Math. Z. 234, 163–189 (2000)

Mathematische Zeitschrift

Sampling expansions associated with Kamke problems

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Received February 16, 1998; in final form March 15, 1999

Dedicated to Professor Paul L. Butzer on the occasion of his seventieth birthday

Abstract. The present paper is devoted to the derivation of sampling expansions for entire functions which are represented as integral transforms where a differential operator is acting on the kernels. The situation generalizes the results obtained in sampling theory associated with boundary value problems to the case when the differential equation has the form $N(y) = \lambda P(y)$, where N and P are two differential expressions of orders n and p respectively, $n > p$ and λ is the eigenvalue parameter. Both self adjoint and non self adjoint cases will be considered with examples in which the boundary conditions are strongly regular.

*Mathematics Subject Classification (1991):*41A05, 34B05, 34L10

1. Introduction

In his note, Weiss [55], was looking for a class of functions which can be sampled (recovered) from their values at a discrete sequence of points. He showed that one of the possibilities is to take a set of integral transforms whose kernels are certain solutions of some Sturm-Liouville eigenvalue problems. In fact let us consider the boundary-value problem (BVP)

$$(1.1) \qquad -y''(x) + q(x)y(x) = \lambda y(x), \quad -\infty < a \leq x \leq b < \infty,$$

⋆ M.H. Annaby wishes to thank Alexander von Humboldt foundation for the grant IV-1039259. He also thanks the staff of Lehrstuhl A für Mathematik, RWTH-Aachen, for cooperation during his stay there.

$$y(a)\cos\alpha + y'(a)\sin\alpha = 0, \tag{1.2}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \tag{1.3}$$

where $q(\cdot)$ is continuous and real valued on $[a,b]$ and $\alpha,\ \beta \in [0,\pi)$. Let $u(\cdot,\lambda)$ be the solution of (1.1) determined by the initial conditions

$$u(a,\lambda) = \sin\alpha, \quad u'(a,\lambda) = -\cos\alpha, \quad \text{for all } \lambda \in \mathbb{C}. \tag{1.4}$$

Hence the set of all integral transforms

$$f(\lambda) = \int_a^b g(x)u(x,\lambda)\,dx, \quad \lambda \in \mathbb{C},\ g(\cdot) \in L^2(a,b) \tag{1.5}$$

can be recovered via the sampling expansion

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda_k)\, \frac{\int_a^b u(x,\lambda)\overline{u(x,\lambda_k)}\,dx}{\int_a^b |u(x,\lambda_k)|^2\,dx}, \tag{1.6}$$

where $\{\lambda_k\}_{k=0}^{\infty}$ is the (real) sequence of the simple eigenvalues, cf. [52]. The series (1.6) converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$. Two years after Weiss' note, Kramer [33] obtained the following sampling theorem for a more general family of integral transforms.

1.1. Theorem. *Let $\{\lambda_k\}_{k=-\infty}^{\infty}$ be a sequence of real numbers. Let $I \subseteq \mathbb{R}$ be an interval and $K(\cdot,\lambda) : I \times \mathbb{C} \longrightarrow \mathbb{C}$ be a function such that $K(\cdot,\lambda) \in L^2(I)$ for all $\lambda \in \mathbb{C}$ and the sequence $\{K(\cdot,\lambda_k)\}_{k=-\infty}^{\infty}$ forms a complete orthogonal set in $L^2(I)$. Then the integral transform*

$$f(\lambda) = \int_I \overline{g}(x)K(x,\lambda)\,dx, \quad g(\cdot) \in L^2(I), \tag{1.7}$$

admits the sampling expansions

$$f(\lambda) = \sum_{k=-\infty}^{\infty} f(\lambda_k)\, \frac{\int_I K(x,\lambda)\overline{K(x,\lambda_k)}\,dx}{\int_I |K(x,\lambda_k)|^2\,dx}. \tag{1.8}$$

Series (1.8) converges uniformly wherever $\|K(\cdot,\lambda)\|$ is bounded.

See [8, 11, 23-25]. A natural question which is motivated by Kramer's theorem is about the prototype situation for which we can find both the kernel $K(\cdot,\lambda)$ and the (sampling) points $\{\lambda_k\}_{k=-\infty}^{\infty}$. As we have seen in Weiss' note, this situation could be the Sturm-Liouville systems. Also Kramer [33] gave an illustration indicating that Weiss' note may be generalized to $n^{th}-$

order self adjoint BVPs. In other words, consider a self adjoint BVP of the form

$$\begin{aligned} \ell(y)(x) &:= \sum_{k=0}^{n} p_k(x) y^{(n-k)}(x) \\ &= \lambda y(x), \quad -\infty < a \le x \le b < \infty,\ \lambda \in \mathbb{C}, \end{aligned} \tag{1.9}$$

$$U_\mu(y) := \sum_{j=1}^{n} \alpha_{j\mu} y^{(j-1)}(a) + \beta_{j\mu} y^{(j-1)}(b) = 0, \quad \mu = 1, 2, \dots, n, \tag{1.10}$$

where $p_k(\cdot)$ are sufficiently smooth functions [41, p. 6] on $[a, b]$; $p_0(x) \neq 0$ for all $x \in [a, b]$, and $\{U_\mu\}_{\mu=1}^n$ are n linearly independent forms of $y^{(j-1)}(a)$, $y^{(j-1)}(b)$ with the complex numbers $\alpha_{j\nu}$, $\beta_{j\nu}$, $1 \le j, \nu \le n$. Kramer showed that his theorem holds if the kernel $K(\cdot, \lambda)$ is taken to be a solution of (1.9) for which the zeros of the entire functions $U_\nu\,(K(\cdot, \lambda))$ are the same, $\nu = 1, 2, \dots, n$. In fact one can prove that his illustration is not true when the eigenvalues are not simple, cf. [2].

As an example of the previous discussion, consider the BVP

$$-iy'(x) = \lambda y(x), \quad -\pi \le x \le \pi, \tag{1.11}$$

$$y(-\pi) = y(\pi). \tag{1.12}$$

This problem is self adjoint with simple eigenvalues $\{\lambda_k = k\}_{k=-\infty}^{\infty}$. The corresponding sequence of eigenfunctions is $\left\{e^{ikx}\right\}_{k=-\infty}^{\infty}$. Hence, applying Kramer's theorem, the integral transform

$$f(\lambda) = \int_{-\pi}^{\pi} \overline{g}(x) e^{i\lambda x}\, dx, \quad g \in L^2(-\pi, \pi) \tag{1.13}$$

has the sampling expansion

$$f(\lambda) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(\lambda - k)}{\pi(\lambda - k)}. \tag{1.14}$$

Series (1.14) converges absolutely on $\mathbb{C}$ and uniformly on both $\mathbb{R}$ and compact subsets of $\mathbb{C}$, [35, p. 261].

The last result is known as Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem [9, 47, 56] and has many applications in signal processing. One may notice from (1.13) that the sampled function $f(\lambda)$ has a compact spectrum, i.e., its Fourier transform $g(\cdot)$ has a compact support. This space is the Paley-Wiener space $B_\pi^2(\mathbb{R})$, which may be defined, in an equivalent way, to be the space of all entire functions of exponential type at most π which belong to $L^2(\mathbb{R})$ when restricted to $\mathbb{R}$, see [7, 42]. Also $B^2_\pi(\mathbb{R})$ is called, in the language of signal processing, the space of bandlimited functions. The WSK sampling theorem, which is a special case of Kramer's one, is also extended in another direction, where the sampling points are not necessarily equidistant. More precisely:

1.2. Theorem. *Let* $\{\lambda_k\}_{k=-\infty}^{\infty}$ *be a sequence of real numbers such that*

$$N := \sup_{k\in\mathbb{Z}} |\lambda_k - k| < \frac{1}{4}. \tag{1.15}$$

Define the entire function

$$G(\lambda) = (\lambda - \lambda_0) \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right)\left(1 - \frac{\lambda}{\lambda_{-k}}\right). \tag{1.16}$$

Let $f(\lambda) \in B^2_\pi(\mathbb{R})$*. Then*

$$f(\lambda) = \sum_{k=-\infty}^{\infty} f(\lambda_k) \frac{G(\lambda)}{(\lambda - \lambda_k)G'(\lambda_k)}, \quad \lambda \in \mathbb{R}. \tag{1.17}$$

The series converges uniformly on compact subsets of $\mathbb{R}$.

If we let, in Theorem 1.2, $\lambda_k = k$, we obtain the WSK sampling theorem. Expansions of the form (1.17) will be called Lagrange-type interpolation expansions. Theorem 1.2 was obtained first by Paley and Wiener [42] who proved the theorem with $N < \frac{1}{\pi^2}$. Then this constant was improved by Levinson [37] and Kadec [26] who gave the exact constant, $1/4$, which cannot be improved. See also [9, 23, 58]. The proof of this theorem was based on the fact that under condition (1.15) the set of the corresponding exponentials, i.e., $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$, is a Riesz basis in $L^2(-\pi, \pi)$. In [4] another approach is derived using second order operator pencils.

The question now is about the relationship between the two extensions of WSK's theorem, i.e., Kramer's theorem and Theorem 1.2. In much recent literature, see for example [1-4, 10, 19-20, 23, 58-60], Kramer's expansions were written in Lagrange-type interpolation forms when the kernel arose from differential/integral operators. This paper is devoted to answering the following questions. Is it possible to define a set of functions (transforms) associated with the BVP

$$N(y) = \lambda P(y), \tag{1.18}$$

$$U_j(y) = 0, \tag{1.19}$$

for which sampling theorems hold? Here both N and P are differential operators such that order N is greater than order P and U_j are two-point boundary conditions, see Sect. 2 below for definitions. When the answer of the above question is yes, what is the type of the obtained sampling expansions? We use the results of [48-49] to define a set of (integrodifferential) transforms associated with problem (1.18)–(1.19) which can be sampled via Lagrange interpolation expansions; we restrict here to that class of BVP (1.18), (1.19) for which the adjoint problem has again boundary conditions which do not depend on λ.

In the rest of this section we give some historical notes on BVPs of the type (1.18)–(1.19). These problems, which are sometimes called Kamke problems, have been studied by many authors. A first systematic treatment for a special self adjoint class has been performed in the 40^{ies} by Kamke [27–31]. This work has been continued and generalized in the 60^{ies} by Schäfke and Schneider [43-46], who introduced the class of S-hermitian problems, and later on by Dijksma, Langer and de Snoo [13-14]. A detailed investigation of non self adjoint BVPs of the form (1.18)–(1.19) was initiated by Eberhard and Freiling in the 70^{ies}, [15-18], [21], who proved asymptotic estimates for fundamental systems of solutions of (1.18) when $|\lambda| \longrightarrow \infty$. These estimates are obtained using the contour integration (Cauchy) method, which generalizes the methods used by Birkhoff [5-6], Stone [50], Tamarkin [51] and many others in the case $P(y) = y$. This work was continued and extended in the 80^{ies} by Minkler [40], Heisecke [22], Mennicken-Möller [38] and Kaufmann [32] who all obtained results on the expandability of given functions in series of eigen and associated functions of (1.18)–(1.19). The main contribution which was obtained recently is due to Shkalikov and Tretter [48-49]; Tretter [54] who constructed subspaces $W^m_{\mathcal{U}}$ of $L^2(0,1)$ in which the system of eigen and associated functions of problem (1.18)–(1.19) is complete, minimal and/or a Riesz basis (in parentheses).

2. Preliminary results and notations

In this section we introduce the notations and summarize the results which we need for the formulation of the sampling theorems. For convenience of the reader we closely follow the notations of [48-49].

We assume that the differential operators N and P of (1.18) have the forms

$$N(y) = y^{(n)} + \sum_{\mu=1}^{n} f_\mu y^{(n-\mu)}, \tag{2.1}$$

$$(2.2) \qquad P(y) = y^{(p)} + \sum_{\mu=1}^{p} g_\mu y^{(p-\mu)},$$

where $n > p \geq 0$ and $f_\mu,\ g_\mu \in W^{n-\mu+\max\{n-1,2(p-1)\}}$. Here W^k denotes the Sobolev space $W^k = W_2^k[0,1] = \{f \in L_2[0,1]\ |f^{(j)} \in L_2[0,1],\ 0 \leq j \leq k\}$. The linear forms U_j in (1.19) are assumed to be given in a normalized form

$$(2.3) \qquad U_j(y) = \sum_{\mu=0}^{l_j} \alpha_{j\mu}^0 y^{(\mu)}(0) + \alpha_{j\mu}^1 y^{(\mu)}(1), \quad 1 \leq j \leq n,$$

where $0 \leq l_1 \leq l_2 \leq \ldots \leq l_n \leq n-1$ and $\alpha_{j\mu}^0,\ \alpha_{j\mu}^1$ are complex numbers for which

$$(2.4) \qquad \left|\alpha_{j\mu}^0\right| + \left|\alpha_{j\mu}^1\right| > 0.$$

The number l_j is called the order of the boundary condition $U_j(y) = 0$, or simply of U_j and the normalization means that $\widetilde{l}_j \geq l_j,\ 1 \leq j \leq n$, for any equivalent system $\left\{\widetilde{U}_j\right\}_{j=1}^{n}$ of linear forms of orders $\widetilde{l}_1 \leq \widetilde{l}_2 \leq \ldots \leq \widetilde{l}_n$.

Set

$$(2.5) \qquad W_U^k = \left\{y \in W^k :\ U_j(y) = 0 \text{ for all } j \text{ with } l_j \leq k-1\right\},$$

$0 \leq k \leq n$. We define the differential operators N and P in $L^2(0,1)$ by the differential expressions (2.1), (2.2) on the domain $D(N) = D(P) = W_U^n$; for convenience we assume that N is invertible, i.e., zero is not an eigenvalue of the pencil $N - \lambda P$. It is proved in [48, Lemma 4.1] that P is closeable and its closure $\overline{P}$ has the domain W_U^p. Moreover, if r is the number of the boundary conditions of order $\leq p-1$, then there exists a system of linear forms $\left\{U_j^*\right\}_{j=1}^{2p-r}$ adjoint to $\{U_j\}_{j=1}^{r}$ with respect to the Lagrange identity for $P(y)$ such that the adjoint operator P^* is defined by the formally adjoint differential expression $P^*(y)$ on the domain

$$(2.6) \qquad D(P^*) = \left\{y \in W^p :\ U_j^*(y) = 0,\ 1 \leq j \leq 2p-r\right\}.$$

Let $\{\phi_j\}_{j=1}^{l}$ be a basis of $\ker\ P^*$ and define the boundary conditions* $\widetilde{U}_j(y)$, $1 \leq j \leq l$, by

$$(2.7) \quad \widetilde{U}_j(y) := \langle N(y), \phi_j\rangle = \widehat{U}_j(y) + \int_0^1 y(x)\overline{(N^*(\phi_j))}(x)\,dx = 0,$$

* The importance of $\widetilde{U}_j(y)$, which were discovered by Shkalikov and Tretter [48-49], is that the eigenfunctions of the problem in question **must** satisfy these new conditions in addition to the boundary conditions. Thus, these new conditions were hidden. This fact indicates how complicated Kamke problems are.

where the linear forms $\widehat{U}_j(y)$ are obtained by integration by parts. Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(0,1)$. The order of $\widetilde{U}_j(y)$ is defined by:

$$\operatorname{ord} \widetilde{U}_j(y) := \begin{cases} \operatorname{ord} \widehat{U}_j(y), & \text{if } \widehat{U}_j(y) \text{ is not identically zero,} \\ -1, & \text{otherwise.} \end{cases}$$

Normalizing the extended system $\{U_j(y)\}_{j=1}^n \cup \left\{\widetilde{U}_j(y)\right\}_{j=1}^l$ we obtain an equivalent system $\{\mathcal{U}_j(y)\}_{j=1}^{\mu}$ of linear forms with $n \le \mu \le n+l$. Now we introduce the subspaces

$$W^k_{\mathcal{U}} := \left\{ y \in W^k : \mathcal{U}_j(y) = 0 \text{ for all } j \text{ with } \operatorname{ord} \mathcal{U}_j(y) \le k-1 \right\}.$$

Substitute $\lambda = -\rho^{n-p}$ and denote by $\omega_1, \ldots, \omega_{n-p}$ to the $(n-p)-$th roots of -1 enumerated such that (for a fixed ν)

$$\Re(\rho\omega_1) \le \Re(\rho\omega_2) \le \ldots \le \Re(\rho\omega_{n-p}), \quad \rho \in S_\nu,$$

where S_ν is the sector in the complex plane determined by

$$S_\nu = \left\{ \rho \in \mathbb{C} : \frac{\nu\pi}{2(n-p)} \le \arg \rho \le \frac{(\nu+1)\pi}{2(n-p)} \right\},$$

if $0 \le \nu \le 4(n-p)-1, \; n-p = 2k-1$ and

$$S_\nu = \left\{ \rho \in \mathbb{C} : \frac{\nu\pi}{(n-p)} \le \arg \rho \le \frac{(\nu+1)\pi}{(n-p)} \right\},$$

when $0 \le \nu \le 2(n-p)-1, \; n-p = 2k$. The following theorem and some of its generalizations have been proved in [18], [32], [38], [53-54].

2.1. Theorem. *For each sector S_ν there exists a fundamental system*

$$\{y_1(\cdot,\lambda), \, y_2(\cdot,\lambda), \ldots, \, y_n(\cdot,\lambda)\}$$

of solutions of the differential equation $N(y) = \lambda P(y)$ with

$$y_\alpha^{(j)}(x, -\rho^{n-p}) = h_\alpha^{(j)}(x) + O\left(\frac{1}{\lambda}\right), \quad 1 \le \alpha \le p, \tag{2.8}$$

$$y_\alpha^{(j)}(x, -\rho^{n-p}) = (\rho\omega_{\alpha-p})^j \, e^{\rho\omega_{\alpha-p}x} \left\{ y_0(x) + O\left(\frac{1}{\rho}\right) \right\}, \tag{2.9}$$
$$p+1 \le \alpha \le n,$$

for $0 \leq j \leq n-1,\ x \in [0,1],\ \rho \in S_\nu$. *Moreover,* $y_0(x) \neq 0$ *for all* $x \in [0,1]$ *and the set* $\{h_1, \ldots, h_p\}$ *is the fundamental system of solutions of* $P(y) = 0$ *defined by the initial conditions*

$$h_\alpha^{(j)}(0) = \delta_{\alpha,j-1}, \quad 1 \leq \alpha, j-1 \leq p.$$

From Theorem 2.1 it follows that, see [18] or [54, Sect. 1.2], for $\rho \in S_\nu$ and $\lambda = -\rho^n$, the characteristic determinant

$$\Delta(\lambda) := \det\left(U_i(y_j)\right)_{1\leq i,\, j\leq n} = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \ldots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \ldots & U_2(y_n) \\ \vdots & \vdots & & \vdots \\ U_n(y_1) & U_n(y_2) & \ldots & U_n(y_n) \end{vmatrix} \tag{2.10}$$

has the asymptotic representation

$$\Delta(-\rho^n) = \rho^{\mathcal{X}_p} e^{\rho\Omega} \begin{cases} [\theta_0] + e^{\rho\omega_k}[\theta_1], & \text{if } n-p = 2k-1, \\ [\theta_0] + e^{\rho\omega_k}[\theta_1] + e^{2\rho\omega_k}[\theta_2], & \text{if } n-p = 2k, \end{cases}$$

where $\mathcal{X}_p := l_{p+1} + \ldots + l_n,\ \Omega := \omega_{k+2} + \ldots + \omega_{n-p}$ and where here and in the sequel $[\theta] := \theta + O\left(\frac{1}{\rho}\right)$.

2.2. Definition. A problem of the type (1.18)–(1.19) is called strongly regular if $2p - n - l_{p+1} \leq 0$ and either

(i) $n - p = 2k - 1$ and $\theta_0\theta_1 \neq 0$, or
(ii) $n - p = 2k,\ \theta_0 \neq 0$ and the quadratic equation $\theta_0 + \theta_1\rho + \theta_2\rho^2 = 0$ has two different roots $\rho_1,\ \rho_2$.

2.3. Theorem. *Each strongly regular problem of the form (1.18)–(1.19) has a sequence* $\{\lambda_\nu\}_{\nu\in\mathbb{N}}$ *of eigenvalues, almost all eigenvalues are simple, moreover, the following asymptotic estimates are fulfilled:*

(a) For $n - p = 2k - 1$ *and* $\nu \in \mathbb{N}$

$$\lambda_{2\nu-1} = (2i\pi\nu)^{n-p}\left\{1 + \frac{\alpha_1}{\nu} + O\left(\frac{1}{\nu^2}\right)\right\} \tag{2.11}$$

and

$$\lambda_{2\nu} = (-2i\pi\nu)^{n-p}\left\{1 + \frac{\alpha_2}{\nu} + O\left(\frac{1}{\nu^2}\right)\right\}, \tag{2.12}$$

with constants $\alpha_1,\ \alpha_2$.

(b) For $n - p = 2k$ *and* $\nu \in \mathbb{N}$

$$\lambda_{2\nu-1} = (2i\pi\nu)^{n-p}\left\{1 + \frac{\alpha_3}{\nu} + O\left(\frac{1}{\nu^2}\right)\right\} \tag{2.13}$$

and

$$\lambda_{2\nu} = (2i\pi\nu)^{n-p}\left\{1 + \frac{\alpha_4}{\nu} + O\left(\frac{1}{\nu^2}\right)\right\}, \tag{2.14}$$

with constants $\alpha_3 \neq \alpha_4$.

See [18, 49]. From now on we assume that problem (1.18)–(1.19) and its adjoint, which will be defined below, are strongly regular. Since under the condition of strong regularity almost all eigenvalues of (1.18)–(1.19) and its adjoint are simple, we assume in the sequel, without any loss of generality, that all eigenvalues are simple. Let ϕ_ν be the normalized eigenfunction corresponding to the eigenvalue λ_ν. Consequently, [49], the adjoint pencil $(N - \overline{\lambda}P)^*$ has also a simple eigenvalue at $\overline{\lambda}_\nu$ with a normalized eigenfunction ψ_ν such that the following biorthogonality relation is fulfilled

$$\langle P\phi_\nu, \psi_j\rangle = -\delta_{\nu j}, \quad \nu, j \in \mathbb{N}. \tag{2.15}$$

See [49, Lemma 5.1 and Remark 5.3]. Thus we have to investigate the adjoint problem. The problem adjoint to (1.18)–(1.19) differs from the problems adjoint to the classical problems discussed in [12, 41], see Sect. 3 below. The adjoint boundary conditions may contain $\lambda-$linear-dependent boundary conditions. More precisely, the problem adjoint to (1.18)–(1.19), [49, p. 290] is given by:

2.4. Definition. The adjoint problem of (1.18)–(1.19) is given by

$$(N - \overline{\lambda}P)^* v = N^*(v) - \lambda P^*(v), \tag{2.16}$$

$$V_j(\lambda)(v) = 0, \quad j = 1, 2, \ldots, n, \tag{2.17}$$

where N^* and P^* are the formal adjoints of N and P respectively; the domain of definition of the linear pencil (2.16) is

$$D((N - \overline{\lambda}P)^*) = \{y \in W^n,\ V_j(\lambda)(v) = 0,\ j = 1, 2, \ldots, n\}.$$

Moreover the forms $V_j(\lambda)$ will be determined as follows. Let r be the number of boundary conditions of (1.19) which have orders less than or equal to $p-1$. Then

$$V_j(\lambda)(v) = \begin{cases} \widehat{V}_j^N(v) - \lambda \widehat{V}_j^P(v), & \text{if } j = 1, 2, \ldots, 2p - r, \\ \widehat{V}_j^N(v), & \text{if } j = 2p - r + 1, \ldots, n. \end{cases} \tag{2.18}$$

The forms $\widehat{V}_j^N(v),\ \widehat{V}_i^P(v),\ 1 \le j \le n,\ 1 \le i \le 2p-r$ are determined by Lagrange's identities

$$(2.19)\quad \langle N(y), v\rangle = \sum_{j=1}^{n} U_j(y)V_j^N(v) + \sum_{j=1}^{n} \widehat{U}_j(y)\widehat{V}_j^N(v) + \langle y, N^*(v)\rangle\,,$$

$$(2.20)\quad \langle P(y), v\rangle = \sum_{j=1}^{r} U_j(y)V_j^P(v) + \sum_{j=1}^{2p-r} \widehat{U}_j(y)\widehat{V}_j^P(v) + \langle y, P^*(v)\rangle\,.$$

Here $\{\widehat{U}_j(y)\}_{j=1}^n$ are n forms such that $\{U_j(y)\}_{j=1}^n \cup \{\widehat{U}_j(y)\}_{j=1}^n$ are linearly independent.

As stated above, the main progress achieved in the last decade for Kamke problems was the description of subspaces of $W_2^k(0,1)$ where the system $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ is a Riesz basis, see [48-49]. For definitions and properties of Riesz bases see e.g. [57].

According to [49, Sect. 6] there is an integer $\kappa \in [-1, p-1]$, depending on (1.18)–(1.19) such that the following theorem holds.

2.5. Theorem. $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ *is a Riesz basis in* $W_{\mathcal{U}}^m$ *if either*

(a) $\max\{l_p, \kappa\} \le m \le n$, *or*
(b) $p \le m \le n,\ m-p-n-l_{p+1} \le 0$ *and* $P : W_{\mathcal{U}}^m \longrightarrow W^{m-p}$ *has a non trivial kernel.*

For each fixed sector S_ν we denote by $W(\cdot,\lambda)$ the Wronskian of the fundamental system $\{y_1(\cdot,\lambda), y_2(\cdot,\lambda), \ldots, y_n(\cdot,\lambda)\}$ considered in Theorem 2.2 above and by $W_j(\cdot,\lambda)$ the algebraic complement of $y_j^{(n-1)}(\cdot,\lambda)$ in $W(\cdot,\lambda)$. Let $g(\cdot,\xi,\lambda)$ be the function

$$(2.21)\quad g(x,\xi,\lambda) := \frac{1}{2}\,\text{sign}\,(x-\xi)\sum_{j=1}^{n} y_j(x,\lambda)\frac{W_j(\xi,\lambda)}{W(\xi,\lambda)}, \quad x,\xi \in [0,1].$$

Green's function of problem (1.18)–(1.19) can be represented in the form

$$(2.22)\quad G(x,\xi,\lambda) := \frac{(-1)^n}{\Delta(\lambda)}\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & \ldots & y_n(x,\lambda) & g(x,\xi,\lambda) \\ U_1(y_1) & U_1(y_2) & \ldots & U_1(y_n) & U_1(g) \\ \vdots & \vdots & & \vdots & \vdots \\ U_n(y_1) & U_n(y_2) & \ldots & U_n(y_n) & U_n(g) \end{vmatrix},$$

where $x,\ \xi \in [0,1]$ and $\lambda \in S_\nu$. From the asymptotic estimates derived in [18], [21] and [48-49], it follows under the above assumptions that for

$\lambda \neq \lambda_\nu$

$$G(x,\xi,\lambda) = \sum_{\nu=1}^{\infty} \frac{\phi_\nu(x)\overline{\psi_\nu}(\xi)}{\lambda_\nu - \lambda}, \quad x,\, \xi \in [0,1]. \tag{2.23}$$

This series (combining [21, Satz 3] and [49, Lemma 5.1]) converges uniformly for each fixed $\lambda \neq \lambda_\nu$.

3. A sampling theorem

In this section we derive interpolation (sampling) expansions associated with problem (1.18)–(1.19) which is not necessarily self adjoint but it is assumed to be strongly regular and all eigenvalues are simple. Here the sampled functions will be integrodifferential transforms whose kernels are solutions of problem (1.18)–(1.19) while in the following section the kernels will be expressed in terms of Green's functions.

The first problem is how to define the set of transforms which we want to sample. To achieve this aim we need to find two functions (kernels) which generate the complete sets of eigenfunctions of problem (1.18)–(1.19) and its adjoint. We discuss the case of problem (1.18)–(1.19) and the case of its adjoint is similar. Let λ be an eigenvalue. Since it is simple, then there is an $(n-1)^{th}-$order minor of $\Delta(\lambda)$ which does not vanish. We assume, without any loss of generality, that for any eigenvalue λ_ν the same $(n-1)^{th}-$order minor, say the first, does not vanish, cf. Remark 3.5 below.

3.1. Lemma. *There are two functions $\phi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$, unique up to a multiplicative constant (entire functions of λ), such that:*

(i) $\phi(\cdot,\lambda)$ *is a solution of (1.18) and* $\psi(\cdot,\lambda)$ *is a solution of (2.16).*
(ii) $U_j(\phi(\cdot,\lambda)) = 0,\ j = 1,\dots,n-1,\ V_j(\psi(\cdot,\lambda)) = 0,\ j = 1,\dots,n-1,\ \forall\lambda \in \mathbb{C}.$
(iii) $U_n(\phi(\cdot,\lambda)) = 0$ *only when* $\lambda = \lambda_\nu$, $V_n(\psi(\cdot,\lambda)) = 0$ *only when* $\lambda = \overline{\lambda}_\nu$.
(iv) $\{\phi(\cdot,\lambda_\nu)\}_{\nu=1}^{\infty}$ *is a set of eigenfunctions of (1.18)–(1.19) and* $\{\psi(\cdot,\overline{\lambda}_\nu)\}_{\nu=1}^{\infty}$ *is a set of eigenfunctions of (2.16)–(2.17).*

Proof. We can see that the functions

$$\phi(x,\lambda) := (-1)^{n-1} \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ U_1(y_1) & U_1(y_2) & \dots & U_1(y_n) \\ \vdots & \vdots & & \vdots \\ U_{n-1}(y_1) & U_{n-1}(y_2) & \dots & U_{n-1}(y_n) \end{vmatrix} \tag{3.1}$$

and

$$\psi(x,\lambda) := (-1)^{n-1}\begin{vmatrix} z_1(x) & z_2(x) & \dots & z_n(x) \\ V_1(z_1) & V_1(z_2) & \dots & V_1(z_n) \\ \vdots & \vdots & & \vdots \\ V_{n-1}(z_1) & V_{n-1}(z_2) & \dots & V_{n-1}(z_n) \end{vmatrix} \tag{3.2}$$

satisfy the lemma, where $y_1,\dots,y_n$ and $z_1,\dots,z_n$ are fundamental solutions of (1.18) and (2.16) respectively. $\square$

From now on we assume that

$$2p - r \leq 0.$$

From Sect. 2 above, $\{\phi(\cdot,\lambda_\nu)\}_{\nu=1}^{\infty},\ \{\psi(\cdot,\overline{\lambda}_\nu)\}_{\nu=1}^{\infty}$ are Riesz bases in $W_{\mathcal{U}}^m$, for some m. Thus, using the biorthogonality relation (2.15), for any $g \in W_{\mathcal{U}}^m$, we have

$$g(x) = \sum_{\nu=1}^{\infty} \frac{\left\langle g(\cdot), P^*(\psi(\cdot,\overline{\lambda}_\nu))\right\rangle}{\left\langle \phi(\cdot,\lambda_\nu), P^*(\psi(\cdot,\overline{\lambda}_\nu))\right\rangle}\phi(x,\lambda_\nu), \tag{3.3}$$

or

$$g(x) = \sum_{\nu=1}^{\infty} \frac{\langle P(\phi(\cdot,\lambda_\nu)), g(\cdot)\rangle}{\left\langle P(\phi(\cdot,\lambda_\nu)), \psi(\cdot,\overline{\lambda}_\nu)\right\rangle}\psi(x,\overline{\lambda}_\nu), \tag{3.4}$$

where the convergence is in the norm of $W_{\mathcal{U}}^m$. This result, the expansion theorem, as well as the corresponding Bessel's inequality and Parseval's equality were obtained also by Kamke [29, p. 269] for a more restrictive class of functions when problem (1.18)–(1.19) is self adjoint. In fact he assumed that the function $g(\cdot)$ which may be expanded is a $C^{(n)}(0,1)-$function which satisfies the boundary conditions (1.19), provided that the problem is self adjoint. Now we state and prove the first sampling result of this paper. We recall that n, p and r are, respectively, the order of N, the order of P and the number of boundary conditions of order less than or equal to $p-1$.

3.2. Theorem. *Let $g(\cdot) \in W_{\mathcal{U}}^m$. Assume that the function $\phi(\cdot,\lambda)$ defined above belongs to $W_{\mathcal{U}}^m$, for all $\lambda \in \mathbb{C}$. Let $f(\lambda)$ be the integrodifferential transform*

$$f(\lambda) = \int_0^1 g(x)\overline{P(\phi(x,\lambda))}\,dx. \tag{3.5}$$

Then, we have the following:

(i) $f(\lambda)$ *admits the pointwise convergent sampling expansions*

$$f(\lambda)=\begin{cases}\sum_{\nu=0}^{\infty} f(\lambda_\nu)\frac{m_\nu(\lambda)}{m_\nu(\lambda_\nu)}\frac{\Delta(\lambda)}{(\lambda-\lambda_\nu)\Delta'(\lambda_\nu)}, & \text{if } r=n,\\ \sum_{\nu=0}^{\infty} f(\lambda_\nu)\frac{\Delta(\lambda)}{(\lambda-\lambda_\nu)\Delta'(\lambda_\nu)}, & \text{if } r<n,\end{cases} \tag{3.6}$$

where $m_\nu(\lambda)=V_n^N(\psi_\nu)-\lambda V_n^P(\psi_\nu)$.

(ii) The series (3.6) converges absolutely on $\mathbb{C}$ *and uniformly on compact subsets of* $\mathbb{C}$.

(iii) $f(\lambda)$ *is an entire function of order not exceeding* $\frac{1}{n-p}$ *and type not exceeding one.*

Proof.
(i) Since $2p-r\le 0$, we have no $\lambda-$dependent boundary conditions. Let $\lambda,\ \lambda'\in\mathbb{C}$. For simplicity when we write $\phi,\ \psi,\ \phi_\nu$ and ψ_ν we mean

$$\phi(\cdot,\lambda),\ \psi(\cdot,\overline{\lambda'}),\ \phi(\cdot,\lambda_\nu),\ \text{and}\ \psi(\cdot,\overline{\lambda}_\nu),\quad \lambda,\ \lambda'\in\mathbb{C},$$

respectively. Since $g(\cdot),\ \phi\in W_{\mathcal{U}}^m$, then they have expansions of the forms (3.3) or (3.4). We prove that

$$\langle\phi_\nu,P^*(\psi_\nu)\rangle=\langle P(\phi_\nu),\psi_\nu\rangle\qquad\text{for } \nu=1,2,\ldots \tag{3.7}$$

Indeed, we have from (2.20)

$$\langle P(\phi),\psi\rangle=\sum_{j=1}^{r}U_j(\phi)V_j^P(\psi)+\langle\phi,P^*(\psi)\rangle\,. \tag{3.8}$$

Since $r\le n$, then, by the properties of $\phi,\ \psi$, cf. Lemma 3.1 above, we have

$$\langle P(\phi),\psi\rangle=\begin{cases}U_n(\phi)V_n^P(\psi)+\langle\phi,P^*(\psi)\rangle\,, & \text{if } r=n,\\ \langle\phi,P^*(\psi)\rangle\,, & \text{if } r<n.\end{cases} \tag{3.9}$$

Thus $\langle P(\phi),\psi\rangle=\langle\phi,P^*(\psi)\rangle$ for all $\lambda,\ \lambda'\in\mathbb{C}$ when $r<n$. When $r=n$, let $\lambda=\lambda_\nu$ in (3.9), we get $\langle P(\phi_\nu),\psi\rangle=\langle\phi_\nu,P^*(\psi)\rangle$ for all $\lambda'\in\mathbb{C}$. Hence (3.7) is proved. Using (3.7), the (Fourier) expansions of $g,\ \phi$ take the forms

$$g(x)=\sum_{\nu=0}^{\infty}\widehat{g}(\nu)\psi_\nu(x) \tag{3.10}$$

and

$$\phi(x,\lambda)=\sum_{\nu=0}^{\infty}\widehat{\phi}(\nu)\phi_\nu(x), \tag{3.11}$$

where

$$\widehat{g}(\nu) = \frac{\langle P(\phi_\nu), g\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle} \quad \text{and} \quad \widehat{\phi}(\nu) = \frac{\langle \phi, P^*(\psi_\nu)\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle}. \tag{3.12}$$

Hence Parseval's identity implies

$$f(\lambda) = \sum_{\nu=0}^{\infty} \frac{\overline{\langle P(\phi_\nu), g\rangle}\, \langle \phi, P^*(\psi_\nu)\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle}. \tag{3.13}$$

By the definition of $f(\lambda)$ we obtain

$$f(\lambda) = \sum_{\nu=0}^{\infty} f(\lambda_\nu) \frac{\langle \phi, P^*(\psi_\nu)\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle}. \tag{3.14}$$

From Lagrange's identity (2.19) and Lemma 3.1, we have

$$\lambda \langle P(\phi), \psi\rangle = \Delta(\lambda) V_n^N(\psi) + \widehat{U}_n(\phi)\widetilde{\Delta}(\overline{\lambda'}) + \lambda' \langle \phi, P^*(\psi)\rangle , \tag{3.15}$$

where $\widetilde{\Delta}(\lambda)$ is the characteristic determinant of problem (2.16)–(2.17) calculated with respect to the system $z_1, \ldots, z_n$. Combining (3.9) and (3.15) we obtain

$$\begin{aligned} &\Delta(\lambda) V_n^N(\psi) + \widehat{U}_n(\phi)\widetilde{\Delta}(\overline{\lambda'}) + \lambda' \langle \phi, P^*(\psi)\rangle \\ &\qquad = \lambda \begin{cases} U_n(\phi) V_n^P(\psi) + \langle \phi, P^*(\psi)\rangle , & \text{if } r = n, \\ \langle \phi, P^*(\psi)\rangle , & \text{if } r < n. \end{cases} \end{aligned} \tag{3.16}$$

Letting $\lambda' \longrightarrow \lambda_\nu$ in (3.16), and noting that $\overline{\lambda}_\nu$ is an eigenvalue of the adjoint problem with the eigenfunction $\psi(x, \overline{\lambda}_\nu) = \psi_\nu$,

$$(\lambda - \lambda_\nu) \langle \phi, P^*(\psi_\nu)\rangle = \begin{cases} \Delta(\lambda) m_\nu(\lambda), & \text{if } r = n, \\ \Delta(\lambda) V_n^N(\psi_\nu), & \text{if } r < n, \end{cases} \tag{3.17}$$

where

$$m_\nu(\lambda) = V_n^N(\psi_\nu) - \lambda V_n^P(\psi_\nu). \tag{3.18}$$

Thus, the relation

$$\langle \phi_\nu, P^*(\psi_\nu)\rangle = \begin{cases} \Delta'(\lambda) m_\nu(\lambda_\nu), & \text{if } r = n, \\ \Delta'(\lambda_\nu) V_n^N(\psi_\nu), & \text{if } r < n \end{cases} \tag{3.19}$$

holds by taking the limit in (3.17) when $\lambda \longrightarrow \lambda_\nu$. Substituting (3.17) and (3.19) in (3.14) we obtain (3.6) where the convergence is pointwise on $\mathbb{C}$.

(ii) We prove the absolute and uniform convergence. First we prove the absolute convergence on $\mathbb{C}$. Let $\lambda \in \mathbb{C}$. Since (3.6) is nothing but (3.13), then using the Cauchy-Schwarz inequality

$$|f(\lambda)| \leq \left(\sum_{\nu=0}^{\infty} \frac{|\langle P(\phi_\nu), g\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}\right)^{1/2} \left(\sum_{\nu=0}^{\infty} \frac{|\langle \phi, P^*(\psi_\nu)\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}\right)^{1/2} < \infty, \tag{3.20}$$

since $\frac{\langle P(\phi_\nu),g\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle}$ and $\frac{\langle \phi, P^*(\psi_\nu)\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle}$ are the Fourier coefficients of g and ϕ respectively. Hence (3.6) converges absolutely on $\mathbb{C}$. To prove the uniform convergence it is sufficient to show that for every compact subset $M \subset \mathbb{C}$ and any natural number $N > 0$, $\lim_{N\to\infty} S_N(\lambda) \longrightarrow 0$ as $N \longrightarrow \infty$ not depending on $\lambda \in M$, where

$$S_N(\lambda) := \left| f(\lambda) - \sum_{\nu=0}^{N-1} \frac{\overline{\langle P(\phi_\nu), g\rangle}\, \langle \phi, P^*(\psi_\nu)\rangle}{\langle \phi_\nu, P^*(\psi_\nu)\rangle} \right|. \tag{3.21}$$

Indeed, for $\lambda \in M$, using the Cauchy-Schwarz inequality

$$S_N(\lambda) \leq \left(\sum_{\nu=N}^{\infty} \frac{|\langle P(\phi_\nu), g\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}\right)^{1/2} \left(\sum_{\nu=N}^{\infty} \frac{|\langle \phi, P^*(\psi_\nu)\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}\right)^{1/2}. \tag{3.22}$$

But the first series on the right hand side of (3.22) approaches zero when $N \longrightarrow \infty$ not depending on λ. Hence to achieve our aim we need to prove that the second series in (3.22) is (uniformly) bounded on M. Again we use Parseval's identity

$$\int_0^1 \phi(x,\lambda)\overline{P(\phi(x,\lambda))}\, dx = \sum_{\nu=0}^{\infty} \frac{|\langle \phi, P^*(\psi_\nu)\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}. \tag{3.23}$$

Since $\phi(\cdot,\lambda),\ P(\phi(\cdot,\lambda))$ are entire functions of λ for each x and continuous in x for each λ, the integral of (3.23) is bounded by

$$C_M = \max_{0\leq x\leq 1,\, \lambda\in M} |\phi(x,\lambda)P(\phi(x,\lambda))|\,. \tag{3.24}$$

Hence

$$S_N(\lambda) \leq C_M \left(\sum_{\nu=N}^{\infty} \frac{|\langle P(\phi_\nu), g\rangle|^2}{|\langle \phi_\nu, P^*(\psi_\nu)\rangle|}\right), \quad \forall \lambda \in M,$$

which proves the uniform convergence on M.

(iii) Since (3.6) converges uniformly on compact subsets of $\mathbb{C}$, then $f(\lambda)$ is analytic on every compact subset of $\mathbb{C}$, i.e., f is entire. The fact that $f(\lambda)$ has order less than or equal to $\frac{1}{n-p}$ and type not exceeding one arises from Theorem 2.1 above. $\square$

3.3. Remark. When $2p - r > 0$ a more complicated sampling expansion may be obtained. In this case we have $(2p - r)\,\lambda-$dependent boundary conditions in the adjoint problem which is not the case of the problem in question, since the boundary conditions (1.19) are independent of λ. So we have lost something of the symmetry of the situation. In fact in this case, in general, we do not have the useful relation (3.7) and hence Parseval's identity will be more complicated. The reason is that the $\lambda-$dependent boundary conditions are obtained from two different Lagrange's identities. Calculations similar to those of the above theorem lead to

$$(\lambda - \lambda_\nu)\,\langle \phi, P^*(\psi)\rangle = \begin{cases} \Delta(\lambda)m_\nu(\lambda) + \sum_{j=1}^{2p-r} \widehat{U}_j(\phi)\beta_{j,\nu}(\lambda), & \text{if } r < n, \\ \Delta(\lambda)V_n^N(\psi_\nu) + \sum_{j=1}^{2p-r} \widehat{U}_j(\phi)\beta_{j,\nu}(\lambda), & \text{if } r = n, \end{cases} \tag{3.25}$$

where

$$\beta_{j,\nu}(\lambda) = \left[\widehat{V}_j^N(\psi_\nu) - \lambda V_j^P(\psi_\nu)\right]$$

and

$$\langle \phi_\nu, P^*(\psi_\nu)\rangle = \begin{cases} \Delta'(\lambda_\nu)m_\nu(\lambda_\nu) - \sum_{j=1}^{2p-r} \widehat{U}_j(\phi_\nu)V_j^P(\psi_\nu), & \text{if } r < n, \\ \Delta'(\lambda_\nu)V_n^N(\psi_\nu) - \sum_{j=1}^{2p-r} \widehat{U}_j(\phi)V_j^P(\psi_\nu), & \text{if } r = n, \end{cases} \tag{3.26}$$

while

$$(\lambda - \lambda_\nu)\,\langle P(\phi), \psi_\nu\rangle = \begin{cases} \Delta(\lambda)m_\nu(\lambda), & \text{if } r < n, \\ \Delta(\lambda)V_n^N(\psi_\nu), & \text{if } r = n, \end{cases} \tag{3.27}$$

and

$$\langle \phi_\nu, P^*(\psi)\rangle = \begin{cases} \Delta'(\lambda_\nu)m_\nu(\lambda_\nu), & \text{if } r < n, \\ \Delta'(\lambda_\nu)V_n^N(\psi_\nu), & \text{if } r = n. \end{cases} \tag{3.28}$$

3.4. Remark. If the $(n-1)^{th}-$ minors which do not vanish in $\Delta(\lambda)$ when $\lambda = \lambda_\nu$ depend on λ_ν, i.e., they are not the same for all ν, we redefine $\phi(\cdot, \lambda)$ to be $\widetilde{\phi}(\cdot, \lambda)$, where

$$\widetilde{\phi}(x, \lambda) := \sum_{k=1}^{n} (-1)^{k-1} \alpha_k \Delta_k(x, \lambda), \tag{3.29}$$

where $\Delta_k(\cdot, \lambda)$ is the algebraic complement of $y_1, \ldots, y_n$ in $\Delta(\lambda)$ and α_k are constants chosen such that $\widetilde{\phi}(\cdot, \lambda_\nu)$ is not identically zero for all ν. Obviously $\widetilde{\phi}(\cdot, \lambda)$ has the following properties:

(i) $\widetilde{\phi}(\cdot, \lambda)$ is a solution of (1.19).
(ii) For all λ we have

$$U_j(\widetilde{\phi}) = \sum_{k=1}^{n} (-1)^k \alpha_k U_j(\Delta_k(x, \lambda)) = \alpha_j \Delta(\lambda).$$

Hence $U_j(\widetilde{\phi}) = 0$ if and only if $\lambda = \lambda_\nu$.
(iii) From (i), (ii), $\widetilde{\phi}(\cdot, \lambda_\nu),\ \nu = 0, 1, \ldots$ are eigenfunctions of problem (1.18)–(1.19).

Similarly we define $\widetilde{\psi}(\cdot, \lambda)$ and derive a theorem similar to Theorem 3.2 above.

In the following we give an example illustrating the above theorem. Since the situation here is more complicated than that when P of (1.18) is the identity operator, examples cannot be easily derived. For instance we can see that, [12, 41, 52], the problem

$$-y'' = \lambda y, \quad 0 \le x \le 1, \tag{3.30}$$

$$U_1(y) = y'(0) = 0, \quad U_2(y) = y'(1) = 0 \tag{3.31}$$

is self adjoint and the boundary conditions (3.31) are strongly regular. If we change equation (3.30) into

$$-y'' = -i\lambda y', \quad 0 \le x \le 1, \tag{3.32}$$

everything becomes worse from many aspects. First the boundary conditions, with respect to (3.32), will not be even regular since the solution $y(x, \lambda) = 1,\ \lambda \in \mathbb{C}$ satisfies (3.32) and (3.31). After some calculations with the aid of Lagrange's identities (2.19)–(2.20), the adjoint problem to (3.32)–(3.31) is

$$-y'' = -i\lambda y', \quad 0 \le x \le 1, \tag{3.33}$$

$$V_1(y) = y'(0) - \lambda y(0) = 0, \tag{3.34}$$

$$V_2(y) = y'(1) - \lambda y(1) = 0. \tag{3.35}$$

Thus the problem is not self adjoint. Moreover $\lambda_0 = 0$ is the only eigenvalue with the eigenfunction $\phi_0(x) = \text{const} \neq 0$.

3.5. Example. Let $D := d/dx$. Consider the boundary value problem, [49, pp. 279–281]

$$(D^2 - \alpha^2)^2 y - i\alpha R(u(D^2 - \alpha^2)y - D^2 uy) = -\lambda(D^2 - \alpha^2)y, \tag{3.36}$$

$$y(0) = 0,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = 0, \tag{3.37}$$

where $\alpha,\ R$ are constants and $u \in C^{(2)}[0,1]$. The above problem arises from hydrodynamics and equation (3.36) is known as Orr-Sommerfeld equation, cf. [48-49]. Problem (3.36)–(3.37) is not self adjoint and its adjoint, which is given precisely in [49, p. 280], takes the form

$$(D^2 - \alpha^2)^2 y + i\alpha R((D^2 - \alpha^2)(uy) - D^2 uy) = -\lambda(D^2 - \alpha^2)y, \tag{3.38}$$

$$y(0) = 0,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = 0. \tag{3.39}$$

Problem (3.36)–(3.37) and its adjoint are strongly regular [49, p. 281]. In the notations of the previous section, $n = 4,\ p = 2,\ r = 4,\ 2p - r = 0 \leq 0$. The extended boundary conditions will include, in addition to the (3.37), the following two integral-boundary conditions

$$\begin{aligned} \mathcal{U}_1(y) &= y'''(0) + \alpha \coth \alpha y''(0) - \frac{\alpha}{\sinh \alpha} y''(1) \\ &\quad - \frac{2i\alpha^2 R}{\sinh \alpha} \int_0^1 y(x) u'(x) \cosh(\alpha(x-1))\, dx, \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \mathcal{U}_2(y) &= y'''(1) + \alpha \coth \alpha y''(1) - \frac{\alpha}{\sinh \alpha} y''(0) \\ &\quad - \frac{2i\alpha^2 R}{\sinh \alpha} \int_0^1 y(x) u'(x) \cosh(\alpha x)\, dx. \end{aligned} \tag{3.41}$$

Now we have the following theorem taken from [49].

3.6. Theorem. *Let $W_{\mathcal{U}}^k$ denote the subspaces mentioned above. Let the eigenvalues of problem (3.36)–(3.37) be simple with the system of eigenfunctions $\{\phi_\nu\}_{\nu=1}^\infty$. Then:*

(i) $\{\phi_\nu\}_{\nu=1}^\infty$ *is complete and minimal in* $W_{\mathcal{U}}^k$ *when* $k = 0,\ 1,\ 2,\ 3,\ 4,\ W_{\mathcal{U}}^0 = L^2(0,1)$.

(ii) $\{\phi_\nu\}_{\nu=1}^{\infty}$ *is a Riesz basis in* $W_{\mathcal{U}}^m$ *when* $m = 1, 2, 3, 4$, *but not in* $W_{\mathcal{U}}^0 = L^2(0,1)$.

(If finitely many eigenvalues are not simple, then the system of eigen- and associated functions *of problem (3.36)–(3.37) has the same basis properties).*

Let $y_1(\cdot,\lambda), \dots, y_4(\cdot,\lambda)$ and $z_1(\cdot,\lambda), \dots, z_4(\cdot,\lambda)$ be fundamental sets of solutions of (3.36) and (3.38) respectively. Define the functions $\phi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$ as in Lemma 3.1 above. Then, $\phi(\cdot,\lambda) \in W_{\mathcal{U}}^1$ for all $\lambda \in \mathbb{C}$ and the transform

$$f(\lambda) = \int_0^1 g(x)\overline{(\phi''(x,\lambda) - \alpha^2\phi(x,\lambda))}\, dx, \quad g \in W_{\mathcal{U}}^1 \tag{3.42}$$

has the expansion

$$f(\lambda) = \sum_{\nu=1}^{\infty} f(\lambda_\nu) \frac{m_\nu(\lambda)\Delta(\lambda)}{m_\nu(\lambda_\nu)(\lambda-\lambda_\nu)\Delta'(\lambda_\nu)}, \tag{3.43}$$

where $\{\lambda_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues of (3.36)–(3.37), $\Delta(\lambda)$ is the characteristic determinant calculated with respect to $y_1(\cdot,\lambda), \dots, y_4(\cdot,\lambda)$ and $m_\nu(\lambda)$ can be derived as in Theorem 3.2.

A concrete case. The sampling expansion of the above example is not given explicitly since the fundamental system $y_1(\cdot,\lambda), \dots, y_4(\cdot,\lambda)$ cannot be derived in the general case. So it may be useful to calculate a concrete case of the above example. Indeed, let $\alpha = 0$. Then we have the self adjoint problem

$$y'''' = -\lambda y'', \tag{3.44}$$

$$y(0) = 0,\ y(1) = 0,\ \ y'(0) = 0,\ y'(1) = 0. \tag{3.45}$$

In this case the fundamental set of solutions is

$$\begin{aligned} &y_1(x,\lambda) = 1,\ \ y_2(x,\lambda) = x,\ \ y_3(x,\lambda) = \cos\sqrt{\lambda}x,\\ &y_4(x,\lambda) = \tfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}. \end{aligned} \tag{3.46}$$

Since the problem is self adjoint, then

$$\phi(x,\lambda) = \psi(x,\lambda) = -\begin{vmatrix} 1 & x & \cos\sqrt{\lambda}x & \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \\ 1 & 0 & 1 & 0 \\ 1 & 1 & \cos\sqrt{\lambda} & \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \\ 0 & 1 & 0 & 1 \end{vmatrix}.$$

Thus

$$\begin{aligned}\phi(x,\lambda) = \psi(x,\lambda) &= \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 1\right)\left(\cos\sqrt{\lambda}x - 1\right) \\ &\quad + \left(\cos\sqrt{\lambda} - 1\right)\left(x - \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right).\end{aligned} \tag{3.47}$$

Hence

$$\begin{aligned}\phi''(x,\lambda) = -\lambda\left(1 - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)\cos\sqrt{\lambda}x \\ -\sqrt{\lambda}\left(\cos\sqrt{\lambda} - 1\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.\end{aligned} \tag{3.48}$$

The characteristic determinant has the form

$$\begin{aligned}\varDelta(\lambda) &= -U_4(\phi(x,\lambda)) = \phi'(1,\lambda) \\ &= -\sqrt{\lambda}\left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 1\right)\sin\sqrt{\lambda} - \left(\cos\sqrt{\lambda} - 1\right)^2 \\ &= \sqrt{\lambda}\sin\sqrt{\lambda} - 2(1 - \cos\sqrt{\lambda}).\end{aligned} \tag{3.49}$$

The eigenvalues, the sampling points, are the zeros of $\varDelta(\lambda)$ and $\lambda = 0$ is not an eigenvalue. Hence the eigenvalues are $\lambda_\nu = t_\nu^2$, where t_ν are the positive solutions of the equation

$$t\sin t = 2(1 - \cos t). \tag{3.50}$$

These zeros form two sequences of points $\lambda_\nu,\ \lambda'_\nu \in \mathbb{R}$, where

$$\lambda_\nu = 2\pi\nu, \qquad \lim_{\nu\to\infty}\left|\lambda_\nu - \lambda'_\nu\right| = 0.$$

Reordering the two sequences $\lambda_\nu,\ \lambda'_\nu \in \mathbb{R}$ in one sequence $\{\lambda_\nu\}_{\nu=1}^{\infty}$ we have the following lemma:

3.7. Lemma. *Let* $g(\cdot) \in W^1_{\mathcal{U}} = \{y \in W^1,\ y(0) = 0,\, y(1) = 0\}$. *Let*

$$f(\lambda) = \int_0^1 g(x)\overline{\phi''(x,\lambda)}\,dx, \tag{3.51}$$

where $\phi''(\cdot,\lambda)$ *is given in (3.48). Then*

$$f(\lambda) = \sum_{\nu=1}^{\infty} f(\lambda_\nu)\frac{\sqrt{\lambda}\sin\sqrt{\lambda} - 2(1 - \cos\sqrt{\lambda})}{(\lambda - \lambda_\nu)(\lambda_\nu\cos\sqrt{\lambda_\nu} - (\lambda_\nu - \frac{1}{2})\frac{\sin\sqrt{\lambda_\nu}}{\sqrt{\lambda_\nu}})}. \tag{3.52}$$

Proof. It is sufficient to apply Theorem 3.1 above and to show that $m_\nu(\lambda) = V_4^N(\phi(\cdot,\lambda_\nu))$, i.e., $V_4^P(\phi(\cdot,\lambda_\nu)) = 0$ for all ν. Indeed, using integration by parts and that $P^* = P$, then for $v,\ w \in C^{(2)}[0,1]$, we have

$$\langle P(v), w\rangle = \left[v'\overline{w} - v\overline{w'}\right]_0^1 + \langle v, P(w)\rangle\,. \tag{3.53}$$

Hence $V_4^P(w) = w(1)$. From the definition of $\phi(\cdot,\lambda)$, we have $\phi(1,\lambda) = 0$ for all λ and the lemma is proved. □

4. Sampling using Green's function

In the following we state and prove another sampling theorem using Green's function of problem (1.18)–(1.19). That is instead of $\phi(\cdot,\lambda)$ in (3.5) we define another kernel for which interpolation expansions can be derived. To indicate the reason of such considerations we start with the following example.

4.1. Example. Consider the problem

$$-y'' = -i\lambda y', \quad 0 \le x \le 1, \tag{4.1}$$

$$U_1(y) = y(0) = 0, \quad U_2(y) = y(1) = 0. \tag{4.2}$$

In this case equation (4.1) has the fundamental system

$$y_1(x,\lambda) = 1, \quad y_2(x,\lambda) = e^{-i\lambda x}. \tag{4.3}$$

Thus $\Delta(\lambda) = e^{-i\lambda} - 1$. Hence $\theta_0 = 1,\ \theta_1 = 1$, i.e., $\theta_0\theta_1 \neq 0$, $n - p = 2k - 1 = 1$. Moreover $2p - n - l_{p+1} = 2 - 2 - 0 \le 0$. Then problem (4.1)–(4.2) is strongly regular. If r is the number of boundary conditions of order less than or equal l_{p-1} then $r = 2$. Hence the boundary conditions determined by $P = i\frac{d}{dx}$ is $2p - r = 2 - 2 = 0$. Hence the adjoint of $P,\ P^*$, is defined by

$$P^* : D(P^*) \longrightarrow L^2(0,1) : \quad P^*(y) = P(y),\ D(P^*) = W^1.$$

Since $\ker P^* = \langle\{\phi_1(x) = 1\}\rangle$, where $\langle\{\cdot\}\rangle$ denotes the linear span of the set $\{\cdot\}$, then the following (hidden) condition must be satisfied by the eigenfunctions

$$\widetilde{U}_1(y) := \langle Ny, \phi_1\rangle = -\int_0^1 y''\,dx = y'(0) - y'(1) = \widehat{U}_1(y). \tag{4.4}$$

Notice that $N^*(\phi_1) = 0$. Thus ord $\widetilde{U}_1 = 1$, cf. (2.7). Hence we have the extended (normalized) system of boundary conditions

$$\mathcal{U}_1(y) = y(0), \quad \mathcal{U}_2(y) = y(1), \tag{4.5}$$

$$\mathcal{U}_3(y) = y'(0) - y'(1). \tag{4.6}$$

According to [49, pp. 291–294] the number κ mentioned above is -1. Hence the eigenfunctions of problem (4.1)–(4.2) form a Riesz basis in $W^m_{\mathcal{U}}$ where $0 < m \leq 2$. That is, the eigenfunctions of the above problem form a Riesz basis in $W^1_{\mathcal{U}}$, $W^2_{\mathcal{U}}$, where

$$W^1_{\mathcal{U}} = \left\{y : y \in W^1_2, \quad \mathcal{U}_i(y) = 0,\ i = 1, 2\right\}$$

and

$$W^2_{\mathcal{U}} = \left\{y : y \in W^2_2, \quad \mathcal{U}_i(y) = 0,\ i = 1, 2, 3\right\}.$$

One can also calculate the adjoint problem of problem (4.1)–(4.2) and find that the problem in question is self adjoint, i.e., its adjoint has the form (4.1)–(4.2). Following Lemma 3.1, we obtain

$$\phi(x, \lambda) = \psi(x, \lambda) = 1 - e^{-i\lambda x}. \tag{4.7}$$

We notice that although problem (4.1)–(4.2) is self adjoint, strongly regular and the condition $2p - r \leq 0$ is satisfied, we cannot apply Theorem 3.2. The reason is that $\phi(\cdot, \lambda)$ does not belong to any of $W^m_{\mathcal{U}}$, $m = 1, 2$, for all λ. So we have to look for another kernel.

As in the previous section, we need to construct a kernel where sampled integrodifferential transforms can be defined. This kernel will be expressed in terms of Green's function. So, first we have to remove the (simple) singularities of Green's function at the eigenvalues to obtain an entire kernel. Indeed, as a consequence of Theorem 2.3 the Weierstrass product

$$\omega(\lambda) := \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_\nu}\right) e^{\delta_{1,n-p} \frac{\lambda}{\lambda_\nu}} \tag{4.8}$$

converges and defines an entire function of order $\frac{1}{n-p}$ which has simple zeros only at the eigenvalues, cf. [36]. Now define the function $\Phi(x, \lambda)$ to be

$$\Phi(x, \lambda) := \omega(\lambda) G(x, \xi_0, \lambda) = \sum_{\nu=0}^{\infty} \left(\frac{\omega(\lambda)}{\lambda_\nu - \lambda} \overline{\psi_\nu}(\xi_0)\right) \phi_\nu(x), \tag{4.9}$$

where $\phi_\nu(x),\ \psi_\nu(x),\ \nu = 1, 2, \ldots$, are biorthonormal sets of eigenfunctions of problem (1.18)–(1.19) and its adjoint respectively. The parameter ξ_0 is an element of $[0, 1]$, such that $\psi_\nu(\xi_0) \neq 0$ for all ν.

4.2. Theorem. *Let $g(\cdot) \in W^m_{\mathcal{U}}$. Let $\Phi(\cdot,\lambda) \in W^m_{\mathcal{U}}$. Define $f(\lambda)$ to be the transform*

$$f(\lambda) = \int_0^1 g(x)\overline{P(\Phi(x,\lambda))}\,dx. \tag{4.10}$$

Then $f(\lambda)$ has the interpolation representation

$$f(\lambda) = \sum_{\nu=0}^{\infty} f(\lambda_\nu)\frac{\omega(\lambda)}{(\lambda-\lambda_\nu)\omega'(\lambda_\nu)}. \tag{4.11}$$

Moreover (4.11) converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and $f(\lambda)$ is an entire function of λ of order less than or equal to $\frac{1}{n-p}$ and type not exceeding one.

Proof. Since $\phi_\nu,\ \psi_\nu$ satisfy the biorthonormality relation

$$\langle P(\phi_\nu), \psi_j\rangle = \langle \phi_\nu, P^*(\psi_j)\rangle = -\delta_{\nu j} \quad \text{for all } \nu,\ j,$$

applying Parseval's identity to (4.10), we obtain

$$f(\lambda) = \sum_{\nu=0}^{\infty} \overline{\langle P(\phi_\nu), g\rangle}\,\langle \Phi, P^*(\psi_\nu)\rangle\,. \tag{4.12}$$

Using the definition of $\Phi(\cdot,\lambda)$ we have

$$f(\lambda) = \sum_{\nu=0}^{\infty} \overline{\langle P(\phi_\nu), g\rangle}\,\frac{\omega(\lambda)}{\lambda-\lambda_\nu}\overline{\psi_\nu}(\xi_0). \tag{4.13}$$

Letting $\lambda \longrightarrow \lambda_\nu$ in (4.10) and using (2.23) one can see that

$$f(\lambda_\nu) = -\omega'(\lambda_\nu)\overline{\psi_\nu}(\xi_0)\int_0^1 g(x)\overline{P(\phi_\nu(x))}\,dx, \tag{4.14}$$

where we have used

$$\lim_{\lambda\to\lambda_\nu} P(\Phi(x,\lambda)) = P\left(\lim_{\lambda\to\lambda_\nu}\Phi(x,\lambda)\right) = -\omega'(\lambda)\overline{\psi_\nu}(\xi_0)P(\phi_\nu(x)).$$

Relation (4.11) results by combining (4.13) and (4.14) where the convergence is pointwise. The rest of the proof can be established as in Theorem 3.2 above. □

Now we return to Example 4.1 above to establish a sampling representation associated with problem (4.1)–(4.2).

4.3. Example. Consider the boundary value problem (4.1)–(4.2) above. Since the choice of ξ_0 depends on the eigenfunctions of the problem, we calculate the eigenvalues and the eigenfunctions. The characteristic determinant of the problem is $\Delta(\lambda) = e^{-i\lambda} - 1$. Hence noting that zero is not an eigenvalue, the eigenvalues of the above problem are the zeros of $\Delta(\lambda)$, i.e., $\lambda_\nu = 2\pi\nu$, $\nu \in \mathbb{Z} - \{0\}$. The eigenvalues are all simple and the corresponding eigenfunctions are

$$\phi_\nu(x) := \phi_\nu(x, 2\nu\pi) = e^{-i2\pi\nu x} - 1, \quad \nu \in \mathbb{Z} - \{0\}.$$

Now, in the view of (2.21) and (2.22), we calculate Green's function of the problem. According to the fundamental set (4.3) we have

$$\Delta(\lambda) = e^{-i\lambda} - 1; \tag{4.15}$$

the Wronskian $W(\xi) := W(y_1, y_2)(\xi)$ is

$$W(\xi) = -i\lambda e^{-i\lambda\xi}. \tag{4.16}$$

Hence the function $g(\cdot, \xi, \lambda)$ takes the form

$$g(x, \xi, \lambda) = \frac{1}{2i\lambda} \begin{cases} 1 - e^{-i\lambda(x-\xi)}, & x \geq \xi, \\ -(1 - e^{-i\lambda(x-\xi)}), & x \leq \xi. \end{cases} \tag{4.17}$$

Then Green's function will be

$$G(x, \xi, \lambda) = \frac{1}{i\lambda(1 - e^{-i\lambda})} \begin{cases} (e^{-i\lambda x} - 1)(e^{-i\lambda(1-\xi)} - 1), & \text{for } x \geq \xi, \\ (e^{-i\lambda\xi} - 1)(e^{-i\lambda(1-x)} - 1), & \text{for } x \leq \xi. \end{cases} \tag{4.18}$$

We also have [36, p. 27]

$$\omega(\lambda) = \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda^2}{(2\pi\nu)^2}\right) = \frac{2}{\pi} \frac{\sin\frac{\lambda}{2}}{\lambda}. \tag{4.19}$$

Hence for a ξ_0 chosen as indicated above we have

$$\Phi(x, \lambda) = \frac{2}{\pi} \frac{\sin\frac{\lambda}{2}}{i\lambda^2(1 - e^{-i\lambda})} \begin{cases} (e^{-i\lambda x} - 1)(e^{-i\lambda(1-\xi_0)} - 1), & \text{for } x \geq \xi_0, \\ (e^{-i\lambda\xi_0} - 1)(e^{-i\lambda(1-x)} - 1), & \text{for } x \leq \xi_0. \end{cases} \tag{4.20}$$

Since $\Phi(0, \lambda) = \Phi(1, \lambda) = 0$, for all $\lambda \in \mathbb{C}$, then $\Phi(\cdot, \lambda) \in W^1_{\mathcal{U}}$. If $g(\cdot) \in W^1_{\mathcal{U}}$ and

$$f(\lambda) = \int_0^1 g(x)\overline{P(\Phi(x, \lambda))}\, dx, \tag{4.21}$$

then

$$f(\lambda) = 8\pi \sum_{\nu=-\infty,\, \nu\neq 0}^{\infty} (-1)^\nu f(2\nu\pi) \frac{\nu \sin\frac{\lambda}{2}}{\lambda(\lambda - 2\pi\nu)}, \tag{4.22}$$

which may be written equivalently as

$$(4.23) \qquad f(\lambda) = 8\pi \sum_{\nu=-\infty,\, \nu\neq 0}^{\infty} f(2\nu\pi) \frac{\nu \sin \frac{1}{2}(\lambda - 2\pi\nu)}{\lambda(\lambda - 2\pi\nu)}.$$

4.4. Remark. The condition $\psi_\nu(\xi_0) \neq 0$ which we assumed on the parameter ξ_0 to define $\Phi(\cdot, \lambda)$ is not necessary. In fact we can define $\Phi(\cdot, \lambda) := \Phi_{\xi_0}(\cdot, \lambda)$ according to any $\xi_0 \in [0, 1]$. That is we can choose ξ_0 such that $\psi_\nu(\xi_0) = 0$ for some ν or even for all ν. For example in Example 4.3 we take $\xi_0 = 0$, then $\psi_\nu(\xi_0) = \phi_\nu(\xi_0) = 0$ for all ν. In this case we will obtain the trivial case, i.e., $f(\lambda) \equiv 0$ on $\mathbb{C}$. If we choose ξ_0 such that $\psi_\mu(\xi_0) = 0$ for some μ then we will obtain the same results with $f(\lambda_\mu) = 0$.

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