Open Loop Stackelberg Equilibria in Linear-Quadratic Differential Games

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Abstract: We present existence and uniqueness results for a hierarchical or Stackelberg equilibrium in a two player differential game with open loop information structure. There is known a convexity condition ensuring existence of a Stackelberg equilibrium which was derived by Simaan and Cruz [19] . This condition applys to games with a rather non-conflicting structure of their cost criteria. By another approach we obtain new sufficient existence conditions for an open loop equilibrium in feedback synthesis in terms of solvability of two symplectic (symmetric) Riccati differential equations and a coupled system of Riccati matrix differential equations. The latter coupled system also appears as a necessary condition. In case that the convexity condition holds, both symplectic equations are of standard type and admit globally a positive semidefinite solution. But the conditions also apply to conflicting situations like zero-sum or nearly zero-sum games. Then the corresponding Riccati differential equations may be of H_∞-type.

We also obtain different unicity conditions using a Lyapunov-type functions approach. The case of time invariant parameters is discussed in more detail and we present a numerical example.

1 Introduction

In non-cooperative game theory the concept of hierarchical or Stackelberg games is eminently important, as it was pointed out for instance in [19] . Different hierarchical structures as well as different information patterns have been investigated (see [16], [5], [17], [15], [20], [13], [2]).

The purpose of this paper is to study a two player Stackelberg differential game under open loop information pattern, where the performance criteria of the players are of *quadratic* type and where a linear differential equation describes the constraints to the state vector. In the case of two players, which we assume for reasons of simplicity, we are given a differential equation in a fixed time interval $[t_0, t_f]$

$$\dot{x} = A(t)x + B_1(t)u_1 + B_2(t)u_2, \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t), x_0 \in \mathbf{R}^n, A(t) \in \mathbf{R}^{n\times n}, B_i(t) \in \mathbf{R}^{n\times m_i}, u_i(t) \in \mathbf{R}^{m_i}, i = 1, 2$; and the performance criteria are

$$J_i = \frac{1}{2}x^T(t_f)K_{if}x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^T(t)Q_i(t)x(t) + \sum_{j=1}^{2} u_j^T(t)R_{ij}(t)u_j(t))\,dt \quad i = 1, 2; \tag{1.2}$$

with the matrices $K_{if}, Q_i(t), R_{ij}(t), 1 \le i, j \le 2$, being symmetric and of appropriate size. Throughout this paper we assume that $R_{ii}^{-1}(t)$ exists for $t \in [t_0, t_f]$.
In order to apply the approach, as presented in [19], $x(t)$ has to be absolutely continuous, the controls $u_i \in L_2^{m_i}([t_0, t_f])$, while the matrices $A, B_i, Q_i, R_{ij}, 1 \le i, j \le 2$ can be considered continuous. Later on we will also deal with constant coefficient matrices, since this comprises important practical cases and is also of sufficient mathematical simplicity to obtain valuable results.
We also assume the player 2 to be the leader, i.e. he is seeking a strategy $u_2^*(t)$, a function of time only in open loop information structure, that he announces before the game starts knowing how the follower reacts to any of his choices. The follower will then calculate his strategy $u_1^*(t)$ also as a function of time only.
More formally, if we define the sets Γ_1, Γ_2 of admissible controls for (1.1) and (1.2), i.e. sets of functions such that (1.1) is solvable for each $u_1 \in \Gamma_1$ and $u_2 \in \Gamma_2$ and that J_1, J_2 exists, respectively, then a Stackelberg equilibrium is defined as follows:
The optimal reaction set of player 1 (the follower) to a control $u_2 \in \Gamma_2$ is

$$R_1(u_2) = \{\gamma \in \Gamma_1 \mid J_1(\gamma, u_2) \le J_1(u_1, u_2) \quad \text{for} \quad \text{all} \quad u_1 \in \Gamma_1\}.$$

If player 2 is leading then $u_2^* \in \Gamma_2$ is called a Stackelberg equilibrium for player 2 if for all $u_2 \in \Gamma_2$

$$sup_{\gamma \in R_1(u_2^*)} J_2(\gamma, u_2^*) \le sup_{\gamma \in R_1(u_2)} J_2(\gamma, u_2).$$

Then

$$u_1^* \in R_1(u_2^*)$$

is an optimal Stackelberg strategy for the follower.
From [19],page 545, Proposition 4.2, we cite

Theorem 1.1 *In the differential game, as given by (1.1), (1.2), let player 2 be the leader. If the following convexity condition*

$$(C) \begin{cases} R_{11}(t) > 0, R_{22}(t) > 0, & t \in [t_0, t_f] \\ R_{21}(t) \ge 0, Q_1(t) \ge 0, Q_2(t) \ge 0, & t \in [t_0, t_f] \\ K_{1f} \ge 0, K_{2f} \ge 0 \end{cases}$$

holds then there exists a unique open loop Stackelberg equilibrium u_1^*, u_2^* in the set of admissible controls $L_2^{m_1}([t_0, t_f]), L_2^{m_2}([t_0, t_f])$, respectively.

Remark 1.2 Though conditions (C) are sufficient for the existence of a unique Stackelberg equilibrium they are rather restrictive, as it was pointed out in [19]. For instance, in zero sum games or "nearly" zero sum games (C) will not hold.
From [19], page 544 one also can derive weaker, but unfortunately implicit, conditions for existence of a unique Stackelberg equilibrium.
If the operators S_1 or S_2 in [19], pages 543, 544 are not invertible, i.e the so called decision operator (see [14], p. 100 for Nash-Games) is not invertible, then there could exist nonunique Stackelberg equilibria. This is only possible if Q_1, Q_2 or R_{21} cease to be positive semidefinite (assuming $K_{1f}, K_{2f} \ge 0$) since we always can normalize to $R_{11}, R_{22} > 0$.

Therefore we are interested to obtain conditions for uniqueness of equilibria as well as existence conditions.
To our knowledge there are no other explicit conditions than (C) for existence of Stackelberg equilibria. In Section 3 we present a different approach to obtain existence results.

Under additional assumptions it can be shown that the Stackelberg equilibrium controls u_1^*, u_2^* can be realized in feedback form, i.e.

$$u_i^*(t) = -R_{ii}^{-1}(t)B_i^T(t)K_i(t)x(t),\ i = 1, 2,$$

where $K_1(t), K_2(t)$ is a solution of a certain coupled system of matrix Riccati differential equations.
In the sequel we use the abbreviations

$$S_1 = B_1R_{11}^{-1}B_1^T,\ S_2 = B_2R_{22}^{-1}B_2^T,\ S_{21} = B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^T,$$

and again from [19],page 549, formulae (48) - (53), we cite

Theorem 1.3 *Let all coefficient matrices in the game be such that (C) holds for all* $t \in [t_0, t_f]$.
If the coupled system of matrix Riccati equations

$$\dot{K}_1 = -A^TK_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2, \tag{1.3}$$

$$\dot{K}_2 = -A^TK_2 - K_2A - Q_2 + Q_1P + K_2S_1K_1 + K_2S_2K_2, \tag{1.4}$$

$$\dot{P} = AP - PA + PS_1K_1 + PS_2K_2 + S_1K_2 - S_{21}K_1, \tag{1.5}$$

admits a unique solution in $[t_0, t_f]$ *with the boundary values* $K_1(t_f) = K_{1f},\ K_2(t_f) = K_{2f} - K_{1f}P(t_f),\ P(t_0) = 0$, *then the Stackelberg equilibrium is given by*

$$\begin{array}{l} u_1^*(t) = -R_{11}^{-1}B_1^TK_1(t)x(t) \\ u_2^*(t) = -R_{22}^{-1}B_2^TK_2(t)x(t) \end{array} \tag{1.6}$$

where $x(t)$ *is a solution of the "closed loop" equation*

$$\dot{x} = (A - S_1K_1(t) - S_2K_2(t))x,\ \ x(t_0) = x_0, \tag{1.7}$$

in $[t_0, t_f]$.

Remark 1.4 Instead of condition (C) in Theorem 1.3 it suffices to impose any condition which ensures the existence of an open loop Stackelberg equilibrium.

Some of the necessary but tedious calculations in the proof are given in more detail in [4]. The main remaining problem is to determine conditions on the coefficients ensuring that the *boundary value problem* (1.3) - (1.5) admits a unique solution.

There have been obtained sufficient conditions for the existence of solutions of an associated *terminal value problem* in the Nash-game-case (see [3] and [11]). These methods could easily be extended also to terminal value problems for Riccati differential equations as given in (1.3) - (1.5). But for the boundary value problem (1.3) - (1.5), which turns out to be more complicated, there are only particular results (see [11]). In [13] there was proposed another approach to proof the existence of a unique Stackelberg equilibrium.
If the Stackelberg game is attacked directly by variational methods then for instance we obtain from [5],p.413 (see also [13]) the following

Theorem 1.5 *If $u_1^*(t), u_2^*(t)$ provides an open loop Stackelberg equilibrium and $x^*(t), t \in [t_0, t_f]$ denotes the corresponding state trajectory, there exist continuously differentiable functions $\gamma : [t_0, t_f] \to \mathbf{R}$, $\psi_1 : [t_0, t_f] \to \mathbf{R}$, $\psi_2 : [t_0, t_f] \to \mathbf{R}$, such that the following relations are satisfied in $[t_0, t_f]$: (for simplicity of notation we suppress here $*$ and write x, u_1, u_2 instead of x^*, u_1^*, u_2^*)*

$$\begin{array}{ll} \dot{x} = A(t)x - S_1(t)\psi_1 - S_2(t)\psi_2, & x(t_0) = x_0, \\ \dot{\gamma} = A(t)\gamma + S_1(t)\psi_2 + S_2(t)\psi_1, & \gamma(t_0) = 0, \\ \dot{\psi}_1 = -Q_1(t)x - A^T(t)\psi_1, & \psi_1(t_f) = K_{1f}x(t_f), \\ \dot{\psi}_2 = -Q_2(t)x - A^T\psi_2 + Q_1\gamma, & \psi_2(t_f) = K_{2f}x(t_f) - K_{1f}\gamma(t_f), \end{array} \tag{1.8}$$

such that the controls can be written as

$$u_i(t) = -R_{ii}^{-1}(t)B_i^T(t)\psi_i(t), \quad i = 1, 2. \tag{1.9}$$

Remark 1.6 Defining

$$\psi_1(t) = K_1(t)x(t), \quad \psi_2(t) = K_2(t)x(t), \quad \gamma(t) = P(t)x(t),$$

we obtain a solution of the boundary value problem (1.8) if K_1, K_2, P is a solution of the boundary value problem (1.3) - (1.5). Together with (1.9) this would yield the possibility to synthesize the open loop Stackelberg equilibrium in feedback-form (1.6) where again $x(t)$ is an solution of (1.7).

Notice that the boundary value problem (1.8) can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \gamma \\ \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A & 0 & -S_1 & -S_2 \\ 0 & A & -S_{21} & S_1 \\ -Q_1 & 0 & -A^T & 0 \\ -Q_2 & Q_1 & 0 & -A^T \end{pmatrix}(t)\begin{pmatrix} x \\ \gamma \\ \psi_1 \\ \psi_2 \end{pmatrix} \tag{1.10}$$

with the boundary conditions.

$$\begin{pmatrix} K_{1f} & 0 \\ K_{2f} & -K_{1f} \end{pmatrix}\begin{pmatrix} x(t_f) \\ \gamma(t_f) \end{pmatrix} = \begin{pmatrix} \psi_1(t_f) \\ \psi_2(t_f) \end{pmatrix}, \begin{pmatrix} x(t_0) \\ \gamma(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \tag{1.11}$$

Similar to the standard control-theoretic case, we can try to solve this boundary value problem by solving an associated terminal value problem and an initial value problem for a Riccati differential equation. In this Stackelberg-game case, however, we have to deal

here with nonsymmetric Riccati differential equations.
The next section will be devoted to the solution of the boundary value problem (1.10),(1.11). If we obtain conditions for unique solvability of this boundary value probelm then, since (1.10), (1.11) represents a necessary condition, the associated Stackelberg problem has at most one equilibrium. Together with the existence result, like in Theorem 1.1 we obtain a representation of this Stackelberg equilibrium by this solution of (1.10), (1.11) via formula (1.9).

2 Uniqueness of Stackelberg equilibria

Here we will obtain sufficient conditions for unique solvability of the boundary value problem (1.10),(1.11).

Theorem 2.1 *(Implicit uniqueness condition for Stackelberg equilibria)*
Let $\tilde{K}_f = \begin{pmatrix} K_{1f} & 0 \\ K_{2f} & -K_{1f} \end{pmatrix}$, *with* K_{1f}, K_{2f} *as in (1.2), and let*

$$\Phi(t, t_0) = \begin{pmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{pmatrix}, \tag{2.1}$$

with $\Phi(t_0, t_0) = I_{4n},\ \Phi_{ik}(t, t_0) \in \mathbf{R}^{2n \times 2n}$, *denote the transition matrix of the system (1.10) in* $[t_0, t_f]$.
If the matrix

$$\Phi_{22}(t_f) - \tilde{K}_f \Phi_{12}(t_f) \tag{2.2}$$

is regular then the Stackelberg game admits at most one equilibrium.

Proof. The solvability of the boundary value problem (1.10), (1.11) is necessary for the existence of a Stackelberg equilibrium. We are going to prove that (2.2) ensures the unique solvability of this boundary value problem.
The general solution of (1.10) is given as

$$\begin{pmatrix} x(t) \\ \gamma(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \Phi(t, t_0) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

for some $c_1, c_2, c_3, c_4 \in \mathbf{R}^n$.
At $t = t_0$ we obtain immediately with (2.1) and (1.11) that

$$c_1 = x_0, \quad c_2 = 0. \tag{2.3}$$

At $t = t_f$ the boundary condition yields

$$\begin{aligned} \begin{pmatrix} x(t_f) \\ \gamma(t_f) \end{pmatrix} &= \Phi_{11} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \Phi_{12} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \\ \begin{pmatrix} \psi_1(t_f) \\ \psi_2(t_f) \end{pmatrix} &= \Phi_{21} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \Phi_{22} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \tilde{K}_f \Phi_{11} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \tilde{K}_f \Phi_{12} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \end{aligned}$$

hence,

$$(\Phi_{22} - \tilde{K}_f\Phi_{12})\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = (\tilde{K}_f\Phi_{11} - \Phi_{21})\begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \tag{2.4}$$

If the matrix in (2.4) is regular as assumed in (2.2) we obtain consequentely the unique solution of (1.10),(1.11) in the form

$$\begin{pmatrix} x(t) \\ \gamma(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \Phi(t,t_0)\begin{pmatrix} I_{2n} \\ (\Phi_{22}(t_f,t_0) - \tilde{K}_f\Phi_{12}(t_f,t_0))^{-1}(\tilde{K}_f\Phi_{11}(t_f,t_0) - \Phi_{21}(t_f,t_0)) \end{pmatrix}\begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \tag{2.5}$$

If the matrix in (2.2) is not regular then either there could be no Stackelberg equilibrium or infinitely many equilibria, according to nonsolvability or nonunique solvability of equation (2.4), respectively. □

In this Theorem 2.1 we made use of the solutions of the system. In order to obtain uniqueness in a more direct manner, we propose another approach, based on Riccati differential equations.
We apply Riccati matrix differential equations similarly to the classical control theoretic case. Notice that, in contrary to this case, here neither the occurring Riccati differential equation nor the terminal data will be symmetric. Hence, standard arguments for solvability will not apply.

Theorem 2.2 *Let*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \tilde{S} = \begin{pmatrix} S_1 & S_2 \\ S_{21} & -S_1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} Q_1 & 0 \\ Q_2 & -Q_1 \end{pmatrix}, \tilde{K}_f = \begin{pmatrix} K_{1f} & 0 \\ K_{2f} & -K_{1f} \end{pmatrix}.$$

$A, S_{21}, S_i, Q_i, K_{if},\ i = 1, 2,$ *as in (1.10),(1.11) or Theorem 1.3, respectively.*
If the Riccati differential equation

$$\dot{V} = \tilde{A}V + V\tilde{A}^T + V\tilde{Q}V - \tilde{S}, \qquad V(t_0) = 0 \tag{2.6}$$

admits a solution in $[t_0, t_f]$, *and if furthermore*

$$\det(I_{2n} - \tilde{K}_f V(t_f)) \neq 0, \tag{2.7}$$

then there exists at most one Stackelberg equilibrium.

Proof. Here again we prove unique solvalability of the boundary value probelm (1.10),(1.11) and then obtain the desired result as in the proof of Theorem 2.1.
If (2.6) has a solution V in $[t_0, t_f]$ then we denote by $Y(t) \in \mathbf{R}^{2n\times 2n}$ the solution of

$$\dot{Y} = (-\tilde{A}^T - \tilde{Q}V)Y, \;\; Y(t_0) = I \tag{2.8}$$

in $[t_0, t_f]$ and define

$$X = VY. \tag{2.9}$$

Then it is straightforward to check that $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a solution of the linear system (1.10). Notice that from (2.8) follows $\det Y(t) \neq 0$ for $t \in [t_0, t_f]$, hence $\begin{pmatrix} X \\ Y \end{pmatrix}$ consists of $2n$ linearly independent solutions to (1.10) with

$$\begin{pmatrix} X \\ Y \end{pmatrix}(t_0) = \begin{pmatrix} V(t_0)Y(t_0) \\ Y(t_0) \end{pmatrix} = \begin{pmatrix} 0_{2n} \\ I_{2n} \end{pmatrix}.$$

Let

$$\Psi(t) = \begin{pmatrix} \psi_{11} & X \\ \psi_{21} & Y \end{pmatrix}(t), \quad \Psi(t_0) = \begin{pmatrix} I_{2n} & 0_{2n} \\ * & I_{2n} \end{pmatrix},$$

denote a fundamental matrix to (1.10) then again the general solution of (1.10) is

$$\begin{pmatrix} x \\ \gamma \\ \psi_1 \\ \psi_2 \end{pmatrix}(t) = \Psi(t) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

and together with (1.11) we infer $c_1 = x_0, \;\; c_2 = 0$ and

$$[\tilde{K}_f V(t_f) - I_{2n}]Y(t_f) \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = [\psi_{21}(t_f) - \tilde{K}_f \psi_{11}(t_f)] \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \tag{2.10}$$

Since $Y(t_f)$ is regular we infer from (2.10) together with (2.7) that (1.10), (1.11) is uniquely solvable, hence the desired result. □

In order to achieve global existence for solutions of the nonsymmetric problem (2.6) , here we propose an approach using Lyapunov-type functions, as we already did for Riccati differential equations in Nash games (see [10]).

Theorem 2.3 *If for some matrices $C, D \in \mathbf{R}^{2n\times 2n}, C^T = C, C > 0,$ with*

$$L(t) = \begin{pmatrix} -C\tilde{A}^T(t) - D\tilde{S}(t) & -C\tilde{Q}(t) - \tilde{A}(t)D + D\tilde{A}(t) \\ 0 & -\tilde{Q}(t)^T D \end{pmatrix}$$

it holds for all $t \in [t_0, t_f]$

$$L(t) + L^T(t) \geq 0 \tag{2.11}$$

then the solution $V(t)$ of (2.6) with $V(t_0) = 0$ exists for all $t \in [t_0, t_f]$.

For the proof see ([10], Theorem 4.1), observing to use the theorem from [10] forward in time.

Notice, that another existence result could also be derived by another choice of the parameter matrices C, D. Global existence can be achieved, for instance, if $C < 0$ and

$$L(t) + L^T(t) \leq 0 \quad \text{in} \quad [t_0, t_f].$$

For utilization of Theorem 2.2 we further present another Lyapunov-type approach by characterizing right invariant domains. It is then obvious that if the initial value $V(t_0) = 0$ in (2.6) is in such an invariant set that no blow-up of the solution can occur.

Theorem 2.4 *If for $\beta, \gamma > 0$ and a positive definite matrix $C \in \mathbf{R}^{2n\times 2n}$ and every $t \in [t_0, t_f]$ holds*

$$(\eta_1^T, \eta_2^T) \begin{pmatrix} \beta(\tilde{A}(t) + \tilde{A}^T(t)) & -\tilde{S}^T(t)C + \beta\tilde{Q}(t) \\ -C\tilde{S}(t) + \beta\tilde{Q}^T(t) & +C\tilde{A}(t) + \tilde{A}^T(t)C \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \leq -\gamma(|\eta_1|^2 + |\eta_2|^2) \qquad (2.12)$$

for all $\eta_1, \eta_2 \in \mathbf{R}^{2n}$, then the set $||V||_C < \sqrt{\beta}$ is positive invariant for (2.6). ($||V||_C^2 = \max\{\text{spect}(V^TCV)\}$).

For the proof of this theorem see again ([10], Theorem 3.1 and Corollary 3.2) with the appropriate modifications for positive invariance.
Notice, that the initial value $V(t_0) = 0$ always is contained in the invariant "ball".

If the *coefficients $\tilde{A}, \tilde{Q}, \tilde{S}$ are constant* and the time horizon is "somehow large" then including the knowledge of the asymptotic behavior of solutions of such Riccati differential equations (see [8]) the conditions of Theorem 2.3 should be slightly changed. If we assume that the "dichotomic solution" V^+ of the associated algebraic Riccati equation

$$0 = \tilde{A}V + V\tilde{A}^T + V\tilde{Q}V - \tilde{S} \qquad (2.13)$$

exists then this dichotomic solution is "attracting" almost all initial values $V_0 \in \mathbf{R}^{2n\times 2n}$, as $t \to \infty$. This attraction actually could happen through infinity, i.e. continuously on a Graßmann manifold.
For that reason it makes more sense to study right invariant domains "around" V^+. The transformation (assuming that (2.13) has a dichotomic solution V^+)

$$Y(t) = V(t) - V^+$$

yields

$$\begin{aligned} \dot{Y} &= (\tilde{A} + V^+\tilde{Q})Y + Y(\tilde{A}^T + \tilde{Q}V^+) + Y\tilde{Q}Y \\ Y(t_0) &= V(t_0) - V^+ = -V^+. \end{aligned} \qquad (2.14)$$

with the dichotomic solution of the associated algebraic equation (as $t \to \infty$) $Y^+ = 0$.

Applying now Theorem 2.4 to equation (2.14), we obtain

Corollary 2.5 *Let $\tilde{A}, \tilde{Q}, \tilde{S} \in \mathbf{R}^{2n\times 2n}$ be constant and let the dichotomic solution V^+ of (2.13) exist.*
If for $\beta > 0$ and a positive definite matrix $C \in \mathbf{R}^{2n\times 2n}$

$$\begin{pmatrix} \beta(\tilde{A} + \tilde{A}^T + \tilde{Q}V^+ + (\tilde{Q}V^+)^T) & \beta\tilde{Q} \\ \beta\tilde{Q}^T & C(\tilde{A} + V^+\tilde{Q}) + (\tilde{A}^T + (V^+\tilde{Q})^T)C \end{pmatrix} < 0 \qquad (2.15)$$

then the set $||V_0 - V^+||_C < \sqrt{\beta}$ is positive invariant for (2.6) or (2.14), respectively.

Summing up the assertions of Theorems 2.2, 2.3, 2.4 and Corollary 2.5 we obtain a direct approach to uniqueness of Stackelberg equilibria.

Theorem 2.6 *(Direct uniqueness conditions for Stackelberg equilibria)*
Let for the Riccati differential equation (2.6) hold either (2.11), (2.12) or (2.15) (in case of constant coefficients) and (2.7), then the Stackelberg game admits at most one equilibrium.

Proof. From Theorems 2.3, 2.4 or Corollary 2.5 we infer that the initial value problem (2.6) has a solution in $[t_0, t_f]$. Together with (2.7) and Theorem 2.2 we then obtain the desired result. □

3 Existence of Stackelberg equilibria

To our knowledge the Hilbert space method - as applied in [19] - was so far the sole method to obtain existence conditions for a Stackelberg equilibrium. The convexity condition (C) is seen to be sufficient for existence. There is obtained also a general necessary and sufficient condition for existence in terms of invertibility of an operator in Hilbert space, but, this condition cannot be applied directly.
In linear quadratic problems there exists also another approach using a value function which can be obtained by an appropriate guess.
For Nash games this is nicely presented in the papers [6],[7], [14]. The sufficient conditions turn out to be more general than (C) but are restricted to feedback controls.

Theorem 3.1 *Let the solution of the Riccati Differential equation*

$$\dot{E}_1 = -E_1 A - A^T E_1 - Q_1 + E_1 S_1 E_1, \quad E_1(t_f) = K_{1f} \tag{3.1}$$

exist on $[t_0, t_f]$.
For any given admissible open loop control u_2 of the leader define $e_1(t), d_1(t)$ by

$$\dot{e}_1 = E_1 S_1 e_1 - A^T e_1 - 2E_1 B_2 u_2, \quad e_1(t_f) = 0 \tag{3.2}$$

$$\dot{d}_1 = -u_2^T R_{12} u_2 - e_1^T B_2 u_2 + \frac{1}{4} e_1^T S_1 e_1, \quad d_1(t_f) = 0. \tag{3.3}$$

Then the following identity holds:

$$2\ J_1(u_1, u_2) = x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0) + \int_{t_0}^{t_f} ||z_1||^2_{R_{11}}(t)\ \ dt, \tag{3.4}$$

where $||z_1||^2_{R_1}(t)\ = z_1^T(t) R_{11}(t) z_1(t)$ and

$$z_1 = u_1 + R_{11}^{-1} B_1^T (E_1 x + \frac{1}{2} e_1) \tag{3.5}$$

with x a solution of (1.1).

Proof. We try a "quadratic" guess for the value function, i.e. we start with

$$
\begin{aligned}
&\tfrac{d}{dt}(x^T E_1 x + x^T e_1 + d_1) = 2x^T E_1(Ax + B_1 u_1 + B_2 u_2) + x^T \dot{E}_1 x + \\
&+ e_1^T(Ax + B_1 u_1 + B_2 u_2) + x^T \dot{e}_1 + \dot{d}_1 = \\
&= x^T[E_1 A + A^T E_1 + \dot{E}_1 + Q_1 - E_1 S_1 E_1]x - x^T Q_1 x - u_1^T R_{11} u_1 - u_2^T R_{12} u_2 + \\
&+ x^T[\dot{e}_1 + A^T e_1 - E_1 S_1 e_1 + 2E_1 B_2 u_2] + \\
&+ \dot{d}_1 + u_2^T R_{12} u_2 - \tfrac{1}{4} e_1^T S_1 e_1 + e_1^T B_2 u_2 + ||z_1(t)||^2_{R_{11}} = \\
&- [x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2] + ||z_1(t)||^2_{R_{11}}.
\end{aligned}
$$

Integrating this identity from t_0 to t_f and observing (3.1), (3.2) and (3.3) yields together with (1.2) the result in (3.4). □

From (3.4) we infer that for any fixed open loop control u_2

$$J_1(u_1, u_2) \geq \frac{1}{2}(x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0)) = J_{10}(u_2)$$

hence, the minimal costs $J_{10}(u_2)$ are attained if and only if $z_1(t) \equiv 0$.
With this remark we obtain:

Theorem 3.2 *Let the solution E_1 of (3.1) exist on $[t_0, t_f]$. Then the unique optimal response of the follower to the leaders open loop strategy u_2 is given by*

$$u_1(t) = -R_{11}^{-1}(t) B_1^T(t)(E_1(t)x(t) + \frac{1}{2}e_1(t)), \tag{3.6}$$

where e_1 is a solution of (3.2) and $x(t)$ is a solution of

$$\dot{x} = (A - S_1 E_1)x - \frac{1}{2} S_1 e_1 + B_2 u_2, \quad x(t_0) = x_0. \tag{3.7}$$

The minimal costs then are

$$J_{10}(u_2) = \frac{1}{2}(x_0^T E_1(t_0) x_0 + x_0^T e_1(t_0) + d_1(t_0)), \tag{3.8}$$

where $d_1(t_0)$ is obtained by integrating (3.3).

Remark 3.3 Notice that $J_{10}(u_2)$ is not depending on u_1. This, in fact is only true if we consider open loop information structure, since, otherwise u_2 would depend on the trajectory x and hence, via (1.1), also on u_1.

In open loop Stackelberg games now the leader (i.e. player 2) tries to find an optimal open loop control u_2 in order to minimize $J_2(u_1(u_2), u_2)$ while $u_1(u_2)$ is defined by (3.6).

Theorem 3.4 *Let the solutions of the Riccati differential equations (3.1) and*

$$\dot{E}_2 = -E_2 H - H^T E_2 - Q + E_2 S E_2, \quad E_2(t_f) = \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

exist in $[t_0, t_f]$, *where*

$$H = \begin{pmatrix} A & -S_1 \\ -Q_1 & -A^T \end{pmatrix}, \quad Q = \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix}, \quad S = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.10}$$

For any given admissible control u_2 *of the leader define the functions* $v_1(t), \ x(t) \in \mathbf{R}^n$ *in* $[t_0, t_f]$ *by the following initial value problems:*

$$\dot{v}_1 = -Q_1 x - A^T v_1, \quad v_1(t_0) = v_{10}, \tag{3.11}$$

$$\dot{x} = Ax - S_1 v_1 + B_2 u_2, \quad x(t_0) = x_0. \tag{3.12}$$

Then we obtain with

$$u_1 := -R_{11}^{-1} B_1^T v_1 \tag{3.13}$$

the following identity:

$$2J_2(u_1, u_2) = (x_0^T, v_{10}) E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix} + \int_{t_0}^{t_f} ||z_2||_{R_{22}}^2(t)\, dt, \tag{3.14}$$

where $||z_2||_{R_{22}}^2(t) := z_2^T(t) R_{22}(t) z_2(t)$, *with*

$$z_2 = u_2 + (R_{22}^{-1} B_2^T, 0_{m_2 \times n}) E_2 y \tag{3.15}$$

and with $y = \begin{pmatrix} x \\ v_1 \end{pmatrix}$ *and* $0_{m_2\times n}$ *the* $m_2 \times n$*-dimensional zero matrix.*

Remark 3.5 Notice that in the term

$$J_{20} = \frac{1}{2}(x_0^T, v_{10}^T) E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix}, \tag{3.16}$$

$x_0, E_2(t_0)$, do not depend on the choice of u_1, u_2. Since we shall study the situation for player two when player one applies his optimal response control defined in (3.6) we have to set $v_1 = E_1 x + \frac{1}{2} e_1$. From (3.2) we can see that $v_1(t_0) = v_{10}$ depends on $e_1(t_0)$ and, hence, via (3.2) also on u_2.
It also should be pointed out that then (3.11) coincides with the third equation from (1.8).

Proof of Theorem 3.4. With $y = \begin{pmatrix} x \\ v_1 \end{pmatrix}$ we define

$$\Phi(t) = y^T(t) E_2(t) y(t).$$

Observe that by (3.11), (3.12) y is a solution of

$$\dot{y} = Hy + Bu_2, \quad y(t_0) = \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix}, \tag{3.17}$$

where $B = \begin{pmatrix} B_2 \\ 0_{n\times m_2} \end{pmatrix} \in \mathbf{R}^{2n\times m_2}$.
Now we proceed as in the proof of Theorem 3.1:

$$\begin{aligned} \frac{d\Phi}{dt} &= 2y^T E_2 \dot{y} + y^T \dot{E}_2 y \\ &= 2y^T E_2 (Hy + Bu_2) + y^T \dot{E}_2 y \\ &= y^T (E_2 H + H^T E_2 + \dot{E}_2) y + 2y^T E_2 B u_2. \end{aligned}$$

With $u_1 = -R_{11}^{-1} B_1^T v_1, \quad S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T$ we obtain

$$y^T Q y = (x^T, v_1^T) \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix} \begin{pmatrix} x \\ v_1 \end{pmatrix} = x^T Q_2 x + v_1^T S_{21} v_1 = x^T Q_2 x + u_1^T R_{21} u_1.$$

Inserting this in the above identity yields after some manipulations

$$\frac{d\Phi}{dt} = y^T (E_2 H + H^T E_2 + \dot{E}_2 + Q - E_2 S E_2) y - \psi_2(t) + ||z_2)||_{R_{22}}^2(t),$$

where $\psi_2(t) = x^T(t) Q_2(t) x(t) + u_1^T(t) R_{21}(t) u_1(t) + u_2^T(t) R_{22}(t) u_2(t)$.
Using the differential equation (3.9) and integrating from t_0 to t_f, together with (1.2) yields

$$\Phi(t_f) - \Phi(t_0) = -2J_2(u_1, u_2) + x^T(t_f) K_{2f} x(t_f) + \int_{t_0}^{t_f} ||z_2||_{R_{22}}^2(t)\, dt. \tag{3.18}$$

Again it remains to calculate $\Phi(t_f) - \Phi(t_0)$. From (3.9), (3.11), (3.12) and (3.13) we infer

$$\begin{aligned} \Phi(t_f) - \Phi(t_0) &= (x^T(t_f), v_1^T(t_f)) \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t_f) \\ v_1(t_f) \end{pmatrix} - \\ &- (x^T(t_0), v_{10}^T) E_2(t_0) \begin{pmatrix} x(t_0) \\ v_{10} \end{pmatrix}, \end{aligned}$$

which together with (3.18) and $x(t_0) = x_0$ yields the desired result. □

In order to derive from Theorems 3.2, 3.4 sufficient conditions for the existence of a unique Stackelberg equilibrium we must get rid of the u_2-dependence of v_{10} in (3.16). Therefore we propose to restrict the set of admissible controls to functions representable in linear feedback form. This was already discussed in the introductory section.

Theorem 3.6 *Let the solutions $E_1(t) \in \mathbf{R}^{n\times n}, E_2(t) \in \mathbf{R}^{2n\times 2n}$ of (3.1) and (3.9) exist in $[t_0, t_f]$, respectively. Let furthermore, the coupled system*

$$\begin{aligned} \dot{K}_1 &= -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2, \\ \dot{K}_2 &= -A^T K_2 - K_2 A - Q_2 + Q_1 P + K_2 S_1 K_1 + K_2 S_2 K_2, \\ \dot{P} &= AP - PA + PS_1 K_1 + PS_2 K_2 + S_1 K_2 - S_{21} K_1, \end{aligned} \tag{3.19}$$

admit a solution with terminal values $K_1(t_f) = K_{1f},\;\; K_2(t_f) = K_{2f},\;\; P(t_f) = 0$ *in* $[t_0, t_f]$. *Then there exists a unique open loop Stackelberg equilibrium which is given by*

$$\begin{aligned} u_1^*(t) &= -R_{11}^{-1}(t)B_1^T(t)K_1(t)x(t) \\ u_2^*(t) &= -R_{22}^{-1}(t)B_2^T(t)K_2(t)x(t), \end{aligned} \tag{3.20}$$

where x *is a solution of the closed loop equation*

$$\dot{x} = (A - S_1K_1 - S_2K_2)x, \;\; x(t_0) = x_0. \tag{3.21}$$

The minimal costs then for the follower are $J_{10}(u_2^*)$ *in (3.8), together with (3.1),(3.2) and (3.3). The costs for the leader are*

$$J_{20}(u_1^*, u_2^*) \;\; = \;\; \frac{1}{2}(x_0^T(I_n, K_1^T(t_0)))E_2(t_0)\begin{pmatrix} I_n \\ K_1(t_0) \end{pmatrix} x_0. \tag{3.22}$$

Proof. First we prove that

$$v_1 = E_1x + \frac{1}{2}e_1 =: K_1x. \tag{3.23}$$

At t_f we obtain together with (3.1), (3.2) and (3.19) $E_1(t_f)x(t_f) + \frac{1}{2}e_1(t_f) = K_{1f}x(t_f) = K_1(t_f)x(t_f)$. We now prove that $v_1 = K_1x$ solves (3.11), i.e.

$$\dot{v_1} = \dot{K_1}x + K_1\dot{x} = [\dot{K_1} + K_1(A - S_1K_1 - S_2K_2)]x = -Q_1x - A^TK_1x$$

if (3.21) holds. On the other hand also, using $v_1 = E_1x + \frac{1}{2}e_1$, we obtain with (3.1), (3.2)

$$\begin{aligned} \dot{v_1} &= \dot{E_1}x + E_1\dot{x} + \frac{1}{2}\dot{e_1} \\ &= (-E_1A - A^TE_1 - Q_1 + E_1S_1E_1)x \\ &+ E_1(Ax + B_1u_1 + B_2u_2) + \frac{1}{2}(E_1S_1e_1 - A^Te_1 - 2E_1B_2u_2) \\ &= -Q_1x - A^T(E_1x + \frac{1}{2}e_1). \end{aligned}$$

So both functions solve (3.11) and have same terminal values hence (3.23) holds.
From (3.23) we now infer that

$$v_1(t_0) = v_{10} = K_1(t_0)x_0, \tag{3.24}$$

does not depend on u_2.
Hence, the minimal value of J_2 is given by (3.16) and is achieved if and only if $z_2 = 0$, which means

$$u_2^* = -(R_{22}^{-1}B_2^T, 0_{m_2\times n})E_2y. \tag{3.25}$$

Remember that with $y = \begin{pmatrix} x \\ v_1 \end{pmatrix} = \begin{pmatrix} I_n \\ K_1 \end{pmatrix} x$ this is already a feedback representation but it remains to show that we can achieve this control also as a feedback representation with K_2.
We define

$$f(t) = \begin{pmatrix} K_2 \\ P \end{pmatrix} x - E_2y.$$

At $t = t_f$ we obtain together with (3.9), (3.11), (3.19) and (3.23)

$$f(t_f) = \begin{pmatrix} K_{2f} \\ 0_n \end{pmatrix} x(t_f) - \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t_f) \\ v_1(t_f) \end{pmatrix} = 0.$$

The derivative of $f(t)$ is

$$\dot{f}(t) = (\frac{d}{dt} \begin{pmatrix} K_2 \\ P \end{pmatrix})x + \begin{pmatrix} K_2 \\ P \end{pmatrix} \dot{x} - \dot{E}_2 y - E_2 \dot{y},$$

which together with (3.9), (3.10), (3.17), (3.19), (3.21), (3.25) $y = \begin{pmatrix} x \\ v_1 \end{pmatrix}$, and $v_1 = K_1 x$ yields

$$\begin{aligned}
\dot{f} &= (\frac{d}{dt} \begin{pmatrix} K_2 \\ P \end{pmatrix})x + \begin{pmatrix} K_2 \\ P \end{pmatrix} (A - S_1 K_1 - S_2 K_2)x - \dot{E}_2 y - E_2 \dot{y} \\
&= [-\begin{pmatrix} K_2 \\ P \end{pmatrix} (A - S_1 K_1 - S_2 K_2) - \begin{pmatrix} A^T & -Q_1 \\ -S_1 & -A \end{pmatrix} \begin{pmatrix} K_2 \\ P \end{pmatrix} - \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix} \begin{pmatrix} I_n \\ K_1 \end{pmatrix}]x \\
&+ \begin{pmatrix} K_2 \\ P \end{pmatrix} (A - S_1 K_1 - S_2 K_2)x - [-E_2 H - H^T E_2 - Q + E_2 S E_2]y - E_2 H y - E_2 B u_2 \\
&= -H^T(\begin{pmatrix} K_2 \\ P \end{pmatrix} x - E_2 y).
\end{aligned}$$

Finally we therefore obtain

$$\dot{f} = -H^T f.$$

Since $f(t_f) = 0$ we conclude $f = 0$, hence u_2 has the desired feedback representation $u_2 = K_2 x$.

Remark 3.7 The convexity condition (C) is equivalent to $R_{11}, R_{22} > 0$ and $Q_1, S_1, Q, S \geq 0$ and $K_{1f}, E_2(t_f) \geq 0$. Therefore the standard Riccati equations (3.1) and (3.9) by classical results, always have a globally existing solution (see [12], p.218).
By more recent results one also obtains sufficient conditions for global existence in case of indefinite coefficients Q_1, S_1, Q, S as in H_∞-type Riccati differential equations (see [9]).
The existence of solutions of the system (3.21) finally can be proved by methods developped in [10] since here, in contrast to (1.3) - (1.5), we have to deal with a terminal value problem which admits at most one solution. The sufficient conditions for global existence now are easily derived from [10], Theorems 3.1, 4.1 - as we already have shown here in similar situations in Theorems 2.4, 2.5 - since the coupled system (3.21) can be written as a single nonsymmetric Riccati matrix differential equation of the form

$$\begin{aligned}
\frac{d}{dt} \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} &= -\begin{pmatrix} Q_1 \\ Q_2 \\ 0 \end{pmatrix} - \begin{pmatrix} A^T & 0 & 0 \\ 0 & A^T & -Q_1 \\ S_{21} & -S_1 & -A \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} A \\
&+ \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} (S_1, S_2, 0) \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}, \quad \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} (t_f) = \begin{pmatrix} K_{1f} \\ K_{2f} \\ 0 \end{pmatrix}. \qquad (3.26)
\end{aligned}$$

Remark 3.8 Notice that the Riccati equations of symplectic type (3.1) and (3.9) reflect nicely the "double" Hamiltonian structure of Stackelberg games, which also can be seen in the system matrix (4.1) of the next section.

4 Representation of solutions in time invariant problems

In [13] there was used an explicit representation of solutions to obtain a numerical algorithm calculating a Stackelberg equilibrium, if it exists. Since the system (1.8) may also be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \psi_1 \\ \psi_2 \\ \gamma \end{pmatrix} = \begin{pmatrix} A & -S_1 & -S_2 & 0 \\ -Q_1 & -A^T & 0 & 0 \\ -Q_2 & 0 & -A^T & Q_1 \\ 0 & -S_{21} & S_1 & A \end{pmatrix} \begin{pmatrix} x \\ \psi_1 \\ \psi_2 \\ \gamma \end{pmatrix} = H_{Sta} \begin{pmatrix} x \\ \psi_1 \\ \psi_2 \\ \gamma \end{pmatrix} \tag{4.1}$$

we recognize that the system matrix H_{Sta} is hamiltonian, hence, the spectrum of H_{Sta} is symmetric with respect to the imaginary axis.
For simplicity we assume that H_{Sta} is a constant matrix (time invariant) and that all eigenvalues are simple and not purely imaginary.
To the hamiltonian matrix H_{Sta} there exists a symplectic matrix

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix}, a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbf{C}^{n \times n},\ 1 \le i,\ j \le 2$$

such that

$$V^{-1} H_{Sta} V = J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & -J_1^* & \\ & & & -J_2^* \end{pmatrix} \tag{4.2}$$

where J_1, J_2 are diagonal matrices having all its eigenvalues in the left halfplane. Representing all solutions of (1.10) in the form

$$\begin{pmatrix} x \\ \psi_1 \\ \psi_2 \\ \gamma \end{pmatrix} = V \begin{pmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & e^{-J_1^* t} & \\ & & & e^{-J_2^* t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{4.3}$$

where

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = V^{-1} C, \quad C \in \mathbf{R}^{4n \times 1},$$

C related to given data.
Now we easily can find conditions such that Z or C, respectively, can be determined from the data as given in (1.11).

At $t_0 = 0$ (which is no restriction of generality, since the coefficients are time invariant) we obtain a system of linear equations

$$M_1 Z = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \tag{4.4}$$

$$M_1 = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix} \in \mathbf{C}^{2n\times 4n}.$$

At $t = t_f$ analogously we have from (1.11)

$$M_2 Z = 0, \tag{4.5}$$

where

$$M_2 = \left(\begin{matrix} (a_{21} - K_{1f}a_{11})e^{J_1 t_f} & (a_{22} - K_{1f}a_{12})e^{J_2 t_f} \\ (c_{11} - K_{2f}a_{11} + K_{1f}c_{21})e^{J_1 t_f} & (c_{12} - K_{2f}a_{12} + K_{1f}c_{22})e^{J_2 t_f} \end{matrix} \right.$$

$$\left. \begin{matrix} (b_{21} - K_{1f}b_{11})e^{J_1^* t_f} & (b_{22} - K_{1f}b_{12})e^{J_2^* t_f} \\ (d_{11} - K_{2f}d_{11} + K_{1f}d_{21})e^{J_1^* t_f} & (d_{12} - K_{2f}b_{12} + K_{1f}d_{22})e^{J_2^* t_f} \end{matrix} \right).$$

From these calculations together with Theorem 1.5 we obtain

Theorem 4.1 *If the system*

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} Z = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution, i.e. $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \in \mathbf{D}^{4n\times 4n}$ *is regular, then there exists at most one Stackelberg equilibrium. If it exists, then it is given by (1.9) and with*

$$\Psi_1(t) = a_{21}e^{J_1 t}z_1 + a_{22}e^{J_2 t}z_2 + b_{21}e^{-J_1^* t}z_3 + b_{22}e^{-J_2^* t}z_4$$
$$\Psi_2(t) = c_{11}e^{J_1 t}z_1 + c_{12}e^{J_2 t}z_2 + d_{11}e^{-J_1^* t}z_3 + d_{12}e^{-J_2^* t}z_4.$$

Notice that in a similar way it can be obtained a solution of the system of coupled matrix Riccati differential equations (1.3)-(1.5) via the linearization as already mentioned in Remark 2.5. But then also further conditions are necessary in order to ensure the existence of $X(t)^{-1}$ in the whole interval $[t_0, t_f]$.

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5 Numerical examples

The following examples are calculated with MAPLE and we also use the MAPLE-style presentation.
Given the following system matrices

$$A := \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \quad S_{11} := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad S_{21} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S_{22} := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad Q_1 := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 := \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Hence, we obtain the matrix of the linear system as defined in (1.10)

$$H_{Sta} := \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 6 & 0 & 0 & 0 & \frac{-1}{2} & 0 & -3 \\ 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 & -1 & 0 & \frac{1}{2} \\ -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -6 & 0 & 0 \\ -4 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -4 & 0 & 2 & 0 & 0 & 0 & -6 \end{bmatrix}$$

with eigenvalues

$$EW := [-7.00000,\ -6.08276,\ -3.85992,\ -2.25854,\ 2.25854,\ 3.85992,\ 6.08276,\ 7.00000].$$

The transformation of H_{Sta} to a diagonal matrix is accomplished by the matrix

$$V := \begin{bmatrix} 0\,,\ 0\,,\ -.3945590280\,,\ .1133900279\,,\ -.4089916454\,,\ -.7836135482\,,\ 0\,,\ 0 \\ -.2195775164\,,\ .0460766408\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ .4986356774\,,\ .9456108576 \\ 0\,,\ 0\,,\ .08867511792\,,\ .2522638826\,,\ -.9099020616\,,\ .1761131259\,,\ 0\,,\ 0 \\ 0\,,\ .09215328139\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ .9972713544\,,\ 0 \\ 0\,,\ 0\,,\ -.4242753540\,,\ .8771433475\,,\ .1920805115\,,\ .2674485655\,,\ 0\,,\ 0 \\ -.4391550328\,,1.113466216\,,0\,,0\,,0\,,0\,,-.08253670075\,,-.1454785935 \\ 0\,,0\,,-.9439044175\,,-.1971334688\,,-.04316910861\,,.5950048243\,,0\,,0 \\ -.8783100655\,,\ .7\,10^{-9}\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -.13\,10^{-9}\,,\ -.2909571870 \end{bmatrix}$$

Applying Theorem 2.2 we first check for global solvability of (2.6). The dichotomic solution of (2.6) exists and is given by

$$W^{-} := \begin{bmatrix} 1.686410585 & 0 & 2.719062266 & 0 \\ 0 & 2. & 0 & 11.08276253 \\ 2.013290037 & 0 & -1.686410587 & 0 \\ 0 & 4. & 0 & -1.999999996 \end{bmatrix}$$

and as proposed after Theorem 2.5 we transform (2.6) as in (2.19) and obtain the associated matrix for the linear system (2.19)

$$\begin{pmatrix} \tilde{A} - \tilde{S}W^- & -\tilde{S} \\ 0_4 & -(\tilde{A}^T - W^-\tilde{S}) \end{pmatrix} = \begin{bmatrix} -3.712990659\,,\,0\,,\,.653758908\,,\,0\,,\,-1.\,,\,0\,,\,-2.\,,\,0 \\ 0\,,-7.00000000\,,0\,,.458618723\,,0\,,-.5000000000\,,0\,,-3. \\ .326879452\,,\,0\,,\,-2.405472853\,,\,0\,,\,-1.\,,\,0\,,\,1.\,,\,0 \\ 0\,,\,0\,,\,0\,,\,-6.08276253\,,\,0\,,\,-1.\,,\,0\,,\,.5000000000 \\ 0\,,\,0\,,\,0\,,\,0\,,\,2.405472851\,,\,0\,,\,.653758904\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,6.08276253\,,\,0\,,\,.458618735 \\ 0\,,\,0\,,\,0\,,\,0\,,\,.326879450\,,\,0\,,\,3.712990661\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,.4\,10^{-8}\,,\,0\,,\,7.00000000 \end{bmatrix},$$

which shows that 0 became the dichotomic solution.
With $\alpha = 1$ and $C_1 = I_4$, the four dimensional unit matrix, the test matrix in (2.22) has eigenvalues

$$[3.18266,\ 5.42518,\ 5.81192,\ 10.0546,\ 10.9993,\ 11.0121,\ 13.1669,\ 17.1534]$$

hence is positive definite. From Corollary 2.6 we infer that the ball with spectral radius 1 around W^- is negative invariant and therefore all solutions of (2.6) having terminal values inside do not blow up as $t \to -\infty$.
The analogous procedure for the dual equation (2.7) leads us to the antidichotomic solution of (2.7) or (2.20)

$$V^+ := \begin{bmatrix} .2027364265 & 0 & .3268794531 & 0 \\ 0 & .04138126518 & 0 & .2293093673 \\ .2420331275 & 0 & -.2027364276 & 0 \\ 0 & .08276253034 & 0 & -.04138126515 \end{bmatrix}.$$

Applying now (2.23), with $\beta = 1$ and $C_2 = I_4$, we obtain the eigenvalues of the matrix in (2.23) as

$$[-14.522,\ -12.562,\ -10.438,\ -8.3987,\ -4.8904,\ -4.2824,\ -1.6767,\ -.2849],$$

hence negative definiteness. From Corollary 2.6 we now infer that the ball of spectral radius 1 around V^+ is positive invariant. Therefore, since $||V^+|| = 0.3917...$ the solution V of (2.7) with initial value 0 exists for all $t \geq t_0$.

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