Monotonicity and convexity properties of matrix Riccati equations

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Abstract. Using a Fréchet derivative based approach some monotonicity, convexity/concavity and comparison results concerning strictly unmixed solutions of continuous and discrete time algebraic Riccati equation are obtained; it turns out that these solutions are isolated and smooth functions of the input data. Similarly, it is proved that the solutions of initial value problems for both Riccati differential and difference equations are smooth and monotonic functions of the input data and of the initial value. They are also convex or concave functions with respect to certain matrix coefficients.

Keywords: Continuous and discrete time algebraic Riccati equation, monotonicity, convexity/concavity and comparison results.

1. Introduction

With each triple $(A, S, Q) \in (\mathbb{C}^{n\times n})^3$ with $S = S^*$ and $Q = Q^*$ we associate

1. the hermitian matrix

$$E(= E(A,S,Q)) = \begin{pmatrix} Q & A^* \\ A & -S \end{pmatrix},$$

2. the continuous-time algebraic Riccati equation

$$R(X,E) := R(X;A,S,Q) := A^*X + XA + Q - XSX = 0, \tag{ARE}$$

and

3. the discrete-time algebraic Riccati equation

$$G(X,E) := G(X;A,S,Q) := A^*X[I_n + SX]^{-1}A + Q - X = 0, \tag{DARE}$$

where $I_n \in \mathbb{C}^{n\times n}$ is the unit matrix.

Similarly, with each triple of piecewise continuous functions

$$A, S, Q : D \to \mathbb{C}^{n\times n}$$

on some interval $D \subset \mathbb{R}$ (or, in the discrete case, $D \subset \mathbb{Z}$) and $t_0 \in D, X_0 = X_0^* \in \mathbb{C}^{n\times n}$ we associate

1. the matrix valued function

$$E = \begin{pmatrix} Q & A^* \\ A & -S \end{pmatrix} : D \to \mathbf{H(2n)},$$

where $\mathbf{H(2n)}$ is the set of all hermitian matrices from $\mathbb{C}^{2n\times 2n}$,

2. the initial value problem

$$\mathcal{R}(X,E) := \mathcal{R}(X;A,S,Q) :=$$
$$\dot{X} + A^*(t)X + XA + Q(t) - XS(t)X = 0, \qquad X(t_0) = X_0, \tag{RDE}$$

for the Riccati differential equation,

3. the terminal value problem

$$G(X,E) := G(X;A,S,Q) :=$$
$$A^*(k)X(k+1)[I_n + S(k)X(k+1)]^{-1}A(k) + Q(k) - X(k), X(k_o) = X_0, \tag{DRDE}$$

for the Riccati difference equation.

It has been shown by Coppel [Coppel65] (under the assumption $S(t) \geq 0$) that the unique solution $X = X(\cdot, t_0, X_0, E)$ of RDE depends monotonically on X_0 and on E. Further it has been proved by Stokes [Stokes74] that the RDE is, for $n > 1$, the only hermitian (matrix) differential equation $\dot{X} = \Phi(t,X)$, $X(t_0) = X_0$ possessing the so-called *order preserving property*, i.e. its solution X depends locally monotonic on X_0.

Under various assumptions the monotonic dependence of the solutions of RDE and DRDE on X_0 and E has been thoroughly investigated by many authors (see [BDK98], [CuRo90], [FrJa96], [BGPK85], [BiGe91], [WiPa92], [ErMy82]). Similar topics but with the accent on the stabilizing solution of ARE and DARE on E where intensively studied as well (see

[Wimmer85],[LRR87], [RaVr88], [CuRo90], [Royden88], [vEHe90]).

Due to their practical relevance in various fields such as: disconjugacy of linear Hamiltonian systems, systems and control, differential geometry and elsewhere, some of these monotonicity and comparison results have been proved or reproved independently several times.

In the discrete- and continuous-time matrix Riccati equations appearing in H^∞-control, the matrix S depends on the attenuation level γ, i.e. $S = S_1 - \frac{1}{\gamma^2} S_2$ and, as has been shown by Scherer [Scherer91], Hewer [Hewer93], Li and Chang [LiCh93], Wredenhagen and Belanger [WrBe93], Gahinet [Gahinet94], Freiling, Jank, Lee and Abou-Kandil [FJLA96] and Takaba and Katayama [TaKa96] and others, the stabilizing and/or the antistabilizing solution to ARE and DARE exibit certain convexity/concavity properties with respect to γ, S and Q.

Recently the authors [FrIo99] have used a Fréchet derivative based approach and the implicit function theorem in order to prove, in an elementary and transparent way, that the solution of DRDE depends (also in the time-varying case) monotonically on E and X_0.

It is the main purpose of this note to show that the same Fréchet derivative based method can be used to prove and partially generalize all above mentioned montonicity and convexity/concavity results for algebraic Riccati equations and Riccati differential and difference equations.

We point out that results for operator Riccati equations could be proved in the same way;

for convenience we confine here to the finite dimensional case; the infinite dimensional versions of most of our results can be easily formulated by the reader.

2. Notations and Preliminaries.

In the sequel we assume that the spaces $\mathcal{C}^0 = C^0(D, \mathbb{C}^{n\times n})$ and $\mathcal{C}^1 = C^1(D, \mathbb{C}^{n\times n})$ of continuous and continuously differentiable functions are equipped with the norms

$$\|X\|_{\mathcal{C}^0} = \max_{t\in D} \|X(t)\| \text{ for } X \in \mathcal{C}^0$$

and

$$\|X\|_{\mathcal{C}^1} = \|X\|_{\mathcal{C}^0} + \gamma \|\dot{X}\|_{\mathcal{C}^0} \text{ for } X \in \mathcal{C}^1,$$

for some $\gamma > 0$ and the Euclidian norm $\|\cdot\|$ on $\mathbb{C}^{n\times n} \simeq \mathbb{C}^{n^2}$; then $\mathcal{C}^0$ and $\mathcal{C}^1$ are Banach spaces and the map

$$\mathcal{C}^1 \to \mathcal{C}^0, X \mapsto \mathcal{R}(X, E)$$

is continuous.

For $X \in \mathbb{C}^{n\times n}$ we define the so-called (continuous and discrete) *closed loop matrices* by

$$A_{c\ell}(X) = A - SX,$$

$$A^d_{c\ell}(X) = [I + SX]^{-1} A$$

and the (continuous and discrete) inertia vectors by

$$\begin{aligned} In(X) &= (\pi(X), \nu(X), \delta(X)), \\ In^d(X) &= (\pi^d(X), \nu^d(X), \delta^d(X)), \end{aligned}$$

where $\pi(X), \nu(X), \delta(X), \pi^d(X), \nu^d(X), \delta^d(X)$ are (counting multiplicities) the number of eigenvalues of X in the open right half-plane $\mathbb{C}^>$, the open left half-plane $\mathbb{C}^<$, the imaginary axis $i\mathbb{R}, \mathbb{C} \setminus \bar{\mathbb{D}}$, $\mathbb{D}$ and $\partial\mathbb{D}$, respectively. Here $\mathbb{D}$ stands for the unit disc.

For two matrices $X_1, X_2 \in \mathbb{C}^{n\times n}$ we write

$$In(X_1) \leq In(X_2) \text{ if } \pi(X_1) \leq \pi(X_2) \text{ and } \nu(X_1) \leq \nu(X_2)$$

and, similary,

$$In^d(X_1) \leq In^d(X_2) \text{ if } \pi^d(X_1) \leq \pi^d(X_2) \text{ and } \nu^d(X_1) \leq \nu^d(X_2).$$

By $\sigma(X)$ we denote the spectrum of X.

A hermitian solution X of ARE is called stabilizing (antistabilizing) if $\sigma(A_{c\ell}(X)) \subset \mathbb{C}^<$ (or $\mathbb{C}^>$, respectively).

A hermitian solution X of DARE is called stabilizing (antistabilizing) if $\sigma(A_{c\ell}^d(X)) \subset \mathbb{D}$ (or $\mathbb{C} \setminus \bar{\mathbb{D}}$, respectively).

A solution X of ARE is called *strictly unmixed* if

$$\sigma(A_{c\ell}(X)) \cap \sigma(-A_{c\ell}(X)^*) = \emptyset. \tag{2.1}$$

A solution X of DARE is called *strictly unmixed* if

$$\lambda\mu \neq 1 \text{ for any } \lambda \in \sigma(A_{cl}^d(X)),\ \mu \in \sigma(A_{cl}^d(X)^*). \tag{2.2}$$

Notice that the stabilizing and antistabilizing solution (if they exist) are strictly unmixed.

A solution of ARE (or DARE) is called *unmixed* if (2.1) holds for all eigenvalues in $\mathbb{C} \setminus i\mathbb{R}$ (provided (2.2) holds for $\lambda, \mu \in \mathbb{C} \setminus \partial\mathbb{D}$, respectively).

As has been shown in [RaRo84b], [RaRo92a] [RaRo92b] if $S = BB^*$ and (A, B) is controllable then

(i) all strictly unmixed solutions of ARE and DARE are isolated and *Lipschitz stable*;

(ii) all isolated solutions of ARE and DARE are stable.

However let us point out that there may exist isolated (stable) solutions which are not unmixed.

It is known from [Delchamps80] and [Rodman80] that the stabilizing solution of ARE depends analytically on the data A, S, Q. As a byproduct of our subsequent results we shall get that *any* strictly unmixed solution of ARE or DARE is isolated and an analytic function of E.

For convenience of the reader we recall here some consequences of results on algebraic Lyapunov and Stein equations (see [LaTi85], Chapter 13 and [LaRo95], Chapter 5).

2.1 Theorem

a) If X is strictly unmixed, then

$$A_{c\ell}(X)^*Y + YA_{c\ell}(X) + W = 0 \tag{2.3}$$

has for any given $W = W^* \in \mathbb{C}^{n\times n}$ a unique hermitian solution Y; if in addition $W \geq 0$ (or $W > 0$) then

$$In(-Y) \leq In(A_{c\ell}(X)) \quad (\text{ or } In(-Y) = In(A_{c\ell}(X)), \text{ respectively }).$$

b) If X is strictly unmixed, then

$$A^d_{c\ell}(X)^*YA^d_{c\ell}(X) - Y + W = 0$$

has for any given $W = W^* \in \mathbb{C}^{n\times n}$ *a unique hermitian solution* Y*; if in addition* $W \geq 0$ *(or* $W > 0$*) then*

$$In^d(Y) \leq In^d(A^d_{c\ell}(X) \quad (or\ In^d(Y) = In(A^d_{c\ell}(X)),\ respectively\ ).$$

Subsequently we use the concept of the Fréchet derivative; for the definitions and notations used here as well as for the proof of basic properties of Fréchet derivatives we refer the reader to [Zeidler95], Chapter 4.

3 Results for ARE

We write the function R, appearing in the definition of ARE, as

$$R(X,E) = \begin{pmatrix} I_n \\ X \end{pmatrix}^* E \begin{pmatrix} I_n \\ X \end{pmatrix}.$$

According to the rules for Fréchet derivatives we get by elementary calculations for $\Delta X = (\Delta X)^*$

$$R_X(X;E)\Delta X = \Delta X A_{c\ell}(X) + A_{c\ell}(X)^*\Delta X = \Delta X(A - SX) + (A - SX)^*\Delta X, \quad (3.1)$$

$$R_E(X;E)\Delta E = \begin{pmatrix} I_n \\ X \end{pmatrix}^* \Delta E \begin{pmatrix} I_n \\ X \end{pmatrix} = (\Delta A)^*X + X\Delta A^* + \Delta Q - X\Delta S X, \quad (3.2)$$

with

$$\Delta E = \begin{pmatrix} \Delta Q & (\Delta A)^* \\ \Delta A & -\Delta S \end{pmatrix} = (\Delta E)^*.$$

Further

$$R_{EE}(X,E) = 0 \quad (3.3)$$

and, for $\Delta X_1 = \Delta X_1^*, \Delta X_2 = \Delta X_2^*$, one gets

$$R_{XX}(X,E)\Delta X_1 \Delta X_2 = \Delta X_1 S \Delta X_2 + \Delta X_2 S \Delta X_1. \tag{3.4}$$

Recall that the second Fréchet derivative $R_{XX}(X,E)$ is a bilinear operator.

Moreover we get for $\Delta Q = (\Delta Q)^*$

$$R_Q(X,E)\Delta Q := R_Q(X;A,Q,S)\Delta Q = \Delta Q \tag{3.5}$$

and

$$R_{QQ}(X,Q) = 0. \tag{3.6}$$

Since each solution $X(E)$ of ARE is given implicitly by an equation of the form

$$R(X(E),E) = 0, \tag{3.7}$$

and since R is a linear (and hence analytic) function of $E = (e_{ij})$ it is natural to employ the implicit function theorem for studying the dependence of $X(E)$ on $E = E^*$.

3.1 Theorem *Assume that for some E_0 there exists a strictly unmixed solution X_0 for which (3.7) is fulfilled. Then there exists $r(E_0) > 0$ such that for E ranging $\|E - E_0\| < r(E_0)$, there exists a unique analytic function of $E \mapsto X(E)$ such that $X(E)$ is a strictly unmixed solution to ARE satisfying $X(E_0) = X_0$ and its Fréchet derivative with respect to E satisfies*

$$In(-X_E(E)\Delta E) \leq In(A_{c\ell}(X(E))); \tag{3.8}$$

for $\Delta E \geq 0$. If $\Delta E > 0$ then the equality (3.8) is strict.

Proof. By differentiation of (3.7) in $E = E_0$ one gets with (3.1) and (3.2)

$$\begin{aligned} 0 &= R_X(X(E_0), E_0)X_E(E_0)\Delta E + R_E(X(E_0), E_0)\Delta E \\ &= YA_{c\ell}(X(E_0)) + A_{c\ell}(X(E_0))^*Y + \begin{pmatrix} I_n \\ X(E_0) \end{pmatrix}^* \Delta E \begin{pmatrix} I_n \\ X(E_0) \end{pmatrix} \end{aligned} \tag{3.9}$$

where $Y := X_E(E_0)\Delta E$.

Since $X(E_0)$ is unmixed, it follows from Theorem 2.1 a) that the operator

$$R_X(X(E_0), E_0) : Y \mapsto YA_{c\ell}(X(E_0)) + A_{c\ell}(X(E_0))^*Y$$

is a bijection. Hence by employing the implicit function theorem and by invoking again Theorem 2.1 a), the validity of all the assertions stated in the theorem follow. □

3.2 Remark As an immediate consequence of Theorem 3.1 and the Taylor formula (see [Zeidler95], Theorem 4.C) it follows that the stabilizing (antistabilizing) solution $X^s(E)$ $(X^a(E))$ of RDE, if it exists, is locally an analytic function of E which is monotonically increasing (decreasing) with respect to E, i.e. $X^s(E) \nearrow$ $(X^a(E) \searrow)$ if $E \nearrow$ because of

$$X_E^s(E_0)\Delta E \geq 0 \qquad (X_E^a(E_0)\Delta E \leq 0) \text{ for } \Delta E \geq 0.$$

All other strictly unmixed solutions have on account of (3.8) at least one eigenvalue which is increasing and at least one eigenvalue which is decreasing if E is strictly increasing. The analytic dependence of $X^s(E)$ on E was already proved independently by Delchamps [Delchamps80] and Rodman [Rodman80].

Next we fix A and S and consider $X(Q)$ as a function of Q only. Differentiating $R(X(Q);A,S,Q)=0$ we get, for $\Delta Q_j=\Delta Q_j^*,\ 1\le j\le 2$, as before

$$0=[X_Q(Q)\Delta Q_1]A_{c\ell}(X(Q))+A_{c\ell}(X(Q))^*[X_Q(Q)\Delta Q_1]+\Delta Q_1 \tag{3.10}$$

and, by one further differentiation,

$$\begin{aligned}0=&[X_Q(Q)\Delta Q_1]S[X_Q(Q)\Delta Q_2]+[X_Q(Q)\Delta Q_2]S[X_Q(Q)\Delta Q_1]\\ &-[X_{QQ}(Q)\Delta Q_1\Delta Q_2]A_{c\ell}(X(Q))-A_{c\ell}(X(Q))[X_{QQ}(Q)\Delta Q_1\Delta Q_2].\end{aligned} \tag{3.11}$$

Notice that if $\Delta Q_1>0$ and $X(Q)$ is strictly unmixed then (3.10) reveals that

$$\det X_Q(Q)\Delta Q_1\neq 0.$$

Notice that following the above procedure it is possible, in principle, to evaluate other derivatives of higher order by solving appropriate Lyapunov equations.

Thus based on the above conclusions (see (3.11)) we get with Theorem 2.1a) as before

3.3. Theorem *Let $X(Q_0)$ be a strictly unmixed solution of ARE. If $S\ge 0,\Delta Q=\Delta Q^*$ then*

$$In(X_{QQ}(Q_0)\Delta Q\Delta Q)\le In(A_{c\ell}(X(Q_0))$$

and, if $S>0,\Delta Q>0$, then

$$In(X_{QQ}(Q_0)\Delta Q\Delta Q)=In(A_{c\ell}X(Q_0)).$$

In particular it follows that the stabilizing (antistabilizing) solution $X^s(Q)(X^a(Q))$ of ARE is locally an analytic, monotonically increasing (decreasing) and concave (convex) function of Q.

In order to study the dependence of an unmixed solution $X(S)$ of ARE on S (for fixed A and $Q \geq 0$) we use the preceding results concerning $X(Q)$ in connection with the *dual* algebraic Riccati equation of ARE which is

$$-PA^* - AP + S - PQP = 0. \qquad \text{(DUAL)}$$

It is quite transparent that P is an invertible solution of DUAL if and only if $X = P^{-1}$ is a solution of ARE.

If $Q > 0$, then it is easy checkable that all hermitian solutions of ARE are invertible.

Hence, in this case, by invoking Theorem 3.3, it follows that $P(S)$ of DUAL have, with respect to S, the same convexity/concavity properties as the solutions $X(Q)$ of ARE have with respect to Q. If $Q \geq 0$, with Ker $Q \neq \{0\}$, one procedes as follows.

Let $X(S)$ be a hermitian solution of ARE with dim (Ker $X(S_0)) = n - k > 0$ and let $U = (u_1, \ldots, u_n)$ be a unitary matrix such that $\mathrm{Im}(u_{k+1}, \ldots, u_n) = \mathrm{Ker}\ X(S)$ (which is contained in the (Q, A)- unobservable subspace and therefore independent on small variations of S). Then we have

$$U^*AU = \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix}, \ U^*QU = \begin{pmatrix} Q_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$U^*SU = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix}, \ U^*X(S)U = \begin{pmatrix} X_{11}(S) & 0 \\ 0 & 0 \end{pmatrix},$$

where $X_{11}(S)$ is clearly an invertible solution of the reduced order algebraic Riccati equation

$$A_{11}^* X_{11} + X_{11} A_{11} + Q_{11} - X_{11} S_{11} X_{11} = 0. \tag{3.12}$$

Thus the situation has been reduced to the previous one, i.e. X_{11}^{-1} will be a solution to the dual of (3.12). Since for an invertible and Fréchet-differentiable $X(S) = P(S)^{-1}$ one gets for $\Delta S = \Delta S^*$

$$\begin{aligned} X_S(S)\Delta S &= -P(S)^{-1}[P_S(S)\Delta S]P(S)^{-1} \\ &= -X(S)[P_S(S)\Delta S]X(S) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} X_{SS}(S)\Delta S\Delta S &= -X(S)[P_{SS}(S)\Delta S\Delta S]X(S) \\ &+ 2X(S)[P_S(S)\Delta S]X(S)[P_S(S)\Delta S]X(S), \end{aligned} \tag{3.14}$$

one concludes from (3.13), (3.14) that $\pm\, X(S) \geq 0$ and $\pm\, P_{SS}(S)\Delta S\Delta S \leq 0$ imply that $\pm\, X_{SS}(S)\Delta S\Delta S \geq 0$.

By combining Theorem 3.3 with the preceding calculations one deduces

3.4 Theorem *Assume that $Q \geq 0$. As long as the stabilizing (antistabilizing) solution of ARE exists and is positive (negative) semidefinite it is an analytic, monotonically decreasing (increasing) and convex (concave, respectively) function of S and, in the case $S = BR^{-1}B^*$, a concave (convex) function of R.*

3.5 Remark There are many quantitave results concerning the sensitivity analysis of algebraic Riccati equations (see [Sun98]). As previously has been shown (see (3.10), (3.11)), the Fréchet derivatives, of any order, of an strictly unmixed solution $X(E)$ of ARE can be calculated successively by solving appropriate algebraic Lyapunov equations. Hence, for any $k \in \mathbb{N}$, it is possible to evaluate the Taylor polynomial $T_k(E)$ and the remainder $R_k(E)$ (for the notations see [Zeidler95], p. 243). Thus

$$X(E + \Delta E) = T_k(E) + R_{k+1}(E) = \sum_{j=0}^{k} X^{(j)}(E)(\Delta E)^j + R_{k+1}(E), \tag{3.15}$$

where $R_{k+1}(E) = \int_0^1 \frac{(1-\tau)^k}{k!} X^{(k+1)}(E + \tau\Delta E)(\Delta E)^{k+1} d\tau$.

Since all terms in (3.15) are known, (3.15) could also be used to derive error estimates for

$$\|X(E + \Delta E) - X(E)\| \text{ or } \|X(E + \Delta E) - T_k(E)\| \text{ for } k \geq 1.$$

4. Results for DARE

The discrete versions of the results proved in Section 3 can be obtained analogously. For $\Delta X = \Delta X^*, \Delta Z = \Delta Z^*, \Delta E = \Delta E^*$ and $\Delta Q = \Delta Q^*$ we get after short calculations

$$G_X(X, E)\Delta X = A_{c\ell}^d(X)^* \Delta X A_{c\ell}^d(X) - \Delta X,$$

$$G_{XX}(X, E)\Delta X \Delta Z = -A_{c\ell}^d(X)^*[\Delta X P \Delta Z + \Delta Z P \Delta X]A_{c\ell}^d(X),$$

$$G_E(X, E)\Delta E = \begin{pmatrix} I_n \\ X A_{c\ell}^d(X) \end{pmatrix}^* \Delta E \begin{pmatrix} I_n \\ X A_{c\ell}^d(X) \end{pmatrix},$$

$$G_Q(X, E)\Delta Q := G_Q(X; A, S, Q)\Delta Q = \Delta Q,$$

$$G_{QQ}(X, E) = 0,$$

where

$$P = S(I_n + XS)^{-1} = P^*.$$

Hence by differentiation of $G(X(E),E)=0$ in E_0 one gets

$$\begin{aligned} 0 &= G_X(X(E_0),E_0)X_E(E_0)\Delta E + G_E(X(E_0),E_0)\Delta E \\ &= A_{c\ell}^d(X(E_0))^* X_E(E_0)\Delta E A_{c\ell}^d(X(E_0)) - X_E(E_0)\Delta E \\ &+ \begin{pmatrix} I_n \\ X(E_0)A_{c\ell}^d(X(E_0)) \end{pmatrix}^* \Delta E \begin{pmatrix} I_n \\ X(E_0)A_{c\ell}^d(X(E_0)) \end{pmatrix}. \end{aligned}$$

Similarly, for fixed A and S, we infer from $G(X(Q),E)=0$ that

$$0 = A_{c\ell}^d(X(Q))^* X_Q(Q)\Delta Q + (X_Q(Q)\Delta Q)A_{c\ell}^d(X(Q)) - X_Q(Q)\Delta Q + \Delta Q,$$

and, after one further derivation,

$$0 = A_{c\ell}^d(X(Q))^*[X_{QQ}(Q)\Delta Q\Delta Q]A_{c\ell}^d(X(Q)) - X_Q(Q)\Delta Q\Delta Q$$

$$-2[(X_Q(Q)\Delta Q)(I+SX)^{-1}A]^* S(I+X(Q)S)^{-1}[(X_Q(Q)\Delta Q)(I+SX)^{-1}A].$$

Using these formulas and Theorem 2.1, b), we obtain analogously to Section 3 the following results

4.1 Theorem *For each strictly unmixed solution $X(E_0)$ of DARE the following statements hold:*

(i) There exist $r>0$ and $\rho>0$ such that for each $E \in U = \{M \in \mathbb{C}^{n\times n} \mid M = M^*,\ \|M - E_0\| < r\}$ there exists a unique unmixed solution $X(E)$ of the DARE $G(X(E),E)=0$ with $\|X(E) - X(E_0)\| < \rho$.

(ii) X is analytic in U with respect to E, and

$$In^d(X_E(E_0)\Delta E) \le (\text{ or } =)\ In^d(A_{c\ell}^d(X(E_0)))\ \ (\textit{for } \Delta E \ge 0 \textit{ for } \Delta E > 0,\ \textit{respectively}).$$

Moreover, the assertions of Theorem 3.3 and Theorem 3.4 remain all valid if in their statements the word ARE is replaced by DARE.

5 Results for RDE

By $X(X_0, E_0)$ we denote the unique solution of the initial value problem

$$(\mathcal{R}(X, E_0(t)) =)\dot{X} + A^*(t)X + XA(t) + Q(t) - XS(t)X = 0, \quad X(t_0) = X_0, \tag{5.1}$$

which is defined in some maximal neighborhood $U(t_0, X_0, E_0)$ of t_0.

Suppressing t, we get by differentiation of $0 = \mathcal{R}(X(X_0, E_0), E_0)$ with respect to (X, E) (here for an accurate evaluations of Fréchet derivatives the norms introduced in Section 2 have been tacitly used)

$$0 = \mathcal{R}_X(X(X_0, E_0), E_0)Y + R_E(X(X_0, E_0), E_0)\Delta E = \tag{5.2}$$

$$= \dot{Y} + YA_{c\ell}(X(X_0, E_0)) + A_{c\ell}(X(X_0, E_0))^*Y + \begin{pmatrix} I \\ X(X_0, E_0) \end{pmatrix}^* \Delta E \begin{pmatrix} I \\ X(X_0, E_0) \end{pmatrix},$$

for $\Delta X_0 = \Delta X_0^*, \Delta E = \Delta E^*$ and $Y := X_{(X,E)}(X_0, E_0)(\Delta X_0, \Delta E_0)$

and for the initial condition $Y(t_0) = \Delta X_0$.

Therefore for $\Delta E \geq 0$ (≤ 0) we have locally

$$\dot{Y} \leq (\geq, \text{ respectively }) - A_{c\ell}(X(X_0, E_0))^*Y - YA_{c\ell}(X(X_0, E_0)).$$

Hence, using [KnKw85], Hilfssatz 10.3, it follows for $\Delta E \geq 0$ and $\Delta X_0 \geq 0$ (or > 0) that

$$Y(t) = X_{(X,E)}(X_0, E_0)(\Delta X_0, \Delta E_0)(t) > 0 \text{ (or } > 0) \text{ for } t \leq t_0$$

and, similarly, for $\Delta E \leq 0$ and $\Delta X_0 \geq 0$ (or $\Delta X_0 > 0$) that

$$Y(t) \geq 0 \text{ (or } > 0) \text{ for } t \geq t_0,$$

as long as $Y(t)$ is defined.

If A and S are fixed and if

$$\mathcal{R}(X(Q_0), Q_0) = 0, \; X(Q_0)(t_0) = X_0,$$

then analogously to (5.2) we get

$$0 = \dot{Z} + ZA_{c\ell}(X(Q_0)) + A_{c\ell}(X(Q_0))^* Z + \Delta Q$$

for $Z := X_Q(Q_0)\Delta Q$ and $\Delta Q = \Delta Q^*$.

After one further derivation one obtains

$$0 = WA_{c\ell}(X(Q_0)) + A_{c\ell}(X(Q_0))^* W - 2ZSZ.$$

for $W := X_{QQ}(Q_0)\Delta Q \Delta Q = W^*$.

Since the initial value problem

$$(\mathcal{R}_X(X_0, E_0)Y =)\dot{Y} + YA_{c\ell}(X(X_0, E_0)) + A_{c\ell}(X(X_0, E_0))^* Y, Y(t_0) = \Delta X_0,$$

is always uniquely solvable, it follows that $\mathcal{R}_X(X_0, E_0)$ is a bijection and we can apply also here the implicit function theorem. Analogously to Section 3, using the preceding calculations, we get

5.1 Theorem *For each* $(\tilde{X}_0, E_0) \in \mathbf{H(n)} \times \mathbf{H(2n)}$ *and* $t_0 \in D$ *there exist an interval* $(t_0 - a, t_0 + a) \subset D$*, a neighborhood* $V = V(\tilde{X}_0, E_0)$ *and a unique infinitly Fréchet-differentiable function* $X : V \to C^1$ *with the following properties:*

(i) $X(\tilde{X}, E)$ *is on* $(t_0 - a, t_0 + a)$ *the unique solution of*

$$\mathcal{R}(X, E) = 0, \; X(t_0) = \tilde{X}.$$

(ii) For given E *the function* $\tilde{X} \mapsto X(\tilde{X}, E)$ *is monotonically increasing.*

For given $\tilde{X}$ *the functions*

$$E \mapsto X(\tilde{X}, E)(t), t \in (t_0 - a, t]$$

and

$$E \mapsto X(\tilde{X}, E)(t), t \in [t_0, t_0 + a)$$

are monotonically increasing and decreasing, respectively.

(iii) Let A *and* S *be fixed and* $X(Q) : D_1 \to \mathbf{H(n)}$ *be a solution of*

$$\dot{X} + A^*(t)X + XA(t) + Q(t) - XS(t)X = 0$$

with $\sigma(A(t) - S(t)X(Q)(t)) \subset \mathbb{C}^<$ *(or* $\mathbb{C}^>$*). Then* $Q \mapsto X(Q)$ *is locally a concave (or convex, respectively) function.*

There is also a discrete version (for DRDE) of Theorem 5.1 . Since the essential part of this discrete version is contained in [FrIo99] we omit here details.

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