BASIC PROPERTIES OF A CLASS OF RATIONAL MATRIX DIFFERENTIAL EQUATIONS

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1 Introduction

In this note we study rational matrix differential equations of the form

$$-\dot{X} = \mathcal{R}(X) \tag{1.1}$$

and the corresponding algebraic equations

$$\mathcal{R}(X) = 0, \tag{1.2}$$

where $\mathcal{R}\colon D(\mathcal{R}) \to \mathcal{H}^n$ with

$$D(\mathcal{R}) := \{X \in \mathcal{H}^n \mid R + \Pi_2(X) > 0\},$$
$$\mathcal{H}^n := \{A \in \mathbb{C}^{n\times n} \mid A^* = A\}$$

and

$$\begin{aligned}\mathcal{R}(X) = A^*X + XA + Q + \Pi_1(X)\\ - [S + XB + \Pi_{12}(X)][R + \Pi_2(X)]^{-1}\\ \times [S + XB + \Pi_{12}(X)]^*\end{aligned} \tag{1.3}$$

is a rational matrix operator with constant matrices A, B, Q, R and S of sizes $n \times n$, $n \times m$, $n \times n$, $m \times m$ and $n \times m$, respectively, such that

$$T := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$$

is hermitian. We assume further that the operator $\Pi\colon \mathcal{H}^n \to \mathcal{H}^{n+m}$ with

$$\Pi(X) := \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{bmatrix}$$

is a positive linear operator, i.e., $X \geq 0$ implies $\Pi(X) \geq 0$.

Equations of this type and certain special cases appear in particular in stochastic control and filtering theory (see [2], [6], [11], [17] and [18]) and can be considered as generalized Riccati-type equations.

Notice that the classes (1.1) and (1.2) of differential and algebraic equations considered here are of very general type. For example, (1.2) reduces to the discrete-time algebraic Riccati equation

$$\begin{aligned}X = A_0^*XA_0 + Q - (S + A_0^*XB_0)\\ \times (R + B_0^*XB_0)^{-1}(S + A_0^*XB_0)^*,\end{aligned}$$

if $A = -\frac{1}{2}I$, $B = 0$, $\Pi_1(X) = A_0^*XA_0$, $\Pi_2(X) = B_0^*XB_0$ and $\Pi_{12}(X) = A_0^*XB_0$.

In the case $\Pi \equiv 0$ (1.1) reduces to the continous-time Riccati differential equation

$$-\dot{X} = A^*X + XA + Q - (S + XB)R^{-1}(S + XB)^*,$$

and for $\Pi_2 \equiv 0$ and $\Pi_{12} \equiv 0$ (1.1) coincides with

$$-\dot{X} = A^*X + XA + Q + \Pi_1(X) - (S + XB)R^{-1}(S + XB)^*.$$

The latter class of linearly perturbed Riccati differential equations appears among others in control problems with stochastically jumping parameters (see [5], [7], [9], [10], [14] and [17]); the corresponding algebraic equations and inequalities play also an important role in the application of the Lyapunov-Krasovskii method to linear time-delay systems (see [12], [15] and the literature cited therein).

First steps concerning the theory of the rational matrix differential equation (1.1) have been performed by Hinrichsen and Pritchard ([11]) and by Chen et al. ([2]) who obtained unter additional assumptions sufficient conditions for the existence of the solutions of (1.1) on a given interval for certain initial values. The algebraic equation (1.2) has been studied recently in detail by Damm and Hinrichsen [4].

In Sections 2 and 3 of this note we recall several notations and preliminary results concerning Schur complements and linearly perturbed Lyapunov equations. In Section 4 we show that the solutions of (1.1) depend in particular monotonically on T and on a given initial or terminal value. As a consequence of Theorem 4.4 we derive two corollaries concerning the existence of the solutions of (1.1) on $(-\infty, t_f]$ which extend the existence result that has been derived recently in [2]. In Section 5 we derive in the time-invariant case a monotonicity property of the solutions of (1.1) and use this fact for the proof of the existence and convergence of certain solutions of (1.1) and of the monotonicity of the minimal (or maximal) positive semidefinite solution of (1.2).

There exist also discrete-time versions of our results – details will be given in [8].

2 Schur complements and rational matrix operators

In this section we present some preliminary results and notations from matrix analysis.

2.1 Definition. The *Moore-Penrose inverse* of a $p \times q$ matrix Z is the unique $q \times p$ matrix Z^+ satisfying the conditions

(i) $Z^+ZZ^+ = Z^+$, $ZZ^+Z = Z$,

(ii) $(Z^+Z)^* = Z^+Z$, $(ZZ^+)^* = ZZ^+$.

If Z is hermitian or positive semidefinite, then so is Z^+ (see [13], Proposition 12.8.3).

2.2 Lemma ([1], Theorem 1). *Let H be a hermitian matrix of size $n+m$ with*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix}$$

where L is $n \times n$ and M is $m \times m$. Then:

(i) H is positive semidefinite if and only if $M \geq 0$, $L \geq NM^+N^*$ and $\operatorname{Ker} M \subseteq \operatorname{Ker} N$.

(ii) If $M > 0$ then H is positive semidefinite if and only if $L \geq NM^{-1}N^*$.

The matrix $H/M := L - NM^+N^*$ is called the Schur complement *of M in H.*

It is obvious that the operator $\mathcal{R}(X)$ is the Schur complement of the so-called *dissipation matrix*

$$\Lambda(X) := \begin{bmatrix} L(X) & N(X) \\ N(X)^* & M(X) \end{bmatrix}$$

with

$$\begin{aligned} L(X) &:= A^*X + XA + Q + \Pi_1(X), \\ M(X) &:= R + \Pi_2(X), \\ N(X) &:= S + XB + \Pi_{12}(X). \end{aligned}$$

Consequently, on $D(\mathcal{R})$ the quadratic matrix inequality $\mathcal{R}(X) \geq 0$ and the linear matrix inequality $\Lambda(X) \geq 0$ are equivalent.

The following lemma provides the basis for the proof of a comparison theorem for rational matrix operators like (2.1). Therefore we introduce another hermitian matrix

$$\tilde{H} = \begin{bmatrix} \tilde{L} & \tilde{N} \\ \tilde{N}^* & \tilde{M} \end{bmatrix}$$

of the same size and structure as H. With these notations we have

2.3 Lemma ([3], Lemma 2.2). *Assume that*

$$\operatorname{Ker} N \supseteq \operatorname{Ker} M = \operatorname{Ker} \tilde{M} \subseteq \operatorname{Ker} \tilde{N}$$

and

$$\operatorname{Ker}(M - \tilde{M}) \subseteq \operatorname{Ker}(N - \tilde{N}).$$

Define $H_d := H - \tilde{H}$, $M_d := M - \tilde{M}$ and

$$K := M^+N^* - \tilde{M}^+\tilde{N}^*.$$

Then

$$H/M - \tilde{H}/\tilde{M} = H_d/M_d - K^*(M^+ - \tilde{M}^+)^+K.$$

2.4 Remark ([16], formula (4.43)). If both M and $\tilde{M}$ are invertible then

$$-M^{-1} + \tilde{M}^{-1} = M^{-1}(M_d + M_d\tilde{M}^{-1}M_d)M^{-1}.$$

3 Lyapunov equations and stability

The following lemma, which will be used in Section 4 to derive some monotonicity results, was proved in [17]:

3.1 Lemma. *Let $X(\cdot)$ be the unique solution of*

$$-\dot{X} = A^*(t)X + XA(t) + Q(t) + \hat{\Pi}(t, X), \quad X(t_f) = X_f,$$

where A, Q and $\hat{\Pi}$ are piecewise continuous, bounded matrix functions with $Q(t) \geq 0$ for all $t \leq t_f$, and $\hat{\Pi}(t, \cdot) : \mathcal{H}^n \to \mathcal{H}^n$ is a positive linear operator for every $t \leq t_f$. If $X_f \geq 0$, then $X(t) \geq 0$ for all $t \leq t_f$.

Till the end of this section we consider now the time-invariant case. Therefore let

$$\mathcal{L}_A \colon \mathcal{H}^n \to \mathcal{H}^n, \;\; X \mapsto A^*X + XA$$

denote the continuous-time Lyapunov operator where A is a given $n \times n$ complex matrix. If all eigenvalues of A lie in the open left half-plane then it is well known that the inverse of $\mathcal{L}_A$ is given by

$$\mathcal{L}_A^{-1}(X) = -\int_0^\infty e^{A^*t} X e^{At}\, dt.$$

For any linear continuous operator $\mathcal{T}$ we denote by $\rho(\mathcal{T})$ the spectral radius of $\mathcal{T}$.

3.2 Theorem ([7]). *The following statements are equivalent:*

(i) All eigenvalues of A lie in the open left half-plane and

$$\rho(\mathcal{L}_A^{-1}(\hat{\Pi})) < 1.$$

(ii) $\mathcal{L}_A + \hat{\Pi}$ is stable, i.e. $\sigma(\mathcal{L}_A + \hat{\Pi}) \subset \mathbb{C}_-$.

(iii) If $Q > 0$ then the linearly perturbed Lyapuov equation

$$\mathcal{L}_A(X) + Q + \hat{\Pi}(X) = 0$$

has a unique solution $X > 0$.

If any one of these conditions is satisfied then A is called stable relative to $\hat{\Pi}$.

Theorem 3.2 is a generalization of Lyapunovs stability theorem – there exist several extensions and modifications of this result (see [4], Corollary 3.8, Proposition 3.10 and Theorem 6.4).

3.3 Lemma ([7]). *Assume that A is stable relative to $\hat{\Pi}$ and let X be the unique solution of the linearly perturbed Lyapunov equation*

$$A^*X + XA + Q + \hat{\Pi}(X) = 0$$

where $\hat{\Pi}$ is a positive linear operator. If $Q \geq 0$ then $X \geq 0$.

3.4 Definition. A pair (A, B) of matrices with sizes $n \times n$ and $n \times m$ is said to be *stabilizable relative to* Π if there is a matrix F such that $A + BF$ is stable relative to $\left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$.

According to Theorem 3.2 (A, B) is stabilizable relative to Π if and only if the inequality

$$(A + BF)^*X + X(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix} < 0$$

is fulfilled by a pair (F, X) with $X > 0$.

3.5 Definition. A pair (C, A) of matrices with sizes $m \times n$ and $n \times n$ is said to be *detectable relative to* $\hat{\Pi}$ if there is a matrix L such that $A + LC$ is stable relative to $\hat{\Pi}$.

It is now possible to prove the following lemma:

3.6 Lemma. *Suppose $Q \geq 0$ and*

$$A^*X + XA + Q + \hat{\Pi}(X) = 0$$

has a solution $X \geq 0$. If (Q, A) is detectable relative to $\hat{\Pi}$ then A is stable relative to $\hat{\Pi}$.

4 Existence and comparison theorems

The proof of the two following lemmata is by straightforward computation and can be found in [4].

4.1 Lemma. *Let X be a hermitian matrix such that $R+\Pi_2(X)$ is invertible. Then*

$$\mathcal{R}(X) = (A + BF)^*X + X(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* [T + \Pi(X)] \begin{bmatrix} I \\ F \end{bmatrix} \quad (4.1)$$

where

$$F = F(X) := -[R + \Pi_2(X)]^{-1}[S + XB + \Pi_{12}(X)]^*.$$

4.2 Lemma. *Let X_1 and X_2 be hermitian matrices such that $R + \Pi_2(X_i)$, $i = 1, 2$, are invertible. For $i = 1, 2$, define*

$$F_i := F(X_i) = -[R + \Pi_2(X_i)]^{-1}[S + X_iB + \Pi_{12}(X_i)]^*.$$

Then

$$\begin{aligned} &\mathcal{R}(X_2) - \mathcal{R}(X_1) \\ &\quad = (A + BF_2)^*(X_2 - X_1) + (X_2 - X_1)(A + BF_2) \\ &\qquad + (F_2 - F_1)^*[R + \Pi_2(X_1)](F_2 - F_1) \\ &\qquad + \begin{bmatrix} I \\ F_2 \end{bmatrix}^* \Pi(X_2 - X_1) \begin{bmatrix} I \\ F_2 \end{bmatrix}. \end{aligned} \quad (4.2)$$

Now we consider another rational matrix operator

$$\tilde{\mathcal{R}} \colon D(\tilde{\mathcal{R}}) := \{X \in \mathcal{H}^n \mid \tilde{R} + \Pi_2(X) > 0\} \to \mathcal{H}^n$$

with

$$\begin{aligned} \tilde{\mathcal{R}}(X) = {} & A^*X + XA + \tilde{Q}Q + \Pi_1(X) \\ & - [\tilde{S} + XB + \Pi_{12}(X)][\tilde{R} + \Pi_2(X)]^{-1} \\ & \times [\tilde{S} + XB + \Pi_{12}(X)]^* \end{aligned} \quad (4.3)$$

in which the matrices Q, R and S of (1.3) are changed to $\tilde{Q}$, $\tilde{R}$ and $\tilde{S}$. We assume that $\tilde{Q}$ and $\tilde{R}$ are hermitian and define the corresponding feedback matrix $\tilde{F}$ by

$$\tilde{F} = \tilde{F}(X) := -[\tilde{R} + \Pi_2(X)]^{-1}[\tilde{S} + XB + \Pi_{12}(X)]^*.$$

With these notations we have

4.3 Lemma. *Let $X \in D(\tilde{\mathcal{R}})$ be given. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix}, \quad (4.4)$$

then

$$X \in D(\mathcal{R}) \quad \textit{and} \quad \mathcal{R}(X) \geq \tilde{\mathcal{R}}(X).$$

Proof. Inequality (4.4) implies, in particular, that $R \geq \tilde{R}$, and consequently

$$R + \Pi_2(X) \geq \tilde{R} + \Pi_2(X) > 0.$$

Define $F_d := F - \tilde{F}$, $Q_d := Q - \tilde{Q}$, $R_d := R - \tilde{R}$ and $S_d := S - \tilde{S}$. Since $T \geq \tilde{T}$ implies $\operatorname{Ker} R_d \subseteq \operatorname{Ker} S_d$, Lemma 2.3 can be applied and together with Remark 2.4 we obtain the identity

$$\begin{aligned} &\mathcal{R}(X) - \tilde{\mathcal{R}}(X) \\ &= Q_d - S_d R_d^+ S_d^* + F_d^*[R + \Pi_2(X)] \\ &\quad \times \left\{ R_d + R_d\, [\tilde{R} + \Pi_2(X)]^{-1} R_d \right\}^+ [R + \Pi_2(X)] F_d \end{aligned} \quad (4.5)$$

which shows that the difference $\mathcal{R}(X) - \tilde{\mathcal{R}}(X)$ is positive semidefinite. □

4.4 Theorem (Comparison theorem). *Let $\mathcal{I} \subset \mathbb{R}$ be some interval and $t_f \in \mathcal{I}$. Assume that X_2 and X_1 are on $\mathcal{I}$ solutions of $-\dot{X}_2 = \mathcal{R}(X_2)$ and $-\dot{X}_1 = \tilde{\mathcal{R}}(X_1)$, respectively, with $X_1(t) \in D(\tilde{\mathcal{R}})$ for $t \in \mathcal{I}$. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix},$$

then $X_2(t_f) \geq X_1(t_f)$ implies that $X_2(t) \geq X_1(t)$ for $t \in \mathcal{I} \cap (-\infty, t_f]$.

Proof. Define $X := X_2 - X_1$. Then, using (4.2) and (4.5) we infer, that X is a solution of the differential equation

$$\begin{aligned} -\dot{X} &= \mathcal{R}(X_2) - \tilde{\mathcal{R}}(X_1) \\ &= \mathcal{R}(X_2) - \mathcal{R}(X_1) + \mathcal{R}(X_1) - \tilde{\mathcal{R}}(X_1) \\ &= \hat{A}^*(t)X + X\hat{A}(t) + \hat{Q}(t) + \hat{\Pi}(t, X), \end{aligned}$$

where

$$\begin{aligned} \hat{A}(t) &= A + BF\big(X_2(t)\big), \\ \hat{Q}(t) &= \left[F\big(X_2(t)\big) - F\big(X_1(t)\big)\right]^* \left[R + \Pi_2\big(X_1(t)\big)\right] \\ &\quad \times \left[F\big(X_2(t)\big) - F\big(X_1(t)\big)\right] + Q_d - S_d R_d^+ S_d^* \\ &\quad + \left[F\big(X_1(t)\big) - \tilde{F}\big(X_1(t)\big)\right]^* \left[R + \Pi_2\big(X_1(t)\big)\right] \\ &\quad \times \left\{R_d + R_d \left[\tilde{R} + \Pi_2\big(X_1(t)\big)\right]^{-1} R_d\right\}^+ \\ &\quad \times \left[R + \Pi_2\big(X_1(t)\big)\right] \left[F\big(X_1(t)\big) - \tilde{F}\big(X_1(t)\big)\right] \end{aligned}$$

and

$$\hat{\Pi}(t, X) = \begin{bmatrix} I \\ F\big(X_2(t)\big) \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F\big(X_2(t)\big) \end{bmatrix}.$$

Therefore Lemma 3.1 yields the statement of the theorem. □

Theorem 4.4 shows that the solutions of (1.1) depend monotonically on $\left(\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right)$ and on the terminal value X_f. If $\Pi_2 \equiv 0$ and $\Pi_{12} \equiv 0$ then the positive semidefinite solutions depend also monotonically on the matrix $\left(\begin{smallmatrix} Q - SR^{-1}S^* & A^* - SR^{-1}B^* \\ A - BR^{-1}S^* & -BR^{-1}B^* \end{smallmatrix}\right)$ (see [9], [10]).

Subsequently we present two corollaries showing how the comparison theorem can be used to derive upper and lower bounds for the solutions of (1.1).

4.5 Corollary. *Assume that $R > 0$ and $Q \geq SR^{-1}S^*$. If $X_f \geq 0$ then the solution X of $-\dot{X} = \mathcal{R}(X)$, $X(t_f) = X_f$, exists on $(-\infty, t_f]$ and fulfills there the inequality*

$$0 \leq X(t) \leq X_u(t)$$

where X_u is the solution of

$$-\dot{X} = A^*X + XA + Q + \Pi_1(X), \quad X(t_f) = X_f. \tag{4.6}$$

Proof. Since (4.6) is a linear differential equation the solution X_u exists on $(-\infty, t_f]$ whereas X exists a priori only on a certain interval $(t^-, t_f]$.

Define $D := X_u - X$. Then D satisfies the differential equation

$$-\dot{D} = A^*D + DA + Q_u + \Pi_1(D), \quad D(t_f) = 0,$$

where

$$\begin{aligned} Q_u &= [S + XB + \Pi_{12}(X)][R + \Pi_2(X)]^{-1} \\ &\quad \times [S + XB + \Pi_{12}(X)]^* \geq 0. \end{aligned}$$

From Lemma 3.1 we obtain now that $D(t) \geq 0$, i.e. $X_u(t) \geq X(t)$, for $t \in (t^-, t_f]$.

With $\tilde{Q} = SR^{-1}S^*$, $\tilde{R} = R$ and $\tilde{S} = S$ the corresponding differential equation $-\dot{X} = \tilde{\mathcal{R}}(X)$ has the trivial solution. Since $\tilde{Q} \geq Q$ and $X(t_f) \geq 0$ it follows from Theorem 4.4 that $X(t) \geq 0$ for $t \in (t^-, t_f]$. Hence, X is bounded from below and above and it follows that $t^- = -\infty$. □

4.6 Corollary. *Let $\mathcal{I} \subset \mathbb{R}$ be some interval and $t_f \in \mathcal{I}$. Assume that X_l and X_u are on $\mathcal{I}$ hermitian solutions of $-\dot{X}_l \geq \mathcal{R}(X_l)$ and $-\dot{X}_u \leq \mathcal{R}(X_u)$, respectively, with $X_l(t) \in D(\mathcal{R})$ for $t \in \mathcal{I}$. Then $X_l(t_f) \leq X_f \leq X_u(t_f)$ implies that the solution X of $-\dot{X} = \mathcal{R}(X)$, $X(t_f) = X_f$, exists on $\mathcal{I} \cap (-\infty, t_f]$ and fulfills there the inequality*

$$X_l(t) \leq X(t) \leq X_u(t). \tag{4.7}$$

Proof. By the hypotheses, there exists a hermitian matrix $Q_l \leq 0$ such that $-\dot{X}_l = \tilde{R}(X_l)$ with $\tilde{Q} = Q - Q_l$, $\tilde{R} = R$ and $\tilde{S} = S$. Since $X_f \geq X_l(t_f)$ we obtain from Theorem 4.4 that $X(t) \geq X_l(t)$ for $t \in \mathcal{I} \cap (-\infty, t_f]$. Substituting Q_l by $Q_u \geq 0$ the right inequality from (4.7) follows analogously. □

4.7 Remark. All the results obtained in Section 4 remain valid if the matrices A, B, Q, R and S depend on t and the assumptions used are valid for all t.

5 Monotonicity and convergence theorems

The following monotonicity property is an immediate consequence of the comparison theorem:

5.1 Theorem. *Let $X(\cdot)$ be a solution of the rational matrix differential equation $-\dot{X} = \mathcal{R}(X)$ on some interval $(t^-, 0]$. Then $\dot{X}(0) \geq 0$ ($\dot{X}(0) \leq 0$) implies $\dot{X}(t) \geq 0$ ($\dot{X}(t) \leq 0$) for $t \in (t^-, 0]$.*

Proof. Differentiating (1.1) we get, using the definition of F,

$$-\ddot{X} = (A + BF)^*\dot{X} + \dot{X}(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(\dot{X}) \begin{bmatrix} I \\ F \end{bmatrix}.$$

The statement of the theorem follows now from Lemma 3.1. □

5.2 Theorem. *Assume that (A,B) is stabilizable relative to Π and that there is a hermitian solution $\hat{X}$ of the inequality $\mathcal{R}(X) \geq 0$ for which $\hat{X} \in D(\mathcal{R})$. Then there exists a hermitian solution X_+ of $\mathcal{R}(X) = 0$ such that $X_+ \geq X$ for every hermitian solution of $\mathcal{R}(X) \geq 0$. In particular, X_+ is the maximal hermitian solution of $\mathcal{R}(X) \geq 0$. Moreover, all the eigenvalues of*

$$A_+ := A - B[R + \Pi_2(X_+)]^{-1}[S + X_+B + \Pi_{12}(X_+)]^* \quad (5.1)$$

are contained in the closed left half-plane.

Proof. By the hypotheses, there exists a hermitian matrix $\hat{X}$ such that $\mathcal{R}(\hat{X}) = Q - \hat{Q}$ where $\hat{Q}$ is a hermitian matrix such that $\hat{Q} \leq Q$.

Since (A,B) is stabilizable relative to Π there exists an F_0 such that $A_0 := A + BF_0$ is stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$. Let X_1 be the unique solution of the linearly perturbed Lyapunov equation

$$X_1A_0 + A_0^*X_1 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_0 \end{bmatrix} = 0.$$

A straightforward computation shows that

$$(X_1 - \hat{X})A_0 + A_0^*(X_1 - \hat{X}) + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* \Pi(X_1 - \hat{X}) \begin{bmatrix} I \\ F_0 \end{bmatrix} + (\hat{F} - F_0)^*[R + \Pi_2(\hat{X})](\hat{F} - F_0) + Q - \hat{Q} = 0,$$

where

$$\hat{F} := -[R + \Pi_2(\hat{X})]^{-1}[S + \hat{X}B + \Pi_{12}(\hat{X})]^*.$$

Since A_0 is stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$, Lemma 3.3 shows that $X_1 \geq \hat{X}$. Consequently, $R + \Pi_2(X_1) \geq R + \Pi_2(\tilde{X}) > 0$.

Continuing as in the proof of Theorem 5.1 in [4] we obtain the statement of the theorem. □

5.3 Theorem. *Let $\mathcal{R}(X)$ and $\tilde{\mathcal{R}}(X)$ be given by (2.1) and (4.3), respectively. Assume that both R and $\tilde{R}$ are positive definite and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \geq 0. \quad (5.2)$$

If the equation $\mathcal{R}(X) = 0$ admits a minimal positive semidefinite solution $X_{\min}$, then $\tilde{R}(X) = 0$ admits also a minimal positive semidefinite solution $\tilde{X}_{\min}$ and $X_{\min} \geq \tilde{X}_{\min}$.

Furthermore, if $\hat{X}$ is the solution of $-\dot{X} = \mathcal{R}(X)$, $X(0) = X_0$, then $0 \leq X_0 \leq X_{\min}$ implies that $\hat{X}(t)$ exists for $t \leq 0$ and $\hat{X}(t) \to X_{\min}$ for $t \to -\infty$.

If in addition $\Pi_2 \equiv 0$ and $\Pi_{12} \equiv 0$ then the hypothesis (5.2) can be replaced by

$$\begin{bmatrix} Q - SR^{-1}S^* & A^* - SR^{-1}B^* \\ A - BR^{-1}S^* & -BR^{-1}B^* \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} - \tilde{S}\tilde{R}^{-1}\tilde{S}^* & \tilde{A}^* - \tilde{S}\tilde{R}^{-1}\tilde{B}^* \\ \tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{S}^* & -\tilde{B}\tilde{R}^{-1}\tilde{B}^* \end{bmatrix} \geq 0.$$

Proof. The solutions X and $\tilde{X}$ of the terminal value problems $-\dot{X} = \mathcal{R}(X)$, $X(0) = 0$ and $-\dot{X} = \tilde{\mathcal{R}}(X)$, $X(0) = 0$, respectively, satisfy $\dot{X}(0) = SR^{-1}S^* - Q \leq 0$ and $\dot{\tilde{X}}(0) = \tilde{S}\tilde{R}^{-1}\tilde{S}^* - \tilde{Q} \leq 0$. Therefore (see Theorem 5.1) X and $\tilde{X}$ are monotonically increasing as t is decreasing where – according to Theorem 4.4 –

$$0 \leq \tilde{X}(t) \leq X(t) \leq X_{\min} \quad \text{for} \quad t \leq 0.$$

Consequently

$$\tilde{X}_{\min} = \lim_{t \to -\infty} \tilde{X}(t) \leq \lim_{t \to -\infty} X(t) = X_{\min}.$$

Since $0 \leq \hat{X}(0) \leq X_{\min}$, we have $X(t) \leq \hat{X}(t) \leq X_{\min}$ for $t \leq 0$ and therefore $\hat{X}(t) \to X_{\min}$ for $t \to -\infty$.

The last statement of the theorem is obtained analogously, using the comparison theorem derived in [10] instead of Theorem 4.4. □

5.4 Remark. The proof of Theorem 5.3 shows how the comparison theorem 4.4 in combination with the monotonicity theorem 5.1 can be used to derive existence, convergence and monotonicity results for the solutions of (1.1) and for the solutions of the corresponding algebraic equation (1.2).

For example we can derive analogously to Theorem 5.3 results on the monotonicity of the maximal solution X_+ of $\mathcal{R}(X) = 0$ (if it exists) on the matrix T and (for $\Pi_2 \equiv 0$ and $\Pi_{12} \equiv 0$) on the matrix $\left(\begin{smallmatrix} Q - SR^{-1}S^* & A^* - SR^{-1}B^* \\ A - BR^{-1}S^* & -BR^{-1}B^* \end{smallmatrix}\right)$.

Further details on the results mentioned above will be presented elsewhere. In particular it can be shown:

(i) If $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ is detectable relative to $\left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]$ then every $X \geq 0$ with $\mathcal{R}(X) = 0$ is stabilizing in the sense that $A + BF(X)$ is stable relative to $\left[\begin{smallmatrix} I \\ F(X) \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F(X) \end{smallmatrix}\right]$.

(ii) If the algebraic equation $\mathcal{R}(X) = 0$ has a stabilizing solution then it is also the maximal solution of $\mathcal{R}(X) \geq 0$. In particular, the stabilizing solution is unique if it exists.

(iii) The assertion on the convergence of $\hat{X}$ of Theorem 5.3 holds for any initial value $X_0 \geq 0$.

6 Conclusions

The differential equation (1.1) and the corresponding algebraic equation (1.2) are generalizations of standard Riccati equations; these equations cannot be treated by transforming them using a transformation $X = YZ^{-1}$ to a linear equation for $\left(\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right)$. Nevertheless we have shown that many of the nice properties of standard Riccati equations remain valid for this more general class of rational matrix equations.

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