Math. Nachr. **229** (2001), 51 – 71

Indefinite Eigenvalue Problems with Several Singular Points and Turning Points

By Walter Eberhard, Gerhard Freiling and Kerstin Wilcken–Stoeber of Duisburg

(Received June 21, 1999; accepted May 25, 2000)

Abstract. We consider a general class of eigenvalue problems with two–point boundary conditions on a finite interval generated by a differential equation with an indefinite weight function which has several zeros and/or poles. As a basic result we derive asymptotic estimates for a special fundamental system of solutions of the corresponding differential equation and determine the asymptotic distribution of the eigenvalues. Finally we prove the uniform convergence of eigenfunction expansions for some class of functions f.

1. Introduction

We investigate the eigenvalue problem generated by the differential equation

$$\ell(v) \;:=\; -v'' + \chi(x)\,v \;=\; \rho^2\phi^2(x)\,v \quad \text{for} \quad x \in I \;:=\; [0,1]\,, \tag{1.1}$$

and two linearly independent two–point boundary conditions

$$U_j(v) \;:=\; U_{j0}(v) + U_{j1}(v) \;=\; 0 \quad (j = 1, 2)\,. \tag{1.2}$$

Here $\lambda = \rho^2$ is the eigenvalue parameter. (We write λ and the weight function as squares in order to simplify the formulas in the following sections.) We assume that the weight function ϕ^2 is real with a finite number of zeros of any order and/or poles of first order in the open interval $(0,1)$, these zeros and poles are the so–called *turning points* of (1.1). Moreover, these turning points are admitted to be poles of first or second order of the function χ.

In Section 2 we determine the asymptotic dependence of the solutions of (1.1) for $|\rho| \to \infty$. Using these asymptotic estimates we derive a formula for the asymptotic distribution of the eigenvalues for a class of regular problems which is a generalization of the class of Birkhoff–regular eigenvalue problems in the definite case. Further we

---

1991 *Mathematics Subject Classification.* 34B24, 34B27, 34L10, 34L20.

Keywords and phrases. Indefinite spectral problems, turning points, weight function, eigenfunction expansions.

prove (under Assumption 6.1) the uniform convergence of the eigenfunction expansion for a function f with the following properties: f has an absolutely continuous derivative f' with $f'' \in L[0,1]$ and satisfies the boundary conditions $U_j(f) = 0$ for $j = 1,2$; a modified version of this theorem is provided in Subsection 6.3.

This paper continues the investigations made by Eberhard, Freiling and Schneider (see [6], [7], [8]) who considered (1.1) in the case of several zeros of ϕ^2 and a bounded and integrable function χ; it contains and extends the main results of the dissertation [24] of the third author, written under the guidance of the two first authors.

Differential equations with turning points arise in various problems of mathematical physics as well as in applications (see [3], [5], [9], [16], [23] for further references). Nevertheless a rigorous treatment of such spectral problems is up to the present time only available in special cases (see [1], [7], [8], [10], [20], [21] and [25] for symmetric problems and for problems with turning points being zeros and also [2], [4], [19] where a Krein/Pontrjagin–space approach is used). The main reason for this gap in the literature is the fact that up to now there existed no adequate asymptotic estimates for differential equations of the general form (1.1); we hope to fill this gap with our results.

2. The fundamental systems in a neighbourhood of one turning point

2.1. Some remarks

If ϕ^2 is definite, i. e. $\pm\phi^2 > 0$, then the theory of Liouville–Green–approximation (see [18], Chapter 6) supplies the existence of two linearly independent solutions of (1.1) with nice asymptotic estimates in I (see Subsection 2.2). Since ϕ^2 is indefinite it is evident that these asymptotic forms cannot be valid in intervals containing a turning point. In the case of one single turning point $x_1 \in I$, the solutions of (1.1) can be expressed using Bessel functions and have been estimated rigorously by R. E. Langer (see [11], [13], [14]). In the case of $m > 1$ turning points in the interval I we obtain estimates for solutions in the whole interval by dividing I into m parts containing only one turning point and by patching together the asymptotic forms appropriate to the neighbouring intervals. However this procedure is not applicable in a straightforward way to the estimates obtained by Langer (see [15]). Therefore we substitute the fundamental system derived by Langer by another fundamental system which is more suitable for the matching process. The idea of the construction comes from [8], where all turning points were zeros of ϕ^2; in the presence of turning points being poles we get asymptotic estimates for a special fundamental system by combining the ideas of [13], [14] and [7]. Since the proofs of the results in Sections 2 – 5 are very technical, we give here only the main idea of the proofs and refer the interested reader to [24] for further details.

2.2. Notations and preliminary results

We suppose that the function ϕ^2 has m turning points x_ν in I of order $l_\nu \in \mathbb{N} \cup \{-1\}$ with $0 < x_1 < \cdots < x_m < 1$.

Notations 2.1. (i) For fixed $\varepsilon > 0$ (sufficiently small) we define the following intervals:

$$\begin{aligned} D_{0,\varepsilon} &:= [0, x_1 - \varepsilon]\,, \\ D_{\nu,\varepsilon} &:= [x_\nu + \varepsilon, x_{\nu+1} - \varepsilon] \quad (1 \le \nu \le m-1)\,, \\ D_{m,\varepsilon} &:= [x_m + \varepsilon, 1]\,, \\ I_{\nu,\varepsilon} &:= D_{\nu-1,\varepsilon} \cup [x_\nu - \varepsilon, x_\nu + \varepsilon] \cup D_{\nu,\varepsilon} \quad (1 \le \nu \le m)\,. \end{aligned}$$

(ii) For $k \in \mathbb{Z}$ we consider the sectors

$$S_k := \left\{ \rho \in \mathbb{C} \;\middle|\; \tfrac{k\pi}{4} \le \arg \rho \le \tfrac{(k+1)\pi}{4} \right\}.$$

For $k \in \mathbb{Z}$ and $\delta > 0$ let

$$\chi^{(k)} := \left\{ \xi \in \mathbb{C} \mid (k - 1 + \delta)\pi \le \arg \xi \le (k + 1 - \delta)\pi \right\};$$

the sectors $\chi^{(k)}$ are understood as sectors of the Riemann surface of log.

Assumption 2.2. (i) The functions

$$\phi_{\nu,0} : I_{\nu,\varepsilon} \longrightarrow \mathbb{R}, \quad \phi_{\nu,0}(x) := (x - x_\nu)^{-l_\nu} \phi^2(x)\,, \quad 1 \le \nu \le m\,,$$

are non–vanishing and real–analytic; denote $\kappa_{\nu 0} := \phi_{\nu,0}(x_\nu)$.

(ii) χ has the form

$$\chi(x) = A_\nu (x - x_\nu)^{-2} + B_\nu (x - x_\nu)^{-1} + C_\nu(x) \quad (x \in I_{\nu,\varepsilon},\ x \ne x_\nu)$$

with constants A_ν, B_ν and a bounded real–analytic function C_ν.

Notations 2.3. (i) We introduce for $1 \le \nu \le m$

$$\begin{aligned} \mu_\nu &:= \frac{1}{l_\nu + 2} \in (0, 1]\,, \\ \eta_\nu &:= \mu_\nu (1 + 4A_\nu)^{1/2} \quad \text{with} \quad \arg \eta_\nu \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right]\,, \\ \sigma_\nu &:= \begin{cases} 1 & \text{if} \quad \mu_\nu > \dfrac{1}{4}\,, \\ 1 - \delta_0 & \text{if} \quad \mu_\nu = \dfrac{1}{4} \quad \text{(with } \delta_0 > 0 \text{ arbitrarily small)}\,, \\ 4\mu_\nu & \text{if} \quad \mu_\nu < \dfrac{1}{4}\,, \end{cases} \end{aligned}$$

and

$$\sigma_0 := \min\{\sigma_\nu \mid 1 \le \nu \le m\}\,.$$

(ii) We recall that there are four different types of zeros of order l_ν and two different types of poles of first order. For $1 \le \nu \le m$

$$T_\nu := \begin{cases} I_N & \text{if } l_\nu \text{ even and } \phi^2(x)(x - x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu,\varepsilon}\,, \\ II_N & \text{if } l_\nu \text{ even and } \phi^2(x)(x - x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu,\varepsilon}\,, \\ III_N & \text{if } l_\nu \text{ odd and } \phi^2(x)(x - x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu,\varepsilon}\,, \\ IV_N & \text{if } l_\nu \text{ odd and } \phi^2(x)(x - x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu,\varepsilon}\,, \\ I_P & \text{if } l_\nu = -1 \text{ and } \phi^2(x)(x - x_\nu) > 0 \text{ in } I_{\nu,\varepsilon}\,, \\ II_P & \text{if } l_\nu = -1 \text{ and } \phi^2(x)(x - x_\nu) < 0 \text{ in } I_{\nu,\varepsilon}\,, \end{cases}$$

is called type of x_ν.

(iii) We use the following abbreviations:

$$\begin{aligned} I_0 &:=]0,x_1[\,,\,, \quad I_\nu :=]x_\nu, x_{\nu+1}[\ (1 \le \nu \le m-1)\,, \quad I_m :=]x_m, 1[\,,\\ R_\nu &:= \int_{I_\nu} |\phi(t)|\,dt \qquad (0 \le \nu \le m)\,,\\ R_0(x) &:= \int_0^x |\phi(t)|\,dt\,, \quad R_\nu(x) := \int_{x_\nu}^x |\phi(t)|\,dt \ (1 \le \nu \le m)\,,\\ \xi_\nu(x,\rho) &:= \rho \int_{x_\nu}^x \phi(t)\,dt \qquad (1 \le \nu \le m)\,. \end{aligned}$$

(iv) Further we set:

$$\begin{aligned} [1] &:= 1 + \mathcal{O}(\rho^{-\sigma_0})\,,\\ [1;\xi] &:= [1] + \mathcal{O}\big(\xi^{-1}\big)\,, \end{aligned}$$

where $\mathcal{O}$ is the Landau symbol.

(v) In the sequel we use the symbol $\overset{(2)}{=}$ instead of $=$ in order to signify that a given formula is valid and also the corresponding formula obtained by formal differentiation with respect to x ignoring the formal differentiation of all error terms.

Now we outline the fundamental rule for the choice of the solutions $w_{\nu,1}^{T_\nu}(\,\cdot\,)$, $w_{\nu,2}^{T_\nu}(\,\cdot\,)$ of (1.1) for $(x,\rho) \in I_{\nu,\varepsilon} \times S_k$ $(1 \le \nu \le m,\ k \in \mathbb{Z})$:

The theory of Liouville–Green–approximation shows that for every sector S_k $(k \in \mathbb{Z})$ and for every interval $D_{\nu,\varepsilon}$ $(0 \le \nu \le m)$ there exist two linearly independent solutions $\widetilde{w}_{\nu,1}$, $\widetilde{w}_{\nu,2}$ of (1.1) satisfying

$$\widetilde{w}_{\nu,j}(x,\rho) \overset{(2)}{=} \phi^{-1/2}(x)\exp\left(\pm\rho\int_{x_\nu}^x \phi(t)\,dt\right)[1] \quad \text{for} \quad (x,\rho) \in D_{\nu,\varepsilon} \times S_k\,.$$

Here and in the following we shall write two formulas in one by the use of double signs together with an index j: the upper signs are to be associated with the value $j = 1$ and the lower signs with $j = 2$.

$\widetilde{w}_{\nu,j}$ is called *dominant* respectively *subdominant* in $D_{\nu,\varepsilon}$ if $\Re\big(\pm\rho\int_{x_\nu}^x \phi(t)\,dt\big)$ is positive respectively negative for $(x,\rho) \in D_{\nu,\varepsilon} \times \overset{\circ}{S}_k$.

Now we define the solutions $w_{\nu,j}^{T_\nu}$ $(j = 1,2)$ in the following way:

$w_{\nu,1}^{T_\nu} : I_{\nu,\varepsilon} \to \mathbb{C}$ is the continuation to $I_{\nu,\varepsilon}$ of a normalized multiple of the subdominant solution of (1.1) in $D_{\nu-1,\varepsilon}$ and $w_{\nu,2}^{T_\nu} : I_{\nu,\varepsilon} \to \mathbb{C}$ is the continuation to $I_{\nu,\varepsilon}$ of a normalized multiple of the subdominant solution (1.1) in $D_{\nu,\varepsilon}$; notice that both solutions can be continuated (in the next step) to the whole interval $I = [0,1]$.

In order to estimate $w_{\nu,1}^{T_\nu}$ and $w_{\nu,2}^{T_\nu}$ we use the results of [14] (see Theorem 2.4); in [14] R. E. Langer proved asymptotic estimates for a fundamental system of solutions of normalized differential equations of the form (1.1) having only one turning point of order $l_0 > -2$. Using this result and some elementary transformations one arrives at the following theorem:

Theorem 2.4. *The differential equation* (1.1) *has for* $1 \le \nu \le m$, $l \in \mathbb{Z}$ *and* $x \in I_{\nu,\epsilon}$ *a fundamental system of solutions* $v_{\nu,l,1}(\,\cdot\,,\rho)$, $v_{\nu,l,2}(\,\cdot\,,\rho)$, *satisfying for a fixed, sufficiently large number* $N > 0$ *the following estimates:*

(i) *For* $|\xi_\nu(x,\rho)| \le N$ *and* $1 \le j \le 2$ *is*

$$\begin{aligned}
&v_{\nu,l,j}(x,\rho)\\
&\overset{(2)}{=} \sqrt{2}\,(x-x_\nu)^{1/4}\left(\tfrac{\pi}{2}\right)^{1/2} e^{\pm(\eta_\nu+l+1/2)i\frac{\pi}{2}}\\
&\quad\times \kappa_{\nu 0}^{\mu_\nu/4}\, 2^{-\mu_\nu(l_\nu+1)/2}\,(x-x_\nu)^{-1/4}\,\psi_\nu(x)\,\xi_\nu^{\mu_\nu/2}(x,\rho)\,H_{\eta_\nu}^{(j)}\big(e^{-l\pi i}\,\xi_\nu(x,\rho)\big)\\
&\quad+ \sqrt{2}\,(x-x_\nu)^{1/4}\kappa_{\nu 0}^{\mu_\nu/4}\, 2^{-\mu_\nu(l_\nu+1)/2}\,(x-x_\nu)^{-1/4}\,\psi_\nu(x)\\
&\quad\times \begin{cases} \xi_\nu^{\mu_\nu/2-\eta_\nu}(x,\rho)\,\dfrac{\mathcal{O}(1)}{\rho_{\mu_\nu}}\,, & \textit{if} \quad \eta_\nu \ne 0\,,\\[2ex] \xi_\nu^{\mu_\nu/2}(x,\rho)\,\log\xi_\nu(x,\rho)\,\dfrac{\mathcal{O}(1)}{\rho_{\mu_\nu}}\,, & \textit{if} \quad \eta_\nu \;=\; 0\,,\end{cases}
\end{aligned}$$

$$\overset{(2)}{=} \sqrt{\pi}\,\kappa_{\nu 0}^{\mu_\nu/4}\, 2^{-\mu_\nu(l_\nu+1)/2}\, e^{\pm(\eta_\nu+l+1/2)i\frac{\pi}{2}}\,\psi_\nu(x)\,\xi_\nu^{\mu_\nu/2}(x,\rho)\,H_{\eta_\nu}^{(j)}\left(\frac{\xi_\nu(x,\rho)}{e^{l\pi i}}\right) \tag{2.1}$$

$$+\,\psi_\nu(x)\begin{cases} \xi_\nu^{\mu_\nu/2-\eta_\nu}(x,\rho)\,\dfrac{\mathcal{O}(1)}{\rho_{\mu_\nu}}\,, & \textit{if} \quad \eta_\nu \ne 0\,,\\[2ex] \xi_\nu^{\mu_\nu/2}(x,\rho)\,\log\xi_\nu(x,\rho)\,\dfrac{\mathcal{O}(1)}{\rho_{\mu_\nu}}\,, & \textit{if} \quad \eta_\nu \;=\; 0\,,\end{cases} \tag{2.2}$$

with $\rho_{\mu_\nu} := \begin{cases} \rho, & \textit{if } \mu_\nu > \frac{1}{2}\,,\\ \rho^{2\mu_\nu}, & \textit{if } \mu_\nu < \frac{1}{2}\,,\end{cases}$ *and where* $H_\eta^{(1)}$ *and* $H_\eta^{(2)}$ *are the Hankel functions* (*i. e. Bessel functions of the third kind*).

(ii) *For* $|\xi_\nu(x,\rho)| > N$ *and* $1 \le j \le 2$ *and* $p \in \mathbb{Z}$ *is*

$$\begin{aligned}
&v_{\nu,2p,j}(x,\rho)\\
&\overset{(2)}{=} \sqrt{2}\,(x-x_\nu)^{1/4}\kappa_{\nu 0}^{\mu_\nu/4}\, 2^{-\mu_\nu(l_\nu+1)/2}\,(x-x_\nu)^{-1/4}\psi_\nu(x)\,\xi_\nu^{(\mu_\nu-1)/2}(x,\rho)\\
&\quad\times\Big\{c_{\nu,j,1}^{(p)}e^{i\xi_\nu(x,\rho)}[1;\xi_\nu] + c_{\nu,j,2}^{(p)}e^{-i\xi_\nu(x,\rho)}[1;\xi_\nu]\Big\} \qquad (2.3)\\
&\overset{(2)}{=} \kappa_{\nu 0}^{\mu_\nu/4}\, 2^{(1-(l_\nu+1)/(l_\nu+2))/2}\,\phi^{-1/2}(x)\,\Phi_\nu^{(1-\mu_\nu)/2}(x)\rho^{(\mu_\nu-1)/2}\,\Phi_\nu^{(\mu_\nu-1)/2}(x)\\
&\quad\times\Big\{c_{\nu,j,1}^{(p)}e^{i\xi_\nu(x,\rho)}[1;\xi_\nu] + c_{\nu,j,2}^{(p)}e^{-i\xi_\nu(x,\rho)}[1;\xi_\nu]\Big\}\\
&\overset{(2)}{=} \kappa_{\nu 0}^{\mu_\nu/4}\, 2^{\mu_\nu/2}\,\rho^{(\mu_\nu-1)/2}\,\phi^{-1/2}(x)\Big\{c_{\nu,j,1}^{(p)}e^{i\xi_\nu(x,\rho)}[1;\xi_\nu] + c_{\nu,j,2}^{(p)}e^{-i\xi_\nu(x,\rho)}[1;\xi_\nu]\Big\}
\end{aligned}$$

where the coefficients are defined by

$$\left\{\begin{array}{l}
\left.\begin{array}{rcl} c_{\nu,1,1}^{(p)} &=& (-1)^{p-s+1}\,\dfrac{\sin(2s-2p-1)\eta_\nu\pi}{\sin\eta_\nu\pi}\\[2ex] c_{\nu,2,1}^{(p)} &=& (-1)^{p-s+1}\,\dfrac{i\sin(2s-2p)\eta_\nu\pi}{\sin\eta_\nu\pi}\end{array}\right\} \textit{for } \xi_{\nu,1}(z,\rho) \in \chi^{(2s-1)} \cup \chi^{(2s)}\,,\\[5ex]
\left.\begin{array}{rcl} c_{\nu,1,2}^{(p)} &=& (-1)^{p-s+1}\,\dfrac{i\sin(2s-2p)\eta_\nu\pi}{\sin\eta_\nu\pi}\\[2ex] c_{\nu,2,2}^{(p)} &=& (-1)^{p-s}\,\dfrac{\sin(2s-2p+1)\eta_\nu\pi}{\sin\eta_\nu\pi}\end{array}\right\} \textit{for } \xi_{\nu,1}(z,\rho) \in \chi^{(2s)} \cup \chi^{(2s+1)}\,.
\end{array}\right. \tag{2.4}$$

Further is

$$\begin{aligned} v_{\nu,2p,1}(x,\rho) &= v_{\nu,2p+1,1}(x,\rho)\,, \\ v_{\nu,2p-1,2}(x,\rho) &= v_{\nu,2p,2}(x,\rho)\,. \end{aligned}$$

(iii) $$W(v_{\nu,l,1},v_{\nu,l,2}) \;=\; -\,\kappa_{\nu 0}^{\mu_\nu/2}\,2^{\mu_\nu+1}\,i\rho^{\mu_\nu}[1] \quad (1\le\nu\le m,\ l\in\mathbb{Z})\,.$$

Unfortunately it is not possible to prove adequate asymptotic estimates of the continuation of the solutions $v_{\nu,l,j}(\,\cdot\,,\rho)$ to the whole interval $I=[0,1]$ (although this continuation exists), therefore we have to replace the solutions $v_{\nu,l,1}(\,\cdot\,,\rho)$, $v_{\nu,l,2}(\,\cdot\,,\rho)$ in the next step $\big($for $x\in I_{\nu,\varepsilon}$ and for adequate $l=2p\in\mathbb{Z}$, which depends on the type T_ν and on the sector in the ρ–plane in which we want to estimate $w^{T_\nu}_{\nu,j}(x,\rho)\big)$ by the linear combinations

$$w^{T_\nu}_{\nu,j}(x,\rho) \;=:\; w^{T_\nu}_{\nu,2p,j}(x,\rho) \;=:\; \sigma_{\nu,p,j}v_{\nu,2p,1}(x,\rho)+\tau_{\nu,p,j}v_{\nu,2p,2}(x,\rho)\,, \quad 1\;\le\;\nu\;\le\;m\,, \tag{2.5}$$

which have been defined above and are more appropriate for our purpose.

It turns out that the connection coefficients $\sigma_{\nu,p,j}$, $\tau_{\nu,p,j}$ can be determined with an adequate accuracy by Cramers rule (since they are calculated for a subdominant solution). The corresponding calculations are elementary but voluminous, since we have to take into account all possible combinations of turning points; in particular it turns out that the estimates obtained are independent of p. Notice that p has an influence on the sector in which the estimates of Theorem 2.4, (ii), are valid, therefore we have to choose p in the correct way when we are matching subsequently solutions, defined for x in $I_{\nu,\epsilon}$ and solutions, defined for x in $I_{\nu+1,\epsilon}$, respectively.

For the formulation of the next theorem we introduce some notations:

Notations 2.5. (i) Let ω_1^+, ω_2^+ respectively ω_1^-, ω_2^- be the square roots of $+1$ respectively -1. According to Naimark [17] they can be ordered for each of the sectors S_k in such a way that the following inequalities hold:

$$\Re\big(\rho\omega_1^+\big) \;\le\; \Re\big(\rho\omega_2^+\big) \quad \text{respectively} \quad \Re\big(\rho\omega_1^-\big)\le\Re\big(\rho\omega_2^-\big) \quad \text{for all} \quad \rho\in S_k\,.$$

Notice that this order essentially depends on the sector S_k, we suppress this dependence in our notation.

Now we set for $0\le\nu\le m$

$$\omega_\nu \;:=\; \begin{cases} \omega_2^+ \text{ if } \phi^2 \;<\; 0 \text{ in } D_{\nu,\varepsilon}\,, \\ \omega_2^- \text{ if } \phi^2 \;>\; 0 \text{ in } D_{\nu,\varepsilon}\,. \end{cases}$$

Therefore we have for $\rho\in S_k$ $(-2\le k\le 1)$

$$\omega_\nu \;=\; \begin{cases} \;\;1 \text{ if } \phi^2 \;<\; 0 \text{ in } D_{\nu,\varepsilon} \text{ and } k\in\{-2,-1,0,1\}\,, \\ \;\;i \text{ if } \phi^2 \;>\; 0 \text{ in } D_{\nu,\varepsilon} \text{ and } k\in\{-2,-1\}\,, \\ -i \text{ if } \phi^2 \;>\; 0 \text{ in } D_{\nu,\varepsilon} \text{ and } k\in\{0,1\}\,. \end{cases}$$

For every sector S_k $(-2\le k\le 1)$ is $\Re(-\omega_\nu\rho)\le 0$.

(ii) For $1 \le \nu \le m$ let

$$a_\nu^{T_\nu} := \begin{cases} 1 & \text{if } T_\nu \in \{I_P, II_P\}\,, \\ \dfrac{\sin\frac{1}{2} l_\nu \eta_\nu \pi}{\sin \eta_\nu \pi} & \text{if } T_\nu \in \{I_N, II_N\}\,, \\ \dfrac{\sin\frac{1}{2}(l_\nu+1)\eta_\nu \pi}{\sin \eta_\nu \pi} & \text{if } T_\nu \in \{III_N, IV_N\}\,, \end{cases}$$

$$A_\nu^{T_\nu} := \begin{cases} \dfrac{\sin\frac{1}{2}(l_\nu+4)\eta_\nu \pi}{\sin \eta_\nu \pi} & \text{if } T_\nu \in \{I_N, II_N\}\,, \\ a_\nu^{T_\nu} & \text{otherwise}\,, \end{cases}$$

$$b_\nu^{T_\nu} := \begin{cases} 2\cos \eta_\nu \pi & \text{if } T_\nu \in \{I_P, II_P\}\,, \\ \dfrac{\sin\frac{1}{2}(l_\nu+2)\eta_\nu \pi}{\sin \eta_\nu \pi} & \text{if } T_\nu \in \{I_N, II_N\}\,, \\ \dfrac{\sin\frac{1}{2}(l_\nu+3)\eta_\nu \pi}{\sin \eta_\nu \pi} & \text{if } T_\nu \in \{III_N, IV_N\}\,, \end{cases}$$

$$c_\nu := \begin{cases} 1 & \text{if} \quad T_\nu \in \{I_N, II_N\}\,, \\ e^{i\frac{\pi}{4}} & \text{if} \quad (T_\nu \in \{II_P, III_N\} \text{ and } \rho \in S_{-2} \cup S_{-1}) \\ & \text{or} \quad (T_\nu \in \{I_P, IV_N\} \text{ and } \rho \in S_0 \cup S_1)\,, \\ e^{-i\frac{\pi}{4}} & \text{otherwise}\,. \end{cases}$$

(Pay attention to the dependence of c_ν on the sector S_k!)

Using the preceding notations we obtain the following estimates for the fundamental system $w_{\nu,1}^{T_\nu}(\,\cdot\,,\rho)$, $w_{\nu,2}^{T_\nu}(\,\cdot\,,\rho)$ which replaces in the sequel $v_{\nu,2p,1}(\,\cdot\,,\rho)$, $v_{\nu,2p,2}(\,\cdot\,,\rho)$ and is more appropriate for the subsequent matching process:

Theorem 2.6. *For $1 \le \nu \le m$ the fundamental system $w_{\nu,1}^{T_\nu}(\,\cdot\,,\rho)$, $w_{\nu,2}^{T_\nu}(\,\cdot\,,\rho)$ of Equation* (1.1) *satisfies the following asymptotic estimates:*

(i) *For the sectors S_k $(-2 \le k \le 1)$ we have the following estimates for the subdominant solutions in $D_{\nu-1,\varepsilon} \times S_k$ and $D_{\nu,\varepsilon} \times S_k$ (which are unique up to normalization):*

$$\begin{cases} w_{\nu,1}^{T_\nu}(x,\rho)\,|\phi(x)|^{1/2} \overset{(2)}{=} e^{\omega_{\nu-1}\rho R_\nu(x)}[1] & \text{for} \quad (x,\rho) \in D_{\nu-1,\varepsilon} \times S_k\,, \\ w_{\nu,2}^{T_\nu}(x,\rho)\,|\phi(x)|^{1/2} \overset{(2)}{=} \frac{c_\nu}{b_\nu^{T_\nu}}\, e^{-\omega_\nu \rho R_\nu(x)}[1] & \text{for} \quad (x,\rho) \in D_{\nu,\varepsilon} \times S_k\,. \end{cases}$$

For the continuation of these solutions to the intervals $D_{\nu,\varepsilon}$ and $D_{\nu-1,\varepsilon}$, respectively, we obtain for $(x,\rho) \in D_{\nu-1,\varepsilon} \times S_k$

$$w_{\nu,1}^{T_\nu}(x,\rho)\,|\phi(x)|^{1/2} \overset{(2)}{=} c_\nu b_\nu^{T_\nu} e^{\omega_\nu \rho R_\nu(x)}[1] +$$

$$
+\begin{cases} 0 & \textit{if}\ \ (T_\nu \in \{I_P, II_N, IV_N\}\ \textit{and}\ k \in \{-2,1\}) \\ & \textit{or}\ \ (T_\nu \in \{II_P, I_N, III_N\}\ \textit{and}\ k \in \{-1,0\})\,, \\ ic_\nu a_\nu^{T_\nu} e^{-\omega_\nu \rho R_\nu(x)}[1] & \textit{if}\ \ (T_\nu \in \{I_P, II_N, IV_N\}\ \textit{and}\ k = -1) \\ & \textit{or}\ \ (T_\nu \in \{II_P, I_N, III_N\}\ \textit{and}\ k = 1)\,, \\ -ic_\nu a_\nu^{T_\nu} e^{-\omega_\nu \rho R_\nu(x)}[1] & \textit{if}\ \ (T_\nu \in \{I_P, II_N, IV_N\}\ \textit{and}\ k = 0) \\ & \textit{or}\ \ (T_\nu \in \{II_P, I_N, III_N\}\ \textit{and}\ k = -2)\,, \end{cases}
$$

and for $(x,\rho) \in D_{\nu-1,\varepsilon} \times S_k$

$$
\begin{aligned}
& w_{\nu,2}^{T_\nu}(x,\rho)\,|\phi(x)|^{1/2} \\
\overset{(2)}{=}\ & e^{-\omega_{\nu-1}\rho R_\nu(x)}[1] \\
& +\begin{cases} \dfrac{iA_\nu^{T_\nu}}{b_\nu^{T_\nu}}\, e^{\omega_{\nu-1}\rho R_\nu(x)}[1] & \textit{if}\ \ (T_\nu \in \{I_P, I_N, IV_N\}\ \textit{and}\ k = 1) \\ & \textit{or}\ \ (T_\nu \in \{II_P, II_N, III_N\}\ \textit{and}\ k = -1)\,, \\ -\dfrac{iA_\nu^{T_\nu}}{b_\nu^{T_\nu}}\, e^{\omega_{\nu-1}\rho R_\nu(x)}[1] & \textit{if}\ \ (T_\nu \in \{I_P, I_N, IV_N\}\ \textit{and}\ k = -2) \\ & \textit{or}\ \ (T_\nu \in \{II_P, II_N, III_N\}\ \textit{and}\ k = 0)\,, \\ 0 & \textit{otherwise}\,. \end{cases}
\end{aligned}
$$

The Wronskian satisfies for $\rho \in S_k$

$$
W\big(w_{\nu,1}^{T_\nu}, w_{\nu,2}^{T_\nu}\big) \;=\; -2\omega_{\nu-1}\rho\,[1]\,.
$$

(ii) *In order to obtain estimates for the fundamental system for other sectors* $S_{k\pm4}$ ($k \in \mathbb{Z}$) *we have to substitute the parameter* ρ *by* $-\rho$ *in the estimates for the sector* S_k.

Notice that $w_{\nu,1}^{T_\nu}(x,\rho)$ and $w_{\nu,2}^{T_\nu}(x,\rho)$ can also be estimated for $(x,\rho) \in [x_\nu - \varepsilon, x_\nu + \varepsilon] \times S_k$ by combining the estimates of Theorem 2.4 and the estimates for the connection coefficients (see Lemmas 6.4, 6.5 and 6.8).

3. Connection matrices

For the proof of estimates for the fundamental system of (1.1) on the whole interval I we have to calculate the so–called *connection matrices* $C_\nu(T_\nu, T_{\nu+1})$ with

$$
\begin{pmatrix} w_{\nu,1}^{T_\nu}(x,\rho) \\ w_{\nu,2}^{T_\nu}(x,\rho) \end{pmatrix} \;=\; C_\nu\big(T_\nu, T_{\nu+1}\big) \begin{pmatrix} w_{\nu+1,1}^{T_{\nu+1}}(x,\rho) \\ w_{\nu+1,2}^{T_{\nu+1}}(x,\rho) \end{pmatrix} \qquad (1 \le \nu \le m-1) \tag{3.1}
$$

for $(x,\rho) \in D_{\nu,\varepsilon} \times S_k$. Therefore we consider every possible transition from one turning point x_ν to the next one $x_{\nu+1}$. Using the estimates for the fundamental system $w_{\nu,j}^{T_\nu}$ ($1 \le \nu \le m$, $j = 1,2$) from Theorem 2.6 and Cramer's rule we are able to determine the connection matrices and get the following result (for further details see [24] and [8]):

Theorem 3.1. *The connection matrices satisfy for* $1 \leq \nu \leq m$ *and* $\rho \in S_k$ $(-2 \leq k \leq 1)$

$$C_\nu(T_\nu, T_{\nu+1}) \;=\; F_\nu E_\nu(\omega_\nu \rho) H_\nu(\omega_\nu \rho; \alpha_\nu, \beta_\nu, \gamma_{\nu+1}) c_\nu$$

with

$$F_\nu \;:=\; \begin{pmatrix} b_\nu^{T_\nu} & 0 \\ 0 & \dfrac{1}{b_\nu^{T_\nu}} \end{pmatrix}, \quad E_\nu(z) \;:=\; \begin{pmatrix} e^{zR_\nu} & 0 \\ 0 & e^{-zR_\nu} \end{pmatrix},$$

$$H_\nu(z; \alpha, \beta, \gamma) \;:=\; \begin{pmatrix} [1] + [\alpha] e^{-2zR_\nu} & e^{2zR_{\nu+1}(x_\nu + \varepsilon)} \mathcal{O}\left(\rho^{-\sigma_0}\right) + [\beta] e^{-2zR_\nu} \\ e^{-2zR_{\nu+1}(x_{\nu+1} - \varepsilon)} \mathcal{O}\left(\rho^{-\sigma_0}\right) + [\gamma] & [1] \end{pmatrix}$$

and

$$\alpha_\nu \;:=\; \beta_\nu \gamma_{\nu+1}\,,$$

$$\beta_\nu \;:=\; \begin{cases} 0 & \textit{if} \;\; (\phi^2 < 0 \textit{ in } I_\nu \textit{ and } k \in \{-1, 0\}) \\ & \textit{or} \;\; (\phi^2 > 0 \textit{ in } I_\nu \textit{ and } k \in \{-2, 1\})\,, \\ \dfrac{a_\nu^{T_\nu}}{b_\nu^{T_\nu}} & \textit{otherwise}\,, \end{cases}$$

$$\gamma_{\nu+1} \;:=\; \begin{cases} 0 & \textit{if} \;\; (\phi^2 < 0 \textit{ in } I_\nu \textit{ and } k \in \{-1, 0\}) \\ & \textit{or} \;\; (\phi^2 > 0 \textit{ in } I_\nu \textit{ and } k \in \{-2, 1\})\,, \\ \dfrac{A_{\nu+1}^{T_{\nu+1}}}{b_{\nu+1}^{T_{\nu+1}}} & \textit{otherwise}\,. \end{cases}$$

($H_\nu(z; \alpha, \beta, \gamma)$ *is only determined in first approximation, i. e.* $(\alpha, \beta, \gamma) = (\alpha_1, \beta_1, \gamma_1)$ *does not imply* $H_\nu(z; \alpha, \beta, \gamma) = H_\nu(z; \alpha_1, \beta_1, \gamma_1)$.)

The connection matrices for the sectors $S_{k \pm 4}$ $(k \in \mathbb{Z})$ *can be determined by substituting the parameter* ρ *by* $-\rho$ *in the estimates for the sector* S_k.

4. The fundamental systems in the whole interval

4.1. The fundamental system y_1, y_2

Now we are able to define a normalized fundamental system of solutions of (1.1) which can be estimated on the whole interval $[0, 1]$.

Notations 4.1. (i) For $\rho \in S_k$ $(-2 \leq k \leq 1)$ we set $n_{T_1}(\rho) := e^{\omega_0 \rho R_0}$.

(ii) Let $k \in \{-2, -1, 0, 1\}$ be fixed and $\rho \in S_k$. Then we denote the continuation of the functions $n_{T_1}(\rho) w_{1,1}^{T_1}(\,\cdot\,, \rho)$ and $n_{T_1}^{-1}(\rho) w_{1,2}^{T_1}(\,\cdot\,, \rho)$ to the whole interval $[0, 1]$ by $y_1(\,\cdot\,, \rho)$ and $y_2(\,\cdot\,, \rho)$, respectively.

Since $W\Big(y_1(\,\cdot\,,\rho),y_2(\,\cdot\,,\rho)\Big) = W\Big(w^{T_1}_{1,1}(\,\cdot\,,\rho),w^{T_1}_{1,2}(\,\cdot\,,\rho)\Big) \neq 0$, $y_1(\,\cdot\,,\rho)$, $y_2(\,\cdot\,,\rho)$ is a fundamental system of (1.1) (for every sector S_k). According to (3.1) we have

$$\begin{pmatrix} y_1(x,\rho) \\ y_2(x,\rho) \end{pmatrix} = \begin{pmatrix} n_{T_1}(\rho) & 0 \\ 0 & n^{-1}_{T_1}(\rho) \end{pmatrix} \prod_{l=1}^{\nu-1} C_l(T_l,T_{l+1}) \begin{pmatrix} w^{T_\nu}_{\nu,1}(x,\rho) \\ w^{T_\nu}_{\nu,2}(x,\rho) \end{pmatrix} \tag{4.1}$$

for $(x,\rho) \in I_{\nu,\varepsilon} \times S_k$, $1 \le \nu \le m$.

Combining Theorem 2.6 and Theorem 3.1 we obtain asymptotic estimates of $y_1(x,\rho)$ and $y_2(x,\rho)$ for $x \in [0,1]$, $\rho \in S_k$. We introduce some notations.

Notations 4.2. For $x \in [0,1]$ let

$$\begin{aligned} R_\pm(x) &:= \int_0^x |\phi_\pm(t)|\,dt \quad \text{with} \quad \phi^2_\pm(t) \;:=\; \max\big\{0, \pm\phi^2(t)\big\}\,, \\ K_\pm(x) &:= \prod_{x_\nu < x} c_\nu b^{T_\nu}_\nu\,, \end{aligned}$$

where in $K_\pm(x)$ the upper sign has to be used if $\rho \in S_{-2} \cup S_{-1}$, the lower sign if $\rho \in S_0 \cup S_1$.

Notations 4.3. (i) For $a \in \mathbb{C}$ let

$$[[a]] \;:=\; (a + T(\rho))[1]\,,$$

where $T(\rho)$ is an exponential sum of the form

$$T(\rho) \;:=\; \sum_{j=1}^{l} c_j e^{\rho d_j} \quad \text{with } c_j \in \mathbb{C}\,,\ d_j \in \mathbb{C}\setminus\{0\} \text{ and } \Re\big(\rho d_j\big) \begin{cases} \le 0 & \text{if } \rho \in S_k\,, \\ < 0 & \text{if } \rho \in \overset{\circ}{S}_k\,. \end{cases}$$

(ii) With $E_k(x,\rho)$ for $-2 \le k \le 1$ and $(x,\rho) \in [0,1] \times S_k$ we denote in the sequel an exponential sum of the form

$$E_k(x,\rho) \;=\; \sum_{n=1}^{l(x)} e^{\rho\alpha_k\beta_{kn}(x)} b_{kn}(x)\,,$$

where $\alpha_{-2} = \alpha_1 = -1$, $\alpha_0 = -\alpha_{-1} = i$,

$$0 \;<\; \delta \;\le\; \beta_{k1}(x) \;<\; \cdots \;<\; \beta_{kl(x)}(x) \;\le\; 2\max\{R_+, R_-\}$$

and where the functions l and b_{kn} are constant in every interval $D_{\nu,\varepsilon}$ $(0 \le \nu \le m)$.

Using these abbreviations we get, using (2.2) and Theorem 3.1, by some elementary but lengthy calculations the following theorem (for details see [24] and — for turning points being zeros — [8]):

Theorem 4.4. *For $\rho \in S_k$ and $x \in \bigcup_{\nu=0}^m D_{\nu,\varepsilon}$ the following estimates hold:*

$$\begin{aligned} y_1(x,\rho) &\overset{(2)}{=} |\phi(x)|^{-1/2} e^{\rho(R_-(x) \pm iR_+(x))}\, K_\pm(x) \underbrace{\Big([1] + E_k(x,\rho)[1]\Big)}_{=\,[[1]]}\,, \\ y_2(x,\rho) &\overset{(2)}{=} |\phi(x)|^{-1/2} e^{\rho(R_-(x) \pm iR_+(x))} e^{\rho(\delta_1(\varepsilon) \pm i\delta_2(\varepsilon))}\, \mathcal{O}(\rho^{-\sigma_0})\,, \end{aligned}$$

where $\delta_1(\varepsilon)+\delta_2(\varepsilon)\geq 0$ *and the upper sign holds for* $k\in\{-2,-1\}$, *the lower sign for* $k\in\{0,1\}$.

Since $y_2(x,\rho)=n_{T_1}^{-1}(\rho)\,w_{1,2}^{T_1}(x,\rho)$ for $(x,\rho)\in[0,x_1-\varepsilon]\times S_k$ Theorem 2.6 yields an estimate of $y_2(x,\rho)$ for $(x,\rho)\in[0,x_1-\varepsilon]\times S_k$ which is more precise than that one in Theorem 4.4. In addition it turns out that the estimates obtained for y_1, y_2 are precise enough to determine the asymptotic behaviour of the eigenvalues of a large class of regular eigenvalue problems defined by (1.1) and (1.2). For the proof of expansion theorems with respect to the eigenfunctions of these eigenvalue problems we cannot use the estimate of y_2. Therefore we have to derive asymptotic estimates for another fundamental system which are more adequate for the proof of expansion theorems.

4.2. The fundamental systems $u_1,\ u_2$ and $y_1,\ u_2$

Notations 4.5. (i) For $\rho\in S_k$ $(-2\leq k\leq 1)$ we set $\tilde{n}_{T_m}(\rho):=e^{-\omega_m\rho R_m}$.
(ii) For fixed $k\in\{-2,-1,0,1\}$ and $(x,\rho)\in[0,1]\times S_k$ we set

$$\begin{pmatrix}u_1(x,\rho)\\ u_2(x,\rho)\end{pmatrix}:=\begin{pmatrix}\tilde{n}_{T_m}(\rho) & 0\\ 0 & \tilde{n}_{T_m}^{-1}(\rho)\end{pmatrix}\left(F_m^{T_m}\right)^{-1}c_m^{-1}\begin{pmatrix}w_{m,1}^{T_m}(x,\rho)\\ w_{m,2}^{T_m}(x,\rho)\end{pmatrix},$$

i. e. $u_j(\,\cdot\,,\rho)$ is the continuation of a normalized multiple of $w_{m,j}^{T_m}(\,\cdot\,,\rho)$ to the whole interval I.

With (3.1) we get for $1\leq\nu\leq m-1$ and $(x,\rho)\in I_{\nu,\varepsilon}\times S_k$

$$\begin{aligned}\begin{pmatrix}u_1(x,\rho)\\ u_2(x,\rho)\end{pmatrix} &= \begin{pmatrix}\tilde{n}_{T_m}(\rho) & 0\\ 0 & \tilde{n}_{T_m}^{-1}(\rho)\end{pmatrix}\left(F_m^{T_m}\right)^{-1}c_m^{-1}\\ &\quad\times C_{m-1}^{-1}(T_{m-1},T_m)\cdots C_\nu^{-1}(T_\nu,T_{\nu+1})\begin{pmatrix}w_{\nu,1}^{T_\nu}(x,\rho)\\ w_{\nu,2}^{T_\nu}(x,\rho)\end{pmatrix}.\end{aligned}\tag{4.2}$$

Because of $\det H_\nu(\omega_\nu\rho;\alpha_\nu,\beta_\nu,\gamma_{\nu+1})=[1]$ we can easily determine the inverses of the connection matrices; these matrices and their inverses have a similar structure, therefore we obtain the estimates of u_1, u_2 in an analogous way to the estimates for y_1 and y_2.

Notations 4.6. For $x\in[0,1]$ let

$$\begin{aligned}\widetilde{R}_\pm(x) &:= R_\pm(1)-R_\pm(x)\,,\\ \widetilde{K}_\pm(x) &:= \prod_{x<x_\nu}c_\nu^{-1}b_\nu^{T_\nu}\end{aligned}$$

with the upper sign in $\widetilde{K}_\pm(x)$ in the case of the sectors S_{-2} and S_{-1}, the lower sign for S_0 and S_1.

Analogously to Theorem 4.4 we derive:

Theorem 4.7. *For $\rho \in S_k$ and $x \in \bigcup_{\nu=0}^{m} D_{\nu,\varepsilon}$ the following estimates hold:*

$$\begin{aligned} u_1(x,\rho) &\overset{(2)}{=} |\phi(x)|^{-1/2} e^{\rho(\tilde{R}_-(x) \pm i\tilde{R}_+(x))} e^{\rho(\delta_1(\varepsilon) \pm i\delta_2(\varepsilon))}\, \mathcal{O}(\rho^{-\sigma_0})\,, \\ u_2(x,\rho) &\overset{(2)}{=} |\phi(x)|^{-1/2} e^{\rho(\tilde{R}_-(x) \pm i\tilde{R}_+(x))}\, \widetilde{K}_\pm(x)[[1]]\,, \end{aligned}$$

where $\delta_1(\varepsilon) + \delta_2(\varepsilon) \geq 0$ and the upper sign holds for $k \in \{-2,-1\}$, the lower sign for $k \in \{0,1\}$.

From

$$W(\rho) \;:=\; W(y_1(\,\cdot\,,\rho), u_2(\,\cdot\,,\rho)) \;=\; 2\rho\alpha e^{\rho(R_-(1) \pm iR_+(1))} K_\pm(1)[[1]] \;\neq\; 0 \;\text{ for } \rho \in S_k \backslash C_k$$

with $\alpha = -\tilde{\alpha} \in \{1,-1,i,-i\}$ and where C_k is a countable set of zeros of the term denoted by [[1]], we infer that $y_1(\,\cdot\,,\rho)$, $u_2(\,\cdot\,,\rho)$ is a fundamental system of solutions of (1.1) for $\rho \in S_k \setminus C_k$. On the other hand it turns out that the estimates obtained in Theorems 4.4 and 4.7 for $y_1(\,\cdot\,,\rho)$, $u_2(\,\cdot\,,\rho)$ are sufficient for the application of the contour–integration method and for the proof of expansion theorems (see Section 6). Notice that $y_1(\,\cdot\,,\rho)$, $u_2(\,\cdot\,,\rho)$ are linearly dependent for $\rho \in C_k$, therefore these solutions cannot be used for the investigation of the spectrum of (1.1), (1.2).

5. The distribution of the eigenvalues

We consider the eigenvalue problem generated by (1.1) and normalized boundary conditions (1.2), i.e.

$$\begin{aligned} U_{j0}(y) &= \alpha_j y^{(k_j)}(0) + \sum_{\mu=0}^{k_j-1} \alpha_{j\mu} y^{(\mu)}(0)\,, \\ U_{j1}(y) &= \beta_j y^{(k_j)}(1) + \sum_{\mu=0}^{k_j-1} \beta_{j\mu} y^{(\mu)}(1)\,, \end{aligned}$$

with $|\alpha_j| + |\beta_j| > 0$ and $1 \geq k_1 \geq k_2 \geq 0$ where $k_1 + k_2$ is (without loss of generality) minimal with respect to all equivalent boundary conditions.

Definition 5.1. Let

$$\theta_j \;:=\; \begin{vmatrix} \alpha_1\,|\phi(0)|^{k_1} & \beta_1 \varepsilon_j^{k_1}\,|\phi(1)|^{k_1} \\ \alpha_2\,|\phi(0)|^{k_2} & \beta_2 \varepsilon_j^{k_2}\,|\phi(1)|^{k_2} \end{vmatrix} \;\neq\; 0 \quad (j = 1,2,3)$$

$$\text{with} \quad \varepsilon_1 \;=\; 1\,, \;\; \varepsilon_2 \;=\; i\,, \;\; \varepsilon_3 \;=\; -i\,.$$

The indefinite eigenvalue problem (1.1), (1.2) is called *regular* if

$$\theta_1 \;\neq\; 0 \;\text{ for }\; \phi^2(0)\phi^2(1) \;>\; 0 \quad\text{and}\quad \theta_2\theta_3 \;\neq\; 0 \;\text{ for }\; \phi^2(0)\phi^2(1) \;<\; 0\,.$$

Remarks 5.2. (i) In the definite case one has $\phi^2(0)\phi^2(1) > 0$, and the regularity condition $\theta_1 \neq 0$ is equivalent to the well–known condition of Birkhoff (see [17], § 4.8).

(ii) It can be checked easily that separated, periodic and antiperiodic boundary conditions are always regular.

Theorem 5.3. *Let* (1.1), (1.2) *be a regular problem. Then there exist two sequences* $\left(\lambda_n^+\right)$, $\left(\lambda_n^-\right)$ *of eigenvalues with the asymptotic distribution*

$$\lambda_n^\pm \;=\; \pm\frac{n^2\pi^2}{R_\pm^2(1)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \qquad (n\in\mathbb{N})\,.$$

In the case $R_+ = 0$ *or* $R_- = 0$ *the corresponding sequence* $\left(\lambda_n^+\right)$ *or* $\left(\lambda_n^-\right)$ *has to be considered empty.*

For the proof of this theorem we estimate the characteristic determinant

$$\Delta(\rho) \;:=\; \det\Big(U_j\big(y_i(\,\cdot\,,\rho)\big)\Big)_{i,j=1,2}\,,$$

using the fundamental system $y_1,\ y_2$ of (1.1) defined by (3.1).

For $\rho\in S_{-1}$ $\Delta(\rho)$ can be represented as an asymptotic exponential sum of the form

$$\Delta(\rho) \;=\; h(\rho)\sum_{\mu=1}^{p}\hat c_\mu e^{i\rho\vartheta_\mu}[1] \tag{5.1}$$

with $h(\rho)\neq 0$, $\hat c_\mu\in\mathbb{C}$ and $\vartheta_1<\vartheta_2<\cdots<\vartheta_p$. The proof of (5.1) is performed like the proof of the main result in [6].

A useful tool for the proof of Theorem 5.3 is the following lemma, which is an immediate consequence of Langer [12], Theorem 7:

Lemma 5.4. *Let* $p\in\mathbb{N}\setminus\{1\}$, $\vartheta_1<\vartheta_2<\cdots<\vartheta_p$, $\varepsilon_\mu:\mathbb{C}\to\mathbb{C}$ *with* $\lim_{|\rho|\to\infty}\varepsilon_\mu(\rho)=0$, $\hat c_\mu\in\mathbb{C}$ $(1\le\mu\le p)$ *with* $\hat c_1\neq 0$, $\hat c_p\neq 0$ *and*

$$F(\rho) \;:=\; \sum_{\mu=1}^{p}(\hat c_\mu+\varepsilon_\mu(\rho))e^{i\rho\vartheta_\mu}\;.$$

Then the zeros of F *fulfill the estimates*

$$\rho_n \;=\; \pm\frac{2n\pi}{\vartheta_p-\vartheta_1}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \qquad (n\in\mathbb{N})$$

in the right respectively left half plane.

We use this lemma to estimate the zeros of Δ, i. e. the square roots of the eigenvalues of (1.1), (1.2). For the determination of the numbers ϑ_1 and $\hat c_1$ in (5.1) we have to consider all terms in the estimates of the product $\prod_{\nu=1}^{m-1}C_\nu(T_\nu,T_{\nu+1})$ (which are obtained, using Theorem 3.1) with a factor $e^{-i\rho\int_{x_1}^{x_m}|\phi|}$, and for the determination of ϑ_p and $\hat c_p$ we have to consider all terms with $e^{i\rho\int_{x_1}^{x_m}|\phi|}$. Taking into account every possible transition from one turning point to the next one the multiplication of these matrices yields

$$\vartheta_1 \;=\; -R_+ \;=\; -\vartheta_p \quad\text{and}\quad \tilde c_1 \;=\; h(\rho)\hat c_1 \;\neq\; 0\,,\;\; \tilde c_p \;=\; h(\rho)\hat c_p \;\neq\; 0\,.$$

Applying Lemma 5.4 to (2.4) we obtain (for $\rho \in S_0 \cup S_{-1}$)

$$\rho_n = \frac{2n\pi}{\vartheta_p - \vartheta_1}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) = \frac{n\pi}{R_+}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad (n \in \mathbb{N}),$$

that is

$$\lambda_n^+ = \frac{n^2\pi^2}{R_+^2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad (n \in \mathbb{N}).$$

The discussion of the characteristic determinant Δ in the sectors S_1 and S_2 can be reduced to the discussion in the sectors S_{-1} and S_0 by replacing $\lambda = \rho^2$ by $-\lambda$ and ϕ^2 by $-\phi^2$. Then R_+ and R_- change their places and we get the sequence

$$\lambda_n^- = -\frac{n^2\pi^2}{R_-^2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad (n \in \mathbb{N}).$$

The discussion of Δ in the sector S_k ($k \in \mathbb{Z}$) results from that in the sector $S_{k\pm 4}$ by replacing ρ by $-\rho$.

Therefore we have proved Theorem 5.3 (for further details see [24]).

6. An expansion theorem

To complete the investigation of the indefinite eigenvalue problem (1.1), (1.2) we want to prove an expansion theorem with respect to its eigenfunctions. The method used is similar to that applied by the authors in [7].

6.1. Notations and results

Let $k \in \mathbb{Z}$ and for $\rho \in S_k$ $y_1(\,\cdot\,,\rho)$, $y_2(\,\cdot\,,\rho)$ be a fundamental system of (1.1) in the interval $[0,1]$. As before let

$$W(y_1,y_2) := W(y_1(\,\cdot\,,\rho), y_2(\,\cdot\,,\rho)) = \begin{vmatrix} y_1(\,\cdot\,,\rho) & y_2(\,\cdot\,,\rho) \\ y_1'(\,\cdot\,,\rho) & y_2'(\,\cdot\,,\rho) \end{vmatrix}$$

be the Wronskian of y_1 and y_2.

For $f \in L_2[0,1]$ the solution y of the inhomogeneous equation

$$\ell(y) = \rho^2\phi^2(x)y + f(x)$$

is of the following form if $\lambda = \rho^2$ is no eigenvalue of the considered regular eigenvalue problem:

$$y(x,\rho) = \int_0^1 G(x,t,\rho^2)f(t)\,dt$$

with the *Green's function* G, which is defined by

$$G\big(x,t,\rho^2\big) := \frac{H(x,t,\rho)}{\Delta(\rho)} \quad \text{for} \quad (x,t) \in [0,1]^2, \quad \rho \in S_k,$$

where

$$H(x,t,\rho) \;:=\; \begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & g(x,t,\rho) \\ U_1(y_1) & U_1(y_2) & U_1(g(\cdot,t,\rho))(t) \\ U_2(y_1) & U_2(y_2) & U_2(g(\cdot,t,\rho))(t) \end{vmatrix}$$

and

$$g(x,t,\rho) \;:=\; \frac{1}{W(y_1,y_2)} \begin{cases} y_1(x,\rho)y_2(t,\rho) & \text{if} \quad x \;\le\; t\,, \\ y_2(x,\rho)y_1(t,\rho) & \text{if} \quad x \;>\; t\,. \end{cases}$$

We define the contour integral

$$S_R(f)(x) \;:=\; \frac{1}{2\pi i}\int_0^1 \int_{\Gamma_{R^2}} G(x,t,\lambda)\,d\lambda\,\phi^2(t)f(t)\,dt \quad (x \in [0,1]) \tag{6.1}$$

where $R > 0$ is chosen such that there is no eigenvalue on the circle Γ_{R^2} of radius R^2. Since the eigenvalues $\lambda_n^\pm$ $(n \in \mathbb{N})$ are poles of the Green's function and since $\operatorname{res}_{\lambda_n^\pm} G(x,t,\lambda)$ can be represented by products of eigen– and associated functions (e. a. f.) of (1.1), (1.2) and of the corresponding adjoint problem, which we denote by y_n and z_n, the integral $S_R(f)(x)$ represents a partial sum of the expansion of f into a series in e. a. f. of (1.1), (1.2):

$$S_R(f)(x) \;=\; \sum_{n\in M_R} \alpha_n y_n(x)$$

with

$$\alpha_n \;:=\; \int_0^1 \phi^2(t)f(t)\overline{z_n(t)}\,dt\,, \quad M_R \;:=\; \left\{n \in \mathbb{N} \mid |\lambda_n^\pm| < R^2\right\}.$$

If f has an absolutely continuous derivative f' with $f'' \in L[0,1]$ and if $U_j(f) = 0$ $(j = 1,2)$ then we infer from (6.1) by partial integration and using the properties of the Green's function that

$$S_R(f)(x) \;=\; f(x) + \frac{1}{2\pi i}\int_{\Gamma_{R^2}} \int_0^1 \frac{G(x,t,\lambda)}{\lambda}\,\ell(f)(t)\,dt\,d\lambda\,,$$

provided all integrals exist.

Assumption 6.1. For each $\nu \in \{1,\dots,m\}$ with $l_\nu = -1$ let $\Re\big(\sqrt{1+4A_\nu}\,\big) \le \frac{1}{2}$.

Below we prove that under Assumption 6.1 the solutions of (1.1) are bounded and that there exists a sequence $(R_n)_{n\in\mathbb{N}}$ and constants K, $\mu_0 > 0$ with

$$|G(x,t,\lambda)| \;\le\; K\,|\lambda|^{-\mu_0} \quad \text{for} \quad \lambda \in \bigcup_{n\in\mathbb{N}} \Gamma_{R_n^2} \quad \text{and} \quad (x,t) \in [0,1]^2\,; \tag{6.2}$$

obviously this implies the uniform convergence of $S_{R_n}(f)(x)$ for $R_n \to \infty$ to $f(x)$, i. e. we have proved the following expansion result:

Theorem 6.2. *Let Assumption* 6.1 *hold. If f has an absolutely continuous derivative f' with $f'' \in L[0,1]$ and if $U_j(f) = 0$ $(j = 1,2)$ then f can be represented by a uniformly convergent series in eigen– and associated functions of* (1.1), (1.2).

Since the set of all functions satisfying the assumptions of Theorem 6.2 is dense in $L_2[0,1]$ we obtain as a corollary of the preceding theorem:

Theorem 6.3. *If Assumption* 6.1 *holds then the set of all eigen– and associated functions of* (1.1), (1.2) *is complete in* $L_2[0,1]$.

In the next subsection we prove estimate (6.2); a generalization of (6.2) is proved in the last subsection.

6.2. Green's function

Without loss of generality we restrict ρ to $\bigcup_{k=-2}^{1} S_k$. Let $k \in \{-2,-1,0,1\}$ be fixed and for $\rho \in S_k$ let $y_1(\,\cdot\,,\rho)$, $u_2(\,\cdot\,,\rho)$ be the fundamental system of (1.1) considered in Subsection 4.2.

From the definition of Green's function we derive

$$G\big(x,t,\rho^2\big) = g(x,t,\rho) + \frac{1}{W(y_1,u_2)\Delta(\rho)}\big\{y_1(x,\rho)u_2(t,\rho)D_1(\rho) + y_1(x,\rho)y_1(t,\rho)D_2(\rho) \\ - u_2(x,\rho)u_2(t,\rho)D_3(\rho) - u_2(x,\rho)y_1(t,\rho)D_4(\rho)\big\}$$

for $(x,t) \in [0,1]^2$, $\rho \in S_k$ with

$$\begin{aligned} D_1(\rho) &:= \begin{vmatrix} U_1(u_2) & U_{10}(y_1) \\ U_2(u_2) & U_{20}(y_1) \end{vmatrix}, \quad D_2(\rho) := \begin{vmatrix} U_1(u_2) & U_{11}(u_2) \\ U_2(u_2) & U_{21}(u_2) \end{vmatrix}, \\ D_3(\rho) &:= \begin{vmatrix} U_1(y_1) & U_{10}(y_1) \\ U_2(y_1) & U_{20}(y_1) \end{vmatrix}, \quad D_4(\rho) := \begin{vmatrix} U_1(y_1) & U_{11}(u_2) \\ U_2(y_1) & U_{21}(u_2) \end{vmatrix}. \end{aligned} \tag{6.3}$$

From (4.1), (4.2), the formulas for the connection matrices and LANGER's results we derive

Lemma 6.4. *Require that Assumption* 6.1 *holds and let* $1 \le \nu \le m$ *and* $0 < \varepsilon < \varepsilon_0 := \min\{x_1, x_2 - x_1, \ldots, x_m - x_{m-1}, 1 - x_m\}$. *Then the following estimates hold:*

$$\begin{aligned} y_1(x,\rho) &\overset{(2)}{=} e^{\rho(R_-(x) \pm iR_+(x))}\rho^{(1-\mu_\nu)/2}\,\mathcal{O}(1)\,, \\ u_2(x,\rho) &\overset{(2)}{=} e^{\rho(\tilde{R}_-(x) \pm i\tilde{R}_+(x))}\rho^{(1-\mu_\nu)/2}\,\mathcal{O}(1)\,, \end{aligned} \tag{6.4}$$

for $(x,\rho) \in (x_\nu - \varepsilon, x_\nu + \varepsilon) \times S_k$ *with* $|\xi_\nu(x,\rho)| \le N$, *where* $N > 0$ *denotes a fixed constant. The upper sign holds in the sectors* S_{-2} *and* S_{-1}, *the lower in* S_0 *and* S_1.

Proof. From [14], formulae (39) and (12), we infer that the fundamental systems $v_{\nu,l,1}$, $v_{\nu,l,2}$, considered in Theorem 2.4, satisfy for $|\xi_\nu(x,\rho)| < N$

$$v_{\nu,l,j}(x,\rho) \overset{(2)}{=} \begin{cases} \xi_\nu^{\mu_\nu/2-\eta_\nu}(x,\rho)\left(1 + \dfrac{\psi_\nu(x)}{\rho_{\mu_\nu}}\right)\mathcal{O}(1)\,, & \text{if} \quad \eta_\nu \ne 0\,, \\ \xi_\nu^{\mu_\nu/2}(x,\rho)\log\xi_\nu(x,\rho)\left(1 + \dfrac{\psi_\nu(x)}{\rho_{\mu_\nu}}\right)\mathcal{O}(1)\,, & \text{if} \quad \eta_\nu = 0\,, \end{cases} \tag{6.5}$$

for $x \in (x_{\nu-\varepsilon}, x_{\nu+\varepsilon})$, $1 \le j \le 2$ and $l \in \mathbb{Z}$.

From (2.5) and the proof of Theorem 2.6 we infer that

$$\begin{aligned} w^{T_\nu}_{\nu,l,1}(x,\rho) &= \Big(\tilde a^{T_\nu}_{\nu,l}(\rho)v_{\nu,l.1}(x,\rho) + \tilde b^{T_\nu}_{\nu,l}(\rho)v_{\nu,l,2}(x,\rho)\Big)\kappa_{\nu0}^{-\mu_\nu/4}2^{-\mu_\nu/2}\rho^{(1-\mu_\nu)/2}\,, \\ w^{T_\nu}_{\nu,l,2}(x,\rho) &= \Big(\tilde c^{T_\nu}_{\nu,l}(\rho)v_{\nu,l,1}(x,\rho) + \tilde d^{T_\nu}_{\nu,l}(\rho)v_{\nu,l,2}(x,\rho)\Big)\kappa_{\nu0}^{-\mu_\nu/2}2^{-\mu_\nu/2}\rho^{(1-\mu_\nu)/2}\,, \end{aligned} \tag{6.6}$$

where the coefficients $\tilde a^{T_\nu}_{\nu,l}$, $\tilde b^{T_\nu}_{\nu,l}$, $\tilde c^{T_\nu}_{\nu,l}$, $\tilde d^{T_\nu}_{\nu,l}$ are all of the form $[K_\nu]$ with constants K_ν depending on the type T_ν.

Combining Theorem 2.4 and Theorem 3.1 we get, using (4.1), that

$$\begin{aligned} y_1(x,\rho) &\overset{(2)}{=} K_\pm(x_\nu)e^{\rho(R_-(x_\nu)\pm iR_+(x_\nu))}\Big\{([1]) + E^{(1)}_k(x_\nu,\rho)[1]w^{T_\nu}_{\nu,l}(x,\rho) \\ &\qquad + \Big(\mathcal{O}(\rho^{-\sigma_0}) + E^{(2)}_k(x_\nu,\rho)[1]\Big)w^{T_\nu}_{\nu,2}(x,\rho)\Big\}\,, \\ u_2(x,\rho) &\overset{(2)}{=} \frac{c_m}{b_m^{T_m}}\widetilde K_\pm(x_{\nu-1})e^{\rho(\tilde R_-(x_\nu)\pm i\tilde R_+(x_\nu))} \\ &\quad\times\Big\{\Big(\mathcal{O}(\rho^{-\sigma_0}) + \widetilde E^{(3)}_k(x_\nu,\rho)[1]\Big)\,w^{T_\nu}_{\nu,1}(x,\rho) \\ &\qquad + \Big([1] + \widetilde E^{(4)}_k(x_\nu,\rho)[1]\Big)w^{T_\nu}_{\nu,2}(x,\rho)\Big\}\,, \end{aligned} \tag{6.7}$$

with bounded exponential sums $\widetilde E^{(j)}_k(x_\nu,\rho)$ (like in Notations 4.3). For $|\xi_\nu(x,\rho)| \le N$ is therefore (under Assumption 6.1),

$$v_{\nu,l,j}(x,\rho) \overset{(2)}{=} \mathcal{O}(1) \quad \text{for} \quad |\rho| \to \infty\,,$$

hence

$$w^{T_\nu}_{\nu,l,j}(x,\rho) \overset{(2)}{=} \rho^{(1-\mu_\nu)/2}\mathcal{O}(1) \quad \text{for} \quad |\rho| \to \infty\,.$$

Since the exponential sums in (6.7) are bounded we receive the assertion of the lemma. □

Lemma 6.5. *Formula* (6.4) *is also valid* (*without Assumption* 6.1) *for* $(x,\rho) \in I_{\nu,\varepsilon} \times S_k$ *with* $|\xi_\nu(x,\rho)| > N$.

The proof of this lemma is essentially the same as the proof of [7], Lemma 1, (where $l_\nu \in \mathbb{N}$).

Lemma 6.6. (Tamarkin [22].) *For every exponential sum* $T(\rho)$ (*see Notations* 4.3) *and every* $\delta > 0$ *there exists a* $C(\delta) > 0$ *such that*

$$|1 + T(\rho)| \ \ge\ C(\delta) \quad \textit{for all} \quad \rho \in S_k(\delta)$$

with

$$S_k(\delta) \ :=\ \{\rho \in S_k \mid \operatorname{dist}(\rho,\sigma') \ge \delta\}\,, \quad \sigma' \ :=\ \{\rho \in \mathbb{C} \mid 1 + T(\rho) = 0\}\,.$$

Now we can estimate the different terms in (6.3).

(i) From Section 4 we recall

$$W(\rho) \ =\ 2\rho\alpha e^{\rho(R_-(1)\pm iR_+(1))}K_\pm(1)\big([1] + \widetilde E_k(1,\rho)[1]\big)$$

with an exponential sum $\widetilde{E}_k(1,\rho)$ having a similar form as the exponential sum $E(x,\rho)$, defined in Notations 4.3, (ii), and which is such that the zeros of W have the same density as the zeros of Δ. From Lemma 6.6 we derive the existence of $C_1(\delta_0) > 0$ for every $\delta_0 > 0$ with

$$|W(\rho)| \;\geq\; |\rho|\, e^{\Re(\rho(R_-(1) \pm iR_+(1)))} C_1(\delta_0) \tag{6.8}$$

for $\rho \in S_k^W(\delta_0)$ with $|\rho| > r_{\delta_0}$, where $S_k^W(\delta_0)$ is the set of all $\rho \in S_k$ with $|\rho - \rho_j| \geq \delta_0$ for the zeros ρ_j of $1 + \widetilde{E}_k(1,\rho)$.

(ii) Since $y_1(1,\rho)$ and $u_2(0,\rho)$ grow exponentially for $\rho \in S_k$ with $|\rho| \to \infty$ we get

$$\begin{aligned}\Delta(\rho) \;&=\; \begin{vmatrix} U_{11}(y_1) & U_{10}(u_2) \\ U_{21}(y_1) & U_{20}(u_2) \end{vmatrix} [1] \\ &=\; \rho^{k_1+k_2} e^{2\rho(R_-(1) \pm iR_+(1))} [[c_0]] \quad \text{with} \quad c_0 \;\neq\; 0\,.\end{aligned}$$

Lemma 6.6 shows that there exist a $C_2(\delta_0) > 0$ with

$$|\Delta(\rho)| \;\geq\; |\rho|^{k_1+k_2} e^{\Re(2\rho(R_-(1) \pm iR_+(1)))} C_2(\delta_0) \tag{6.9}$$

for $\rho \in S_k^{\Delta}(\delta_0)$ with $|\rho| > r_{\delta_0}$, where $S_k^{\Delta}(\delta_0)$ is the set of all $\rho \in S_k$ with $|\rho - \rho_j| \geq \delta_0$ for the square roots ρ_j of the eigenvalues of (1.1), (1.2).

(iii) In the same way we infer from (6.3) and Theorems 4.4, 4.7 and 6.5 that

$$D_j(\rho) \;=\; \rho^{k_1+k_2+1/2-\widetilde{\mu}} e^{\rho(R_-(1) \pm iR_+(1))} \mathcal{O}(1) \quad \text{for} \quad \rho \in S_k\,, \quad 1 \;\leq\; j \;\leq\; 4 \tag{6.10}$$

with $\widetilde{\mu} \in (0, \frac{1}{2}]$.

Assume now that Assumption 6.1 holds; then we get finally:

With (6.4) and (6.8) it follows $g(x,t,\rho) = \rho^{-2\tilde{\mu}} \mathcal{O}(1)$ for $(x,t) \in [0,1]^2$, $\rho \in S_k^W(\delta_0)$ with $|\rho| > r_{\delta_0}$. From this and the estimates (6.8), (6.9) and (6.10) we obtain

$$\bigl|G\bigl(x,t,\rho^2\bigr)\bigr| \;\leq\; c(\delta_0)\, |\rho|^{-2\mu_0}$$

for $(x,t) \in [0,1]^2$ and $\rho \in S_k^{\Delta}(\delta_0) \cap S_k^W(\delta_0)$ with $|\rho| > r_{\delta_0}$ and a constant $\mu_0 \in \bigl(0, \frac{1}{2}\bigr]$. From the distribution of the zeros of W and Δ it is obvious that for $\delta_0 > 0$ (sufficiently small) there exist a sequence $(R_n)_{n \in \mathbb{N}}$ with $R_n \to \infty$ $(n \to \infty)$ such that (6.2) is valid.

Notice that we do not have to require that Assumption 6.1 is fulfilled in order to prove that (6.10) holds for $|\rho^2| = R_n$, $n \in \mathbb{N}$.

Remark 6.7. The estimates for the fundamental systems and the connection matrices are also valid if we allow that $l_\nu = 0$ for one or several $\nu \in \{1, \dots, m\}$, i. e. if the weight function ϕ^2 has no turning point in x_ν, but the function χ has a pole of first or second order in x_ν. The asymptotic distribution of the eigenvalues is also preserved if we exclude the cases $\bigl(l_1 = 0$ and $\phi^2(x) > 0$ for $x \in I_{1,\varepsilon}\bigr)$ and $\bigl(l_m = 0$ and $\phi^2(x) > 0$ for $x \in I_{m,\varepsilon}\bigr)$.

6.3. Expansion theorem in the general case

Let in this subsection $\nu_1, \ldots, \nu_\kappa$ denote those indices $\nu \in \{1, \ldots, m\}$ for which $l_\nu = -1$ and for which in addition Assumption 6.1 is not fulfilled. Moreover let N and ε be chosen as in Lemma 6.4.

Then we replace the estimates of Lemma 6.4 for $x\big(x_{\nu_j} - \varepsilon, x_{\nu_j} + \varepsilon\big)$ by the following:

Lemma 6.8. *For* $\nu \in \{\nu_1, \ldots, \nu_\kappa\}$, $|\xi_\nu(x, \rho)| \leq N$ *and* $(x, \rho) \in \big(x_{\nu_j} - \varepsilon, x_{\nu_j} + \varepsilon\big) \times S_k$ *is*

$$\begin{aligned} y_1(x, \rho) \;&\overset{(2)}{=}\; e^{\rho(R_-(x) \pm i R_+(x))} \rho^{\mu_\nu/2 - \eta_\nu}\, \mathcal{O}\Big(|x - x_\nu|^{1/4 - 1/2\, \Re\sqrt{1+4A_\nu}}\Big)\,, \\ u_2(x, \rho) \;&\overset{(2)}{=}\; e^{\rho(\tilde{R}_-(x) \pm i \tilde{R}_+(x))} \rho^{\mu_\nu/2 - \eta_\nu}\, \mathcal{O}\Big(|x - x_\nu|^{1/4 - 1/2\, \Re\sqrt{1+4A_\nu}}\Big)\,, \end{aligned} \tag{6.11}$$

where the upper sign has to be used for the sectors S_{-2}, S_{-1} *and the lower sign for the sectors* S_0, S_1.

Proof. Since $l_\nu = -1$ we have near x_ν

$$\left|\frac{\xi_\nu(x, \rho)}{\rho}\right| \;=\; \mathcal{O}\bigg(\bigg|\int_{x_\nu}^{x} \phi(t)\, dt\bigg|\bigg) \;=\; \mathcal{O}\big(|x - x_\nu|^{1/2}\big)\,.$$

Moreover $l_\nu = -1$ yields $\mu_\nu = 1$ and $\eta_\nu = \sqrt{1 + 4A_\nu}$, where we choose the root such that $-\frac{\pi}{2} < \arg\, \eta_\nu \leq \frac{\pi}{2}$.

Hence near x_ν

$$\left|\left[\frac{\xi_\nu(x, \rho)}{\rho}\right]^{\mu_\nu/2 - \eta_\nu}\right| \;=\; \mathcal{O}\Big(|x - x_\nu|^{1/4 - 1/2 \Re\sqrt{1+4A_\nu}}\Big)\,.$$

Together with (6.5), (6.6) and (6.7) this proves the assertion of the lemma. □

Lemma 6.8 shows together with the representation of the Green's function and the proofs of Lemma 6.4 and Lemma 6.5 that (6.2) is no longer valid if Assumption 6.1 is not fulfilled. Instead one gets in this case, using Lemma 6.8, that there exists a sequence $(R_n)_{n\in\mathbb{N}}$ and constants $K,\ \mu_0 > 0$ with

$$\begin{aligned} |G(x, t, \lambda)| \;\leq\; K h(x, t)\, |\lambda|^{-\mu_0} \quad \text{for} \quad & \lambda \in \bigcup_{n\in\mathbb{N}} \Gamma_{R_n^2} \\ \text{and} \quad & x\,,\, t \in [0, 1] \setminus \{x_{\nu_1}, \ldots, x_{\nu_\kappa}\}\,, \end{aligned} \tag{6.12}$$

where

$$h(x, t) \;=\; \prod_{j=1}^{\kappa} |(x - x_{\nu_j})(t - x_{\nu_j})|^{1/4 - 1/2 \Re\sqrt{1+4A_{\nu_j}}}\,.$$

Let $\delta_0 := \min\big\{\frac{\mu_\nu}{2} - \Re\, \eta_\nu \mid 1 \leq \nu \leq m\big\}$ (which is negative if Assumption 6.1 is not fulfilled).

Then we get with (6.12) and Lemma 6.8 analogously to the proof of Theorem 6.2:

Theorem 6.9. *Let f satisfy the assumptions of Theorem 6.2 and require in addition that the functions*

$$f_{\nu_j} \; : \; [0,1] \longrightarrow \mathbb{R}\,, \quad t \mapsto f(t)\left|t - x_{\nu_j}\right|^{1/4-1/2\Re\sqrt{1+4A_{\nu_j}}}$$

are integrable for $1 \leq j \leq \kappa$. Then

$$\lim_{n\to\infty} S_{R_n}(f)(x) \;=\; f(x)$$

holds uniformly for x in any compact subset $K \subset I \setminus \{x_{\nu_1}, \ldots, x_{\nu_\kappa}\}$; here R_n is chosen as in (6.12).

References

[1] Atkinson, F. V., and Mingarelli, A. B.: Asymptotics of the Number of Zeros and of the Eigenvalues of General Weighted Sturm–Liouville Problems, J. Reine Angew. Math. **375/376** (1987), 380–393

[2] Curgus, B., and Langer, H.: A Krein–Space Approach to Symmetric Ordinary Differential Operators with an Indefinite Weight Function, J. Diff. Equ. **79** (1989), 31–61

[3] Daho, K., and Langer, H.: Sturm–Liouville Operators with an Indefinite Weight Function, Proc. Roy. Soc. Edinburgh **78 A** (1977), 161–191

[4] Dijksma, A., Gohberg, I., Kaashoek, M. A., and Mennicken, R. (Eds.): Contributions to Operator Theory in Spaces with an Indefinite Metric, The Heinz Langer anniversary volume on the occasion of his 60th birthday, Operator Theory: Advances and Applications **106** (1998), Birkhaeuser, Basel

[5] Dorodnicyn, A. A.: Asymptotic Laws of Distribution of the Characteristic Values for Certain Special Forms of Differential Equations of the Second Order, Amer. Math. Soc. Translations **16** (1960), 1–101

[6] Eberhard, W., and Freiling, G.: The Distribution of the Eigenvalues for Second Order Eigenvalue Problems in the Presence of an Arbitrary Number of Turning Points, Results Math. **21** (1992), 24–41

[7] Eberhard, W., and Freiling, G.: An Expansion Theorem for Eigenvalue Problems with Several Turning Points, Analysis **13** (1993), 301–308

[8] Eberhard, W., Freiling, G., and Schneider, A.: Connection Formulas for Second Order Differential Equations with a Complex Parameter and Having an Arbitrary Number of Turning Points, Math. Nachr. **165** (1994), 205–229

[9] Fedoryuk, M. V.: Asymptotic Analysis, Springer–Verlag, Berlin, 1993

[10] Freiling, G., and Yurko, V.: Inverse Spectral Problems for Differential Equations on the Half–Line with Turning Points, Journal of Differential Equations **154** (1999), 419–453

[11] Langer, R. E.: The Boundary Problem Associated with a Differential Equation in Which the Coefficient of the Parameter Changes Sign, Trans. Amer. Math. Soc. **31** (1929), 1–24

[12] Langer, R. E.: On the Zeros of Exponential Sums and Integrals, Bull. Am. Math. Soc. **37** (1931), 213–239

[13] Langer, R. E.: On the Asymptotic Solutions of Ordinary Differential Equations, with an Application to the Bessel Functions of Large Order, Trans. Amer. Math. Soc. **33** (1931), 23–64

[14] Langer, R. E.: On the Asymptotic Solutions of Ordinary Differential Equations, with Reference to the Stokes' Phenomenon about a Singular Point, Trans. Amer. Math. Soc. **37** (1935), 397–416

[15] Langer, R. E.: The Asymptotic Solutions of a Linear Differential Equation of the Second Order with Two Turning Points, Trans. Amer. Math. Soc. **90** (1959), 113–142

[16] McHugh, J.: An Historical Survey of Ordinary Linear Differential Equations with a Large Parameter and Turning Points, Arch. Hist. Exact. Sci. **7** (1970), 277 – 324

[17] Naimark, M. A.: Linear Differential Operators, Part I, Frederick Ungar Publishing Co., 1967

[18] Olver, F. W. J.: Asymptotics and Special Functions, Acad. Press, New York, 1974

[19] Schneider, A., and Vonhoff, R.: Indefinite Boundary Eigenvalue Problems in a Pontrjagin Space Setting, Result. Math. **35** (1999), 325 – 354

[20] Schröder, M.: On the Asymptotic Distribution of the Eigenvalues of Singular Sturm – Liouville Problems with an Indefinite Weight Function, Results in Mathematics **32** (1997), 338 – 351

[21] Stakun, A. A.: Properties of a Differential Operator with a Multiple Turning Point, Diff. Equ. **23** (1987), 667 – 671

[22] Tamarkin, J.: Some General Problems of the Theory of Ordinary Linear Differential Equations and Expansion of an Arbitrary Function in Series of Fundamental Functions, Math. Z. **27** (1928), 1 – 54

[23] Wasow, W.: Linear Turning Point Theory, Springer–Verlag, Berlin, 1985

[24] Wilcken–Stoeber, K.: Über eine Klasse indefiniter Eigenwertprobleme zweiter Ordnung mit einer beliebigen Zahl von Nullstellen und Polen in der Gewichtsfunktion, Dissertation, Cuvillier Verlag Göttingen, 1998

[25] Zyablov, A. A.: On the Basis Properties of the Eigenfunctions of a Model Problem with an Indefinite Weight, Mat. Zametki **54** (1993), 149 – 151 (Translation: Math. Notes **54** (1993), 1079 – 1080)

FB 11 – Mathematik
Universität Duisburg
D–47048 Duisburg
Germany
E – mail:
freiling@math.uni–duisburg.de