Reconstructing parameters of a medium from incomplete spectral information

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1. Problem formulation

The lecture deals with the following system of differential equations

$$\frac{dy_1}{dx} = i\rho\, R(x)\, y_2, \; \frac{dy_2}{dx} = i\rho\, \frac{1}{R(x)}\, y_1, \tag{1}$$

$x \in [0, T]$ with the initial conditions

$$y_1(0, \rho) = 1, \; y_2(0, \rho) = -1.$$

Here $\rho = \sigma + i\tau$ is the spectral parameter, and $R(x)$ is a real function which is called the wave resistance. System (1) describes the wave propagation in a stratified medium and often appears in optics, spectroscopy, in electrodynamic and acoustic problems. Radio engineering problems of the design of directional couplers for non-uniform electronics lines and synthesizing transitions between acoustic wave guides can also be reduced to studying system (1).
Some of these classes of synthesis problems relate to inverse problems of spectral analysis with incomplete spectral information.
Synthesis problems for system (1) were studied first by Tikhonravov and then by the authors of this lecture, using various methods of the inverse spectral theory for Sturm-Liouville problems associated with system (1). In this lecture I shall present one of several possible algorithms for the solution of such classes of inverse problems.

We shall consider three classes of coefficients $R(x)$ and shall say that
$R(x) \in B_0$ if $R(x) \in W_2^2(0, T), \; R(x) > 0, \; R(0) = 1$ and $R'(0) = 0$.
Here we also consider the more general case when $R(x)$ has poles or zeros inside the interval $(0, T)$. We say that $R(x) \in B_0^-$ if $R(x)$ has the form

$$R(x) = \sum_{j=1}^{p} \frac{R_j}{(x - x_j)^2} + R_0(x),$$

$$0 < x_1 \; < \ldots < x_p < T, \; R_j > 0,$$

and $R_0(x) \in W_2^2(0, T), \; R(x) > 0 \; (x \neq x_j), \; R(0) = 1, \; R'(0) = 0$. In particular, if $p = 0$, then $R(x) \in B_0$, moreover we say that $R(x) \in B_0^+$ if $\dfrac{1}{R(x)} \in B_0^-$.

Let us consider the functions

$$f_1(\rho) = \frac{y_1(T, \rho) - R^0 y_2(T, \rho)}{2\sqrt{R^0}},$$

$$f_2(\rho) = \frac{y_1(T, \rho) + R^0 y_2(t, \rho)}{2\sqrt{R^0}}, \; R^0 = R(T)\,,$$

which are called the transmission coefficients.

$$\triangle(\rho) = \frac{1}{\sqrt{R^0}}\, y_1(T, \rho)$$

is called the characteristic function; clearly,

$$\triangle(\rho) = f_1(\rho) + f_2(\rho)\;. \tag{2}$$

Denote $\alpha_j(\sigma) = \; |f_j(\sigma)|\;, \sigma = Re\,\rho$. It is shown below (see (69)) that $\alpha_1^2(k) - \alpha_2^2(k) \equiv 1$.
For a wide class of synthesis problems, the phase is difficult or impossible to measure, while the amplitude is easily accessible to measurement. Such cases lead us to inverse spectral problems with incomplete information. In this paper we study the following incomplete inverse problem:

Inverse Problem 1. Given $\alpha_1(\sigma)$, construct $R(x)$.

An investigation of the classes B_0^- and B_0^+ is completely similar because the replacement $R \to 1/R$ is equivalent to the replacement $(y_1, y_2) \to (-y_2, -y_1)$. Below for definiteness we study the case when $R(x) \in R_0^-$.

The solution of Inverse Problem 1 is divided into two main parts:

PART I: Reconstruction of the characteristic function $\Delta(\rho)$ from the power reflection coefficient $P(\sigma)$

In this part we have to reconstruct the transmission coefficients $[f_j(\rho)$ from their moduli $\alpha_j(\sigma)$ on the real line. As a consequence we are faced with a lack of information which leads to nonuniqueness of the solution. For the construction of the transmission coefficients and we use additional information about their zeros.

PART II: Reconstruction of the wave resistance $R(x)$ from the characteristic function $\Delta(\rho)$

In this second part we can choose one of several known methods from inverse spectral theory for Sturm-Liouville problems. In this lecture we use for convenience the method due to Gelfand/Levitan/Marchenko; alternatively we could use a method based on the properties of the so-called Weyl-function.

2. Reduction to the Sturm-Liouville equation.

We introduce the functions

$$u(x,\rho) = y_1(x,\rho)/(R(x))^{1/2}, \tag{3}$$

$$v(x,\rho) = (R(x))^{1/2}\, y_2(x,\rho), \quad h(x) = R^{'}(x)/(2R(x)).$$

Then the transmission coefficients can be written in the form

$$f_{1/2}(\rho) = \frac{u(T,\rho) \mp v(T,\rho)}{2} . \tag{4}$$

The function $u(x,\rho)$ satisfies the equation

$$-u'' + q(x)u = \lambda u \ , \ \lambda = \rho^2 \ , \tag{5}$$

and the initial conditions

$$u(0,\rho) = 1 \ , \ u'(0,\rho) = -i\rho \ , \tag{6}$$

where

$$q(x) = h^2(x) - h'(x) \ . \tag{7}$$

Lemma 1. *$R(x) \in B_0^-$ if and only if $q(x) \in L_2(0,T)$, $u(T,0) \neq 0$.*

The associated SL-problem Denote

$$p(x) = \begin{cases} q(T-x) & , \quad 0 \le x \le T \\ 0 & , \quad x > T \ , \end{cases}$$

and consider the Sturm-Liouville problem

$$-y'' + p(x)y = \lambda y, \ x > 0, \tag{8}$$

$$y(0) = 0, \ \ y \in L_2(I\!R^+). \tag{9}$$

Let $e(x,\rho)$ be the corresponding Jost-solution of (14) such that $e(x,\rho) \equiv e^{i\rho x}$ for $x \ge T$, and $\delta(\rho) := e(0,\rho)$. Clearly,

$$e(x,\rho) = e^{i\rho T}\, u(T-x,\rho), \ 0 \le x \le T, \tag{10}$$

and

$$\delta(\rho) = e^{i\rho T}\ \Delta(\rho) \ . \tag{11}$$

Consequently the zeros of δ (and Δ) are the eigenvalues of (!!!), (!!!); moreover we can use the inverse Sturm-Liouville theory for reconstructing the coefficients $p(x)$ and $q(x)$ from $\Delta(\rho)$.

3. Properties of the characteristic function.

Denote $\Delta(\rho) = u(T,\rho)$. The function $\Delta(\rho)$ is called the characteristic function. The function $\Delta(\rho)$ is entire in ρ of exponential type. We interested in zeros of $\Delta(\rho)$ in the upper half-plane $\Pi_+ := \{\rho : Im\,\rho > 0\}$.

Theorem 1. *Let $R(x) \in B_0^-$. Then in $\overline{\Pi}_+$ $\Delta(\rho)$ has at most a finite number of zeros $\{\rho_k\}_{k=\overline{1,m}}$, $m \ge 0$. All zeros of $\Delta(\rho)$ in $\overline{\Pi}_+$ are simple and pure imaginary, i.e. $\rho_k = i\tau_k$, $\tau_k > 0$.*
Moreover, the function $\Delta(\rho)$ has no zeros in $\overline{\Pi}_+$ iff $R(x) \in B_0$.

In order to solve Inverse Problem 1 we need to reconstruct the transmission coefficients from their moduli and then reduce Inverse Problem 1 to the inverse problem of recovering $R(x)$ from the characteristic function.

By virtue of (1), (2),

$$u'(x,\rho) + h(x)\,u(x,\rho) = i\rho v(x,\rho)\ .$$

Substituting into (3), we get

$$\left.\begin{aligned} f_1(\rho) &= \frac{1}{2}\,u(T,\rho) - \frac{1}{2i\rho}\,\left(u'(T,\rho) + hu(T,\rho)\right)\ ,\\ f_2(\rho) &= \frac{1}{2}\,u(T,\rho) + \frac{1}{2i\rho}\,\left(u'(T,\rho) + hu(T,\rho)\right)\ , \end{aligned}\right\} \tag{12}$$

where $h := h(T)$.

The functions $f_j(\rho)$ are entire in ρ of exponential type T. It follows from (10) and (67) that

$$f_1(\rho)f_1(-\rho) - f_2(\rho)f_2(-\rho) \equiv 1\ . \tag{13}$$

Since $\overline{f_j(\rho)} = f_j(-\overline{\rho})$, we have from (68) that

$$\alpha_1^2(\sigma) - \alpha_2^2(\sigma) \equiv 1 \tag{14}$$

where $\alpha_j(\sigma) = |f_j(\sigma)|$ as above.

Theorem 1. Let $R\,(x) \in B_0^-$. Then

(i) The characteristic function $\triangle(\rho)$ is entire in ρ, and the following representation holds:

$$\triangle(\rho) = e^{-i\rho T} + \int_{-T}^{T} \tag{15}$$

where

$$\eta\,(t)e^{-i\rho t}\,dt,\ \eta\,(t) \in AC\,[-T,T], \eta'(t) \in L_2(-T,T), \eta\,(-T) = 0,$$

is a real function, and

$$\eta\,(T) = \frac{1}{2}\int_0^T q(t)dt = -\frac{h\,(T)}{2} + \frac{1}{2}\int_0^T h^2(t)dt\ .$$

(ii) For real ρ, $\triangle\,(\rho)$ has no zeros. For $Im\ \rho > 0$, $\triangle(\rho)$ has at most a finite number of simple zeros of the form $\rho_j = i\tau_j$, $\tau_j > 0$, $j = \overline{1,m}$, $m \geq 0$.

(iii) The function $\triangle\,(\rho)$ has no zeros for $Im\ \rho \geq 0$ if and only if $R\,(x) \in B_0$.

Theorem 2. Let $R(x) \in B_0^-$.

(i) The functions $f_1(\rho), f_2(\rho)$ are entire in ρ, and has the form

$$f_1(\rho) = e^{-i\rho T} + \int_{-T}^{T} g_1(t)e^{-i\rho t}dt, \tag{16}$$

$$g_1(t) \in AC[-T,T],\ g_1'(t) \in L_2(-T,T),\ g_1(-T) = 0,$$

$$f_2(\rho) = \int_{-T}^{T} g_2(t) e^{-i\rho t} dt, \tag{17}$$

$$g_2(t) \in AC[-T,T],\ g_2'(t) \in L_2(-T,T),\ g_2(-T) = 0\ ,$$

where $g_j(t)$ are real, and

$$g_1(T) = \frac{1}{2}\int_0^T h^2(t),\ g_2(T) = -\frac{h}{2},\ h := h\,(T)\ . \tag{18}$$

(ii)

$$f_1(\rho) f_1(-\rho) - f_2(\rho) f_2(-\rho) \equiv 1\ . \tag{19}$$

(iii) For real ρ, $f_1(\rho)$ has no zeros. In $\bar{\Pi}_+, f_1(\rho)$ has a finite number of simple zeros of the form $\rho_j^* = i\tau_j^*,\ \tau_j^* > 0,\quad j = \overline{1, m^*},\quad m^* \geq 0$.

(iv) If $R(x) \in B_0$, then $f_1(\rho)$ has no zeros in $\bar{\Pi}_+$, i.e. $m^* = 0$.

Lemma 3. *Suppose that a function $\gamma(\rho)$ is analytic in $\overline{\Pi}_+$, has no zeros in Π_+ and for $|\rho| \to \infty,\ \rho \in \overline{\Pi}_+,\ \gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(\sigma) = |\gamma(\sigma)| e^{-i\beta(\sigma)},\ \sigma = Re\,\rho$. Then*

$$\beta(\sigma) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi - \sigma}\, d\xi\ . \tag{20}$$

In (72) (and everywhere below, where necessary) the integral is understood in the principal value sense.

Proof. First we suppose that $\gamma(\sigma) \neq 0$ for real σ. By Cauchy's theorem, taking into account the hypothesis of the lemma, we obtain

$$\frac{1}{2\pi i}\int_{C_{r,\epsilon}} \frac{\ln\,\gamma(\xi)}{\xi - \sigma}\, d\xi = 0\ , \tag{21}$$

where $C_{r,\epsilon}$ is the closed contour (with counterclockwise circuit) consisting of the semicircles $C_r = \{\xi : \xi = re^{i\varphi}\ ,\ \varphi \in [0,\pi]\},\ \Gamma_\epsilon = \{\xi : \xi - \sigma = \epsilon e^{i\varphi},\ \varphi \in [0,\pi]\}$ and the intervals $\xi \in [-r,r] \setminus [\sigma - \epsilon, \sigma + \epsilon]$ of the real axis. Since

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\Gamma_\epsilon} \frac{\ln\gamma(\xi)}{\xi - \sigma}\, d\xi = -\frac{1}{2}\ln\,\gamma(\sigma)\ ,$$

$$\lim_{r \to \infty} \frac{1}{2\pi i}\int_{C_r} \frac{\ln\,\gamma(\xi)}{\xi - \sigma} d\xi = 0\ ,$$

we get from (73) that

$$\ln\,\gamma(\sigma) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln\,\gamma(\xi)}{\xi - \sigma}\, d\xi\ .$$

Separating here real and imaginary parts, we arrive at (72).

Suppose now that for real ρ, the function $\gamma(\rho)$ has one zero $\rho_0 = 0$ of multiplicity s (the general case is treated in the same way). Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho)\left(\frac{\rho + i\epsilon}{\rho}\right)^s,\ \epsilon > 0;\ \ \tilde{\gamma}(\sigma) = |\tilde{\gamma}(\sigma)| \cdot e^{-i\tilde{\beta}(\sigma)}\ .$$

Then

$$\beta(\sigma) = \tilde{\beta}(\sigma) + s\,\mathrm{arctg}\frac{\epsilon}{\sigma}\ . \tag{22}$$

For the function $\tilde{\gamma}(\rho)$, (72) has been proved. Hence, (74) takes the form

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - \sigma} \, d\xi$$

$$+ \frac{s}{2\pi} \int_{-\infty}^{\infty} \frac{\ln(1 + \frac{\epsilon^2}{\xi^2})}{\xi - \sigma} \, d\xi + s \operatorname{arctg} \frac{\epsilon}{\sigma} \; .$$

As $\epsilon \to 0$, it gives us (72).

Lemma 4. *Suppose that a function $\gamma(\rho)$ is analytic in $\overline{\Pi}_+$, $\gamma(-\overline{\rho}) = \overline{\gamma(\rho)}$, and for $|\rho| \to \infty$, $\rho \in \overline{\Pi}_+$, $\gamma(\rho) = 1 + O(\frac{1}{\rho})$. Let $\gamma(\sigma) = |\gamma(\sigma)| e^{-i\beta(\sigma)}$, $\sigma = Re\, \rho$, and let $\rho_k = \sigma_k + i\tau_k$, $\tau_k > 0$, $k = \overline{1, s}$ be zeros of $\gamma(\rho)$ in Π_+. Then*

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \sigma} \, d\xi + 2 \sum_{k=1}^{s} \; arctg \frac{\tau_k}{\sigma - \sigma_k} \; . \tag{23}$$

Using Lemma 4 one can construct the transmission coefficients from their moduli and information about their zeros in Π_+. For definiteness we confine ourselves to the case $h \neq 0$.

Theorem 9. *Let*

$$f_1(\sigma) = \alpha_1(\sigma)\, e^{-i\delta_1(\sigma)} \; , \tag{24}$$

and let $\rho_k^* = i\tau_k^*, \tau_k^* > 0, k = \overline{1, m^*}$ be the zeros of $f_1(\rho)$ in Π_+.
Then

$$\delta_1(\sigma) = \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - \sigma} \, d\xi + 2 \sum_{k=1}^{m^*} \; arctg \; \frac{\tau_k^*}{\sigma} \; . \tag{25}$$

In particular, if $R(x) \in B_0$, then

$$\delta_1(\sigma) = \sigma T + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - \sigma} \, d\xi \; . \tag{26}$$

Proof. Denote $\gamma(\rho) = e^{i\rho T} f_1(\rho)$. It follows from (70) that the function $\gamma(\rho)$ satisfies the hypothesis of Lemma 4. Using (75) and the relations $|\gamma(\sigma)| = \alpha_1(\sigma)$, $\delta_1(\sigma) = \sigma T + \beta(\sigma)$, we arrive at (79).

Similarly one can reconstruct $f_2(\sigma)$ from $|f_2(\sigma)|$.

3. Synthesis of $R(x)$ from the characteristic function $\triangle(\rho)$.

Theorem 3. For a function $\triangle(\rho)$ of the form (12) to be the characteristic function for a certain $R(x) \in B_0^-$, it is necessary ans sufficient that all zeros of $\triangle(\rho)$ in $\bar{\Pi}_+$ are simple, have the form $\rho_j = i\tau_j, \tau_j > 0, j = \overline{1, m}, m \geq 0$, and

$$\alpha_j := \frac{i\delta(-\rho_j)}{\dot{\delta}(\rho_j)} > 0 \; .$$

$R(x) \in B_0$ if and only if $\triangle(\rho)$ has no zeros in $\bar{\Pi}_+$, i.e. $m = 0$. The specification of the characteristic function $\triangle(\rho)$ uniquely determines $R(x)$.

Theorem 3 gives us the following algorithm for constructing $R(x)$ from the characteristic function $\triangle(\rho)$:

Algorithm 1. Let a function $\triangle(\rho)$ satisfying the hypothesis of Theorem 3 be given. Then

(1) Construct $F(t), t \in (0, 2T)$ from the integral equation

$$\theta(t) + F(t) + \int_t^{2T} \theta(s-t)F(s)ds = 0\,, \quad t \in (0, 2T)\,, \tag{27}$$

where $\theta(t) = \eta(T - t)$.
(2) Find $G(x, t)$ from the integral equation

$$G(x,t) + F(x+t) + \int_x^{2T-t} F(t+s)G(x,s)ds = 0, \tag{28}$$

$0 \le x \le T,\ x < t < 2T - x$.

(3) Calculate $R(x)$ by

$$R(x) \;=\; \frac{1}{e^2(T-x)},\ \ e(x) = 1 + \int_x^{2T-x} G(x,t)dt. \tag{29}$$

Next we provide two other algorithms for the synthesis of the wave resistance from the characteristic function, which sometimes may give advantage from the numerical point of view.

Consider the function $a(x) :=$

$$\sum_{j=1}^{m} \beta_j \,\frac{\cos \rho_j x}{\rho_j^2} + \frac{2}{\pi} \int_0^\infty \cos \rho x \left(\frac{1}{|\triangle(\rho)|^2} - 1 \right) d\rho\,, \tag{30}$$

where

$$\beta_j = -\frac{4i\rho_j^2}{\triangle(-\rho_j)\dot{\triangle}(\rho_j)}\;.$$

$a(x)$ is absolutely continuous and $a'(x) \in L_2$.
The function $R(x)$ can be constructed by the following algorithm:

Algorithm 2. Let $\triangle(\rho)$ be given. Then
(1) Construct $a(x)$ by (45).
(2) Find $Q(x, t)$ from the integral equation

$$f(x,t) + Q(x,t) + \int_0^x Q(x,s)f(s,t)ds = 0\,, \tag{31}$$

$x \ge 0\,, \quad 0 < t < x$, where $f(x,t) = \frac{1}{2}(a(x-t) - a(x+t))$. (3) Calculate $p(x) := 2\dfrac{d}{dx}Q(x,x)$.
(4) Construct

$$R(x) := \frac{1}{e^2(T-x)}\;,$$

where $e(x)$ is the solution of the Cauchy problem $y'' = p(x)y, y(T) = 1, y'(T) = 0$.

Remark 3. If $R(x) \in B_0$, then

$$a(x) = \frac{2}{\pi} \int_0^\infty \cos \rho x \left(\frac{1}{|\triangle(\rho)|^2} - 1 \right) d\rho\,,$$

and for constructing the solution of the inverse problem it is sufficient to specify $|\triangle(\sigma)|$ for $\sigma \ge 0$.