a trajectory of the difference Riccati equation (3.1);*

ii) If $W(k) = Y(k)X^{-1}(k)$ is a trajectory of the difference Riccati equation, (3.1) then the sequence $Z_k = \binom{X_k}{Y_k}_{0\le k\le\nu}$, defined according to the equations

$$X(k+1) = B_{11}^{-1}X(k) - B_{11}^{-1}B_{12}W(K)X(k),\ X(0) = I_n, 0 \le k \le \nu - 1,$$
$$Y(k) = W(k)X(k),\ 0 \le k \le \nu,$$

is a trajectory of (3.2).

Our goal is to establish whether the trajectory of the Riccati difference equation exists for $k = 0, \ldots, \nu$. According to the Lemma this is the case if $\det X_k \neq 0$ along the corresponding solution of the linear difference equation (3.2). We shall apply a modification of the Lyapunov function method of the previous Section to the linear difference equation in order to derive a sufficient condition for existence of a solution of the Riccati difference equation.

Evidently, if $\det X = 0$, then $X\bar{x} = 0$ for some nonzero $\bar{x} \in \mathbf{C}^n$. We will derive now a sufficient condition for the implication

$$\det X(k) \neq 0 \Rightarrow \det X(k+1) \neq 0,\ k \ge 0,$$

to hold for the trajectory $Z(k) = \binom{X(k)}{Y(k)}$ of the linear difference equation (3.2).

To derive this condition let us choose matrices $C \in \mathbf{C}^{n\times n}$, $C^* = C$, $D \in \mathbf{C}^{n\times m}$ and for any $Z = \binom{X}{Y} \in \mathbf{C}^{(n+m)\times n}$ introduce a quadratic form in $\mathbf{C}^n$:

$$V_{X,Y}(x) = x^*(X^*CX + X^*DY + Y^*D^*X)x; \tag{3.3}$$

here $x \in \mathbf{C}^n$.

Let us denote for the sake of brevity the components $X(k), Y(k)$ of $Z(k)$ by X, Y and assuming $\det X \neq 0$ define $X(k+1), Y(k+1)$ (the components of $Z(k+1)$) according to equation (3.2). Computing the function $V_{X(k+1),Y(k+1)}(x)$ we obtain

$$\begin{aligned} V_{X(k+1),Y(k+1)}(x) = x^*(T_{11}X + T_{12}Y)^*C(T_{11}X + T_{12}Y)x + \\ x^*(T_{11}X + T_{12}Y)^*D(T_{21}X + T_{22}Y)x + x^*(T_{21}X + T_{22}Y)^*D^*(T_{11}X + T_{12}Y)x = \\ x^*X^*(T_{11}^*CT_{11} + T_{11}^*DT_{21} + T_{21}^*D^*T_{11})Xx + \\ x^*X^*(T_{11}^*CT_{12} + T_{11}^*DT_{22} + T_{21}^*D^*T_{12} + T_{12}^*CT_{11} + T_{22}^*D^*T_{11} + T_{12}^*DT_{21})Yx + \\ x^*Y^*(T_{12}^*CT_{12} + T_{12}^*DT_{22} + T_{22}^*D^*T_{12})Yx. \end{aligned}$$

With the abbreviations $\xi = Xx \in \mathbf{C}^n,\ \eta = Yx \in \mathbf{C}^m.$ we can write the latter quadratic form as $\xi^*M\xi + \xi^*N\eta + \eta^*N\xi + \eta^*P\eta$ where

$$\begin{aligned} M &= T_{11}^*CT_{11} + T_{11}^*DT_{21} + T_{21}^*D^*T_{11}, \\ N &= T_{11}^*CT_{12} + T_{11}^*DT_{22} + T_{21}^*D^*T_{12}, \\ P &= T_{12}^*CT_{12} + T_{12}^*DT_{22} + T_{22}^*P^*T_{12}. \end{aligned} \tag{3.4}$$

Let us consider now this quadratic form for *arbitrary* values $\xi \in \mathbf{C}^n,\ \eta \in \mathbf{C}^m$. Assume the quadratic form $S(\xi,\eta) = \xi^*M\xi + \xi^*N\eta + \eta^*N^*\xi + \eta^*P\eta$ to be positive definite in $\mathbf{C}^n \times \mathbf{C}^m$. Then for any non-vanishing $\bar{x} \in \mathbf{C}^n:\ X\bar{x} \neq 0$ and hence $V_{X(k+1),Y(k+1)}(\bar{x}) = S(X\bar{x}, Y\bar{x}) \neq 0$. Using (3.3), this in its turn implies $X(k+1)\bar{x} \neq 0$.
Summarizing we obtain the following result:

3.2 Theorem. *If there exist matrices $C \in \mathbf{C}^{n\times n},\ C^* = C,\ \ D \in \mathbf{C}^{n\times m}$ such that the quadratic form $S(\xi,\eta) = \xi^*M\xi + \xi^*N\eta + \eta^*N\xi + \eta^*P\eta$ (where $M,\ N,\ P$ are defined by the equalities (3.4)), is positive definite, then $\det X(k) \neq 0$ along the trajectory $Z_k = \binom{X_k}{Y_k}_{k\geq 0}$ of the linear difference equation (3.2) and therefore the solution W_k of the Riccati difference equation is defined for $k \geq 0$.*
Next we define $X(k),\ Y(k),\ Z(k),\ C,\ D,\ M,\ N,\ P$ and $V_{X,Y}(x)$ as above; moreover we put

$$\begin{aligned} \tilde{M} &= T_{11}^*CT_{11} + T_{11}^*DT_{21} + T_{21}^*D^*T_{11} - C, \\ \tilde{N} &= T_{11}^*CT_{12} + T_{11}^*DT_{22} + T_{21}^*D^*T_{22} - D, \\ \tilde{P} &= T_{12}^*CT_{12} + T_{12}^*DT_{22} + T_{22}^*P^*T_{12}. \end{aligned} \tag{3.5}$$

and consider the quadratic form $\tilde{S}(\xi,\eta) = \xi^*\tilde{M}\xi + \xi^*\tilde{N}\eta + \eta^*\tilde{N}\xi + \eta^*\tilde{P}\eta$ in $\mathbf{C}^n \times \mathbf{C}^m$.
Using these notations we get

3.3 Theorem. *If there exist matrices $C \in \mathbf{C}^{n\times n},\ C^* = C,\ \ D \in \mathbf{C}^{n\times m}$ such that the quadratic form $\tilde{S}(\xi,\eta)$ is positive semidefinite in $\mathbf{C}^n \times \mathbf{C}^m$, and if the initial value $W_0 = W(0)$ of the Riccati difference equation (3.1) satisfies*

$$C + DW_0 + W_0^*D^* > 0, \tag{3.6}$$

then the solution $W(k)$ of (3.1) exists for $k \geq 0$.
Proof. Analogously to the proof of Theorem 3.3 we get by induction that the components of $Z(k) = \binom{X(k)}{Y(k)}$, $k \geq 0$ satisfy
$V_{X(k+1),Y(k+1)} - V_{X(k),Y(k)} \geq 0$ for $k \geq 0$. Therefore (3.6) yields $V_{X(k),Y(k)} > 0$ for $k \geq 0$, and this implies $\det X(k) \neq 0$ for $k \geq 0$. □

Theorem 3.3 is the discrete version of Theorem 2.3, and one could also formulate a discrete version of Corollary 2.4 as well as applications of these theorems which are similar to those presented at the end of Section 2; we omit details.

4. Existence results for perturbed symmetric Riccati differential equations.

In this section we apply modifications of the methods presented in [15] to a class of Riccati differential equations and perturbed Riccati differential equations admitting hermitian or symmetric solutions. In particular we deal with initial value problems of the form

$$\dot{K} = -A^*K - KA - Q + KSK + g(K), \quad K(t_f) = K_f, \tag{4.1}$$

where the coefficient matrices $A = A(t),\ Q = Q(t),\ S = S(t)$ are piecewise continuous and locally integrable in $(-\infty, t_f]$, $K_f \in \mathbf{C}^{n\times n}$, with $Q^* = Q,\ S^* = S,\ K_f^* = K_f$. If furthermore $t \mapsto g(t, K)$ is piecewise continuous and locally integrable on $(-\infty, t_f]$, $g(t, \cdot)$ is a Lipschitz continuous map of $\mathbf{C}^{n\times n}$ into itself and

$$(g(t, K^*))^* = g(t, K), \ \ t \in (-\infty, t_f], \tag{4.2}$$

then (4.1) admits a hermitian solution

$$K(t) = K^*(t)$$

in a neighborhood of t_f.

In the sequel we present conditions for these solutions to exist in $(-\infty, t_f]$. Since we are interested in specific applications, we restrict our considerations to the perturbation terms $g(K)$ which arise from these applications.

The following result holds for hermitian (or real symmetric) solutions of (4.1).

4.1 Theorem. *Assume that in equation (4.1)*

$$g(K) = -g_1(K) + Kg_2(K) + g_3(K)K, \tag{4.3}$$

where $g_1^*(K^*) = g_1(K)$ and $g_2^*(K^*) = g_3(K)$.
If for some positive α and for all $x \in \mathbf{C}^n$

$$x^*(g_1(t, K) + Q(t))x \geq \alpha|x|^2 \tag{4.4}$$

for $t \in (-\infty, t_f]$ and $K \geq 0$ then any solution of (4.1) with positive definite terminal data $K_f > 0$ stays positive definite for $t \leq t_f$.

Proof. as it is $K(t, t_\omega)$ Let $K(t)$ be a hermitian solution of the equation which is *piecewise continuously differentiable* on a finite interval $[t_0, t_f]$. This will mean that $K(t)$ is *continuous* on $[t_0, t_f]$ and there exists a subdivision of $[t_0, t_f]$ into a finite number of intervals such that on each interval the function $K(t)$ is continuously differentiable possessing one-sided finite derivatives at the end-points of these intervals.

The present proof as well as the proof of the next theorem makes use of the following technical result

4.2 Proposition. *If K is piecewise continuously differentiable and hermitian on a finite interval then its minimal eigenvalue λ is an absolutely continuous (in fact a Lipschitzian) function of t. At any point where it is differentiable (therefore almost everywhere on the interval) there holds*

$$\dot{\lambda}(t) = x^*(t)\dot{K}(t)x(t), \tag{4.5}$$

where $x(t)$ is a unit eigenvector of $K(t)$ corresponding to this minimal eigenvalue. The same also holds true for the maximal eigenvalue of K.

We postpone proving this lemma till the end of this section [1] and now complete the proof of Theorem 4.1.

Let $\lambda(t)$ denote the minimal eigenvalue of $K(t, t_f) = K(t)$ with $x(t)$ being a corresponding unit eigenvector. Then (suppressing the variable t and involving (4.3)) we conclude

$$\frac{d\lambda}{dt} = x^*\dot{K}x = -2x^*KAx - x^*Qx + x^*KSKx + x^*g(K)x =$$

$$-2\lambda x^*Ax + \lambda^2 x^*Sx - x^*(g_1(K) + Q)x + 2\lambda x^*(g_2(K))x. \tag{4.6}$$

¿From (4.6) and (4.4) we now obtain

$$\frac{d\lambda}{dt} \leq -\lambda(2x^*Ax - 2x^*g_2(K)x - \lambda x^*Sx) - \alpha|x|^2$$

a.e. in $(-\infty, t_f]$, whenever $K \geq 0$.
It is now evident that for sufficiently small $\lambda(t) > 0$ we have $\frac{d\lambda}{dt} < 0$. ¿From this monotonicity property and the continuity of $\lambda(t)$ we conclude $\lambda(t) > 0$ for all $t < t_f$, whenever $\lambda_f =$

[1]Were the function $x(t)$ always differentiable with respect to t, the formula (4.5) could be obtained by a direct computation. Difficulties arise when multiple eigenvalues occur and one can *not* always choose an eigenvector $x(t)$ depending continuously on t. To overcome this difficulty we use below a technical trick similar to the one utilized (in quite different context) in [16, Ch.1]

$\lambda(t_f) > 0$, where λ_f denotes the minimal eigenvalue of the terminal matrix $K_f > 0$. Hence, the solution of (4.1) remains positive for $t \in (-\infty, t_f]$. □

In the next theorem we describe another type of invariant domains for the equation (4.1).

4.3 Theorem. *Let $P : \mathbf{R}_+ \to \mathbf{R}_+$ be any function. If for some $\varepsilon > 0$, $\alpha, \beta \in \mathbf{R}, \alpha < \beta$ and for all $t \leq t_f$ and $y \in \mathbf{C}^n$, with $\|y\| = 1$ the following inequalities hold*

$$\begin{array}{rcl} y^*(-2\alpha A(t) + \alpha^2 S(t) - Q(t))y & \leq & -P(|\alpha|) - \varepsilon, \\ y^*(-2\beta A(t) + \beta^2 S(t) - Q(t))y & \geq & P(|\beta|) + \varepsilon, \end{array} \tag{4.7}$$

where for g_1, g_2, g_3 as in (4.3) there holds

$$||g_1(K) - Kg_2(K) - g_3(K)K|| \leq P(||K||) \tag{4.8}$$

for all hermitian matrices $K \in \mathbf{C}^{n\times n}$ and all $t \in (-\infty, t_f]$, then the set

$$I(\alpha, \beta) = \{K = K^* \in \mathbf{C}^{n\times n} | \ \alpha I_n \leq K \leq \beta I_n\}$$

is negative invariant.

Proof. As in the proof of Theorem 4.1 let us denote by $\lambda(t)$ either the minimal or maximal eigenvalue of the hermitian solution $K(t)$ to (4.1), respectively, and let $y(t)$ denote a corresponding eigenvector with $|y(t)| = 1$.
As in the previous proof we derive from (4.8)

$$\frac{d\lambda}{dt} = y^*\dot{K}y = -2y^*KAy - y^*Qy + y^*KSKy + y^*g(K)y =$$

$$-2\lambda y^*Ay + \lambda^2 y^*Sy - y^*Qy - y^*(g_1(K) - Kg_2(K) - g_3(K)K)y$$

$$\leq y^*(-2\lambda A + \lambda^2 S - Q)y + P(||K||). \tag{4.9}$$

This together with (4.7) yields

$$\frac{d\lambda}{dt} < 0 \tag{4.10}$$

as λ (here considered to be the minimal eigenvalue) approaches the value α from above. Analogously from the inequality

$$\frac{d\lambda}{dt} \geq y^*(-2\lambda A + \lambda^2 S - Q)y - P(||K||)$$

we obtain together with (4.7)

$$\frac{d\lambda}{dt} > 0 \tag{4.11}$$

as λ (here considered to be the maximal eigenvalue) approaches the value β from below.
¿From inequalities (4.10), (4.11) we conclude similarly as in the proof to Theorem 4.1 that $I(\alpha, \beta)$ is a negative invariant set for equation (4.1). □

As it has already been mentioned in Section 2, a similar method was used in [15] to obtain results on the invariant sets for non-symmetric Riccati matrix differential equations (see Theorem 2.1).

Proof of Proposition 4.2. Since the proof for the maximal eigenvalue follows by a completely analogous argumentation, we restrict ourselves to the case of the minimal eigenvalue. Without loss of generality we may reduce our consideration to a (closed finite) interval on which $K(.)$ is *continuously differentiable.* Let $\lambda(t)$ be the minimal eigenvalue of $K(t)$ and $x(t)$ be a unit eigenvector of $K(t)$ corresponding to this minimal eigenvalue.

Let us introduce a function $\Lambda(s,t) = x^*(t)K(s)x(t)$. Evidently for each t the function $s \mapsto \Lambda(s,t)$ is continuously differentiable and $\lambda(t) = \Lambda(t,t)$. As the minimal eigenvalue also is defined by

$$\lambda(s) = \min_{|x|=1} x^* K(s)x,$$

we infer that $\Lambda(s,t) \geq \lambda(s) = \Lambda(s,s)$, for all s,t.

Then we may conclude for any t', t'':

$$\Lambda(t'',t'') - \Lambda(t',t'') = \lambda(t'') - \Lambda(t',t'') \leq \lambda(t'') - \lambda(t') \leq \Lambda(t'',t') - \lambda(t') = \Lambda(t'',t') - \Lambda(t',t'),$$

deriving from it

$$\int_{t'}^{t''} \frac{\partial \Lambda}{\partial s}(\sigma, t'')d\sigma \leq \lambda(t'') - \lambda(t') \leq \int_{t'}^{t''} \frac{\partial \Lambda}{\partial s}(\sigma, t')d\sigma \qquad (4.12).$$

As far as $\frac{\partial \Lambda}{\partial s}(\sigma,t)$ is continuous with respect to the first argument, then it is bounded on any finite closed interval and we conclude that $\lambda(t)$ is Lipschitzian (and absolutely continuous) on this interval.

To derive equality (4.5) at any point t_0 of differentiability of $\lambda(t)$ let us choose $h > 0$ and first estimate the difference

$$\lambda(t_0+h) - \lambda(t_0) - x^*(t_0)\dot{K}(t_0)x(t_0)h = \lambda(t_0+h) - \lambda(t_0) - \frac{\partial \Lambda}{\partial s}(t_0,t_0)h.$$

Choosing $t' = t_0, t'' = t_0 + h$ and utilizing the upper estimate of (4.12) we obtain

$$\lambda(t_0+h) - \lambda(t_0) - \frac{\partial \Lambda}{\partial s}(t_0,t_0)h \leq \int_{t_0}^{t_0+h} (\frac{\partial \Lambda}{\partial s}(\sigma,t_0)) - \frac{\partial \Lambda}{\partial s}(t_0,t_0))d\sigma.$$

As long as $\sigma \mapsto \frac{\partial \Lambda}{\partial s}(\sigma,t_0)$ is continuous, then the latter integral is $o(h)$, as $h \to +0$ and we conclude $\frac{d\lambda}{dt}(t_0) \leq \frac{\partial \Lambda}{\partial s}(t_0,t_0)$.

Let us now consider the difference $\lambda(t_0) - \lambda(t_0 - h) - \frac{\partial \Lambda}{\partial s}(t_0,t_0)h$. Choosing $t' = t_0 - h, t'' = t_0$ and utilizing the lower estimate of (4.12) we conclude

$$\int_{t_0-h}^{t_0} (\frac{\partial \Lambda}{\partial s}(\sigma,t_0) - \frac{\partial \Lambda}{\partial s}(t_0,t_0))d\sigma \leq \lambda(t_0) - \lambda(t_0-h) - \frac{\partial \Lambda}{\partial s}(t_0,t_0)h.$$

¿From this estimate we derive $\frac{d\lambda}{dt}(t_0) \geq \frac{\partial \Lambda}{\partial s}(t_0,t_0)$ arriving to the equality (4.5). □

5. Applications.

Now we describe some possible applications of the Theorems 4.1 and 4.2.
First we recall the Riccati matrix differential equation from classical *theory of optimal control* (see for instance [24])

$$\dot{K} = -A^*K - KA - Q + KSK, \quad K(t_f) = K_f.$$

Here we clearly have $g_1 = g_2 = g_3 = 0$. From Theorem 4.1 we infer that if $K_0 > 0$ and if $x^*Q(t)x \geq \alpha|x|^2$ then the solution $K(t)$ stays positive for $t \leq t_f$, hence a blow up of the solution only can occur if one of the eigenvalues of $K(t)$ tends to $+\infty$ in finite time. If now also the second condition in (4.7) is fulfilled for some β with $P = 0$ then we obtain boundedness, i.e $K(t) \leq \beta I$ if $K(t_f) \leq \beta I$.

As our next example we discuss coupled Riccati type differential equations as they appear in the theory of *Nash games* with *memoryless feedback* (or closed loop) information structure (see [42]) in the two player case:

$$\dot{K}_1 = -A^TK_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2 + K_2S_2K_1 - K_2S_{12}K_2, \;\; K_1(t_f) = K_{1f},$$

$$\dot{K}_2 = -A^TK_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1 + K_1S_1K_2 - K_1S_{21}K_1, \;\; K_2(t_f) = K_{2f}.$$

With the following notation

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \; \tilde{Q} = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \; \tilde{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},$$

$$S_0 = \begin{pmatrix} S_{12} & 0 \\ 0 & S_{21} \end{pmatrix}, \; K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \; K_f = \begin{pmatrix} K_{1f} & 0 \\ 0 & K_{2f} \end{pmatrix}, \; J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where $A \in \mathbf{R}^{n\times n}$ and $Q, S, S_0 \in \mathbf{R}^{n\times n}$ are symmetric, the coupled system then can be written as a single equation

$$\dot{K} = -\tilde{A}^TK - K\tilde{A} - \tilde{Q} + K\tilde{S}K + JK\tilde{S}JK + KJ\tilde{S}KJ - JKJS_0JKJ,$$

with terminal values $K(t_f) = K_f, \;\; K_f^T = K_f$.
Here we can apply Theorems 4.1,4.2 with

$$g_1(K) = JKJS_0JKJ,$$

where $g_1^T(K) = JK^TJS_0JK^TJ = g_1(K^T)$, and

$$\begin{aligned} g_2(K) &= J\tilde{S}KJ, \\ g_3(K) &= JK\tilde{S}J, \end{aligned}$$

where $g_3^T(K) = J\tilde{S}K^TJ = g_2(K^T)$. Notice that this equation also has been studied in [4] [18] [40].

In *Minimal Cost Variance Control* problems there appears a coupled system of Riccati type differential equations quite similar to that one from memoryless feedback Nash games. We consider the following system (see [37]):

$$\dot{M} = -A^T M - MA - Q + MBR^{-1}B^T M - \gamma^2 VBR^{-1}B^T V,$$

$$\dot{V} = -A^T V - VA + 2\gamma VBR^{-1}B^T V + MBR^{-1}B^T V + VBR^{-1}B^T M - 4ME\tilde{W}E^T M,$$

with $M(t_f) = Q_f, \;\; V(t_f) = 0$.
Introducing $S_1 = BR^{-1}B^T, S_{12} = \gamma^2 BR^{-1}B^T, S_2 = 2\gamma BR^{-1}B^T, S_{21} = 4E\tilde{W}E^T$ and

$$K := \begin{pmatrix} M & 0 \\ 0 & V \end{pmatrix}, \quad K_f = \begin{pmatrix} Q_f & 0 \\ 0 & 0 \end{pmatrix},$$

we represent this system as a single generalized Riccati differential equation

$$\dot{K} = -\tilde{A}^T K - K\tilde{A} - \hat{Q} + K\tilde{S}K + JK\tilde{S}JK + KJ\tilde{S}KJ - JKJS_0JKJ - K\hat{S}JKJ - JKJ\hat{S}K,$$

with terminal values $K(t_f) = K_f, K_f^T = K_f$. Here we used the same notation as above for $\tilde{A}, \tilde{S}, S_0$ and, additionally,

$$\hat{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this situation we can apply the Theorems 4.1, 4.2 with

$$g_1(K) = JKJS_0JKJ - K\hat{S}JKJ - JKJ\hat{S}K, \;\; g_2(K) = J\tilde{S}KJ, \;\; g_3(K) = JK\tilde{S}J,$$

where $g_1^T(K) = JK^TJS_0JK^TJ - JK^TJ\hat{S}K^T - K^T\hat{S}JK^TJ = g_1(K^T)$,
and $g_3^T(K) = J\tilde{S}K^TJ = g_2(K^T)$.

Finally we show the applicability of these Theorems 4.1, 4.2 to coupled generalized Riccati differential equations occurring in *Markovian Jump Linear Quadratic Control* problems.
In [3] we investigated the following equation

$$\dot{K} = -A^T K - KA - Q + KSK - \sum_{i=1}^{N-1} C_i^T K C_i, \quad K(t_f) = 0,$$

where Q, S are symmetric matrices. In order to apply Theorems 4.1, 4.2 we set here

$$g_1(K) = \sum_{i=1}^{N-1} C_i^T K C_i, \quad g_1^T(K) = g_1(K^T),$$

and $g_2 = g_3 = 0$.

References

[1] H. ABOU-KANDIL, G. FREILING and G. JANK, *Necessary conditions for constant solutions of coupled Riccati equations in Nash games* , Systems and Control Letters **21** (1993), 295 - 306.

[2] H. ABOU-KANDIL, G. FREILING and G. JANK, *Asymptotic behavior of Leader-Follower Control: Application to Flexible Structures.* In: Proceedings of ECC, **93**, Groningen (1993), 1014-1019.

[3] H. ABOU-KANDIL, G. FREILING and G. JANK, *Solution and asymptotic behavior of coupled Riccati equations in jump-linear systems* , IEEE Trans. Automat. Contr., **AC - 39** (1994), 1631-1636.

[4] H. ABOU-KANDIL, G. FREILING and G. JANK, *On Global Existence of Solutions to Coupled Matrix Riccati Equations in Closed Loop Nash Games* IEEE Trans. Automat. Contr., **AC - 41** (1996), 264- 269.

[5] R.P. AGARWAL, *Difference Equations and Inequalities: theory methods, applications,* Marcel Dekker N.Y., 1992.

[6] T. BAŞAR and G.J. OLSDER, *Dynamic non-cooperative game theory,* Academic Press, New York, 1995.

[7] D. J. CLEMENTS and B. D. O. ANDERSON, *Polynomial factorization via the Riccati equation*, SIAM J. Appl. Math. **31** (1976), 179-205.

[8] E.A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York,1955.

[9] D. D'ALESSANDRO, *Invariant manifolds and projective combinations of solutions of the Riccati differential equation.* To appear

[10] V. DRAGAN and R. MOROZAN, *Global solution to game-theoretic Riccati equation of stochastic control*, J. Diff. Equ. **138**, (1997), 328-350.

[11] M. FEUCHT, *Differentialspiele und gekoppelte Riccatische Matrixdifferentialgleichungen,* Dissertation Univ. Ulm (1994).

[12] G.FREILING and V.IONESCU, *Non-symmetric discrete-time difference and algebraic Riccati equations: Some representation formulae and comments*, Preprint SM-DU-389, Gerhard Mercator Universität, Duisburg, 1997.

[13] G. FREILING and G. JANK, *Non-symmetric matrix Riccati equations*, Journal Analysis Appl. **14** (1995), 259-284.

[14] G. FREILING, G. JANK, S.-R. LEE and A. SARYCHEV, *Sufficient conditions for the existence of open loop Nash and Stackelberg strategies.* In: Proceedings of Controlo, **96**, (1996).

[15] G. FREILING, G. JANK and A. SARYCHEV, *Lyapunov-type functions and invariant sets for Riccati matrix differential equations*, Proceedings of ECC 1997, Bastin, van Dooren, Eds., Brussels, (1997).

[16] R.GAMKRELIDZE, *Principles of Optimal Control Theory*, Plenum Press, New York, 1978.

[17] M. GEWERT, *Some properties of solutions of Riccati matrix differential equations*, Fasciculi Math. **24**, (1994), 43-53.

[18] G. JANK, and G. KUN, *Solutions of Generalized Riccati Differential Equations and their Approximation*, Computational Methods in Function Theory (CMFT'97), N. Papamichael, St. Ruscheweyh and E. B. Saff (Eds.), World Scientific Publ.,(1998) 1-18.

[19] L. JODAR, *Rational approximate solution and error bounds for the non-symmetric Riccati matrix differential equation*, Appl. Math. Comp. **50**, (1992), 23-31.

[20] L. JODAR and E. PONSODA, *Non-autonomous Riccati-type matrix differential equations: existence interval, construction of continuous numerical solutions and error bounds*, IMA J. Numerical Anal., **15** (1995), 61-74.

[21] J. JUANQ and M. T. LEE, *Comparison theorems for the matrix Riccati equation*, Lin. Alg. Appl., **196**, (1994), 183-191.

[22] J. JUANQ, *Existence of algebraic matrix Riccati equations arising in transport theory*, Lin. Alg. Appl., **230**, (1995), 89-100.

[23] H. W. KNOBLOCH, *Differential inequalities and maximum principles for second order differential equations*, Nonlinear Analysis, **25** (1995), 1003-1016.

[24] H. W. KNOBLOCH and H. KWAKERNAAK, *Lineare Kontrolltheorie*, Springer-Verlag Berlin, Heidelberg, New York, Tokio, 1985.

[25] H. W. KNOBLOCH and M. POHL, *On Riccati matrix differential equations*, Results Math. **31** (1997), 337-364.

[26] H.J. KUIPER, *Positive invariance and asymptotic stability of solutions to certain Riccati equations*, Dynamics and Stability of Systems **9** (1994), 331-344.

[27] V. LAKSHMIKANTHAM and D. TRIGIANTE, *Theory of Difference Equations: numerical methods and applications,* Academic Press, Boston, 1988.

[28] J. MEDANIC, *Geometric properties and invariant manifolds of the Riccati equation*, IEEE Trans. Automat. Contr. **AC-27** (1982), 670-677.

[29] M. PAVON and D. D'ALESSANDRO, *Families of solutions of matrix Riccati differential equations* SIAM J. Contr. **35** (1997), 194-204.

[30] R. REDHEFFER, *Inequalities for a matrix Riccati equation*, J. Math. Mech. **8** (1959), 349-367.

[31] R. REDHEFFER, *Supplementary Note to [30]* J. Math. Mech. **9** (1960), 745-748.

[32] R. REDHEFFER, *Matrix differential equations*, Bull. Amer. Math. Soc. **81** (1975), 485-488.

[33] R. REDHEFFER and P. VOLKMANN, *Differential equations in operator algebras*, Math. Ann. **239** (1979), 97-110.

[34] R. REDHEFFER and P. VOLKMANN, *Differential equations in operator algebras II. Invariance of the order cone*, J. Diff. Equat. **130** (1996), 356-367.

[35] W. T. REID, *Properties of solutions of a Riccati matrix differential equation, J. Math. Mech.* **9** (1960), 749-770.

[36] W. T. REID, *Riccati Differential Equations*, Academic Press, New York, (1972).

[37] M.K. SAIN, C.H. WON and B.F. SPENCER, Jr. *Cumulants in Risk-Sensitive-Control: The Full-State-Feedback Cost Variance Case*, *Proceedings 34th IEEE Conference on Decision and Control*, pp. 1036 - 1041, (1995).

[38] M. A. SHAYMAN, *Phase portrait of the matrix Riccati equation*, SIAM J. Contr. Opt. **24** (1986), 1-65.

[39] M. A. SHAYMAN, *A geometric view of the matrix Riccati equation*, The Riccati Equation, Eds.: S. Bittanti, A. Laub and J. C. Willems, Springer Verlag, Berlin, (1991).

[40] M. SIMAAN and Jr. J.B. CRUZ. *Sampled-data Nash controls in non-zero-sum differential games*, *Int. J. Control* **17**(*6*):1201–1209, 1973.

[41] M. SORINE and P. WINTERNITZ, *Superposition laws for solutions of differential matrix Riccati equations arising in control theory*, IEEE Trans. Automat. Contr. **AC-30** (1985), 266-272.

[42] A.W. STARR and Y.C. HO. *Nonzero-sum differential games*, *J. Optimization Theory & Applications* **3**(**3**):184–206, 1969.

[43] D.D. THOMPSON and R.A. VOLZ, *The linear quadratic cost problem with linear state constraints and the non symmetric Riccati equation*, SIAM J. Contr., **13** (1975), 110-145.

[44] M. I. ZELIKIN, *On the Theory of the Matrix Riccati Equation*, Math. USSR Sbornik, **73** (1992), 341- 354.

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