On the determination of differential equations with singularities and turning points

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ABSTRACT: The inverse problem of synthesizing parameters of differential systems having a finite number of arbitrary order singularities and turning points is investigated. We establish properties of the spectral characteristics, prove a uniqueness theorem and provide a prodcedure for constructing the solution of the inverse problem.

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1. Introduction

We consider the following system of differential equations

$$\frac{dy_1}{dx} = ipR(x)y_2,\ \frac{dy_2}{dx} = ip\frac{1}{R(x)}y_1,\ x \in [0,T] \tag{1}$$

with the initial conditions $y_1(0,\rho) = 1, y_2(0,\rho) = -1$. Here $\rho = \sigma + i\tau$ is the spectral parameter, and R is a real function which is called the wave resistance.

System (1) is a canonical form for many problems in natural sciences. For example, system (1) describes the wave propagation in a stratified medium and often appears in optics, spectroscopy, in electrodynamic and acoustic problems. Radioengineering problems of the design of directional couplers for non-uniform electronic lines and synthesizing transitions between acoustic wave guides can also be reduced to studying system (1) (see $[1-3]$ and references therein).

The main goal of this paper is to study the inverse problem of synthesizing the wave resistance R with desirable spectral characteristics in the case when the system has arbitrary order singularities and turning points inside the interval. More precisely, we will suppose that

$$R(x) = \prod_{j=1}^{N} |x-\gamma_j|^{p_j-1} R_0(x), \tag{2}$$

where $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N < T$, p_j are real numbers, $R_0(x) \in W_2^1[0,T], R_0(x) > 0$, $R(0) = 1$ and $R'(0) = 0$.

Some aspects of synthesis problems for system (1) with a positive $R > 0$ (i.e. $N = 0$) were studied in $[3-4]$ and other works. The central role there was played by the so-called transformation operator method (see [5] for details). The presence of arbitrary order singularities and turning points produces essential qualitative difficulties in the investigation of the inverse problem. The transformation operator method in this case is not suitable for the solution of the inverse problems. In this paper we use another method connected with ideas of the contour integral method. In our method an important role is played by the special fundamental systems of solutions (FSS's) which help us matching together solutions at the singular points and give us an opportunity to obtain the asymptotic behavior of the corresponding Stokes multipliers and to study the so-called Weyl function.

As the main spectral characteristics we introduce the amplitude reflection coefficient

$$r(\rho) = \frac{y_1(T,\rho) + a y_2(T,\rho)}{y_2(T,\rho) - a y_2(T,\rho)}, \ a := R(T),$$

the transmission coefficients

$$f_j(\rho) = \frac{1}{2\sqrt{a}} \, (y_1(T,\rho) + (-1)^j a y_2(T,\rho)), \ j = 1, 2,$$

and the characteristic function

$$\Delta(\rho) = \frac{1}{\sqrt{a}} \, y_1(T,\rho).$$

Clearly,

$$r(\rho) = \frac{f_2(\rho)}{f_1(\rho)}, \tag{3}$$

$$\Delta(\rho) = f_1(\rho) + f_2(\rho). \tag{4}$$

In this paper we will study the following inverse problem: Given $r(\rho)$, construct $R(x)$. In Section 2 we study properties of the spectral characteristics, in Section 3 a uniqueness theorem is proved, and in Section 4 we provide a constructive procedure for the solution of the inverse problem.

We note that inverse problems for Sturm-Liouville equations with turning points have been studied in [6]. Inverse problems for higher-order differential operators with singularities were considered in $[7-8]$. Some aspects of the turning point theory and a number of applications are described in $[9-11]$. In [12] the authors investigated system (1) for a very particular case of singularities which can be treated by the same way as for positive R with adequate modifications. We also note that in many cases of practical interest only the amplitude $|r|$ is accesible to measurement. Problems of reconstruction of analytic functions from their moduli have been studied in many works (see $[13-15]$ and the references therein).

2. Properties of spectral characteristics

2.1. We transform (1) by means of the replacement

$$y_1(x,\rho) = \sqrt{R(x)}\, u(x,\rho),\ y_2(x,\rho) = \frac{1}{\sqrt{R(x)}} v(x,\rho) \tag{5}$$

to the system

$$u' + h(x)u = i\rho v,\ v' - h(x)v = i\rho u,\ x \in [0,T] \tag{6}$$

with the conditions $u(0,\rho) = 1,\ v(0,\rho) = -1$, where

$$h(x) = (2R(x))^{-1}R'(x). \tag{7}$$

System (6) after elimination of v reduces to the equation

$$-u'' + q(x)u = \lambda u,\ \lambda = \rho^2, \tag{8}$$

and the conditions

$$u(0,\rho) = 1,\ u'(0,\rho) = -i\rho, \tag{9}$$

where

$$q(x) = h^2(x) - h'(x). \tag{10}$$

It follows from (2), (7) and (10) that $q(x)$ has the form

$$q(x) = \sum_{j=1}^{N} \frac{a_j}{(x-\gamma_j)^2} + q_0(x),$$

where $a_j := (\frac{p_j}{2})^2 - \frac{1}{4}$. For definiteness, we shall assume that $\frac{p_j}{2} \notin \mathbb{Z}$ (small modifications have to be introduced for the other cases). We also assume that

$$q_0(x) \cdot \prod_{j=1}^{N} |x-\gamma_j|^{1-|p_j|} \in L(0,T).$$

The amplitude reflection coefficient r, the characteristic function Δ and the transmission coefficients f_j take the form

$$r(\rho) = \frac{u(T,\rho)+v(T,\rho)}{u(T,\rho)-v(T,\rho)},\ \Delta(\rho) = u(T,\rho),\ f_j(\rho) = \frac{u(T,\rho)+(-1)^j v(T,\rho)}{2}. \tag{11}$$

2.2. Let $v_j = \frac{|p_j|}{2},\ \mu_{k_j} = (-1)^k \nu_j + \frac{1}{2},\ \omega_j = (\gamma_j, \gamma_{j+1}),\ \gamma_0 := 0,\ \gamma_{N+1} := T$. Denote

$$C_{kj}(x,\lambda) = (x-\gamma_j)^{\mu_{kj}} \sum_{m=0}^{\infty} c_{kmj}(\rho(x-\gamma_j))^{2m},\ k = 1,2,\ j = \overline{1,N},$$

where

$$c_{10j}c_{20j} = (2\nu_j)^{-1},\ c_{kmj} = (-1)^m c_{k0j} \cdot (\prod_{s=1}^{m} (2s+\mu_{kj})(2s+\mu_{kj}-1) - a_j))^{-1}.$$

Here and in the sequel $z^\mu = \exp(\mu(\ln|z| + i \arg z)),\ \ \arg z \in (-\pi,\pi]$. For $x \in \omega_j \cup \omega_{j-1}$ the functions $C_{kj}(x,\lambda)$ are solutions of the equation

$$-y'' + \frac{a_j}{(x-\gamma_j)^2}\, y = \lambda y,$$

and

$$\det[C_{kj}^{(m-1)}(x,\lambda)]_{k,m=1,2} \equiv 1.$$

Let $S_{kj}(x,\lambda),\ j=\overline{1,N},\ k=1,2$ be solutions of the following integral equation

$$S_{kj}(x,\lambda) = C_{kj}(x,\lambda) + \int_{\gamma_j}^{x} g_j(x,t,\lambda)(q(t) - \frac{a_j}{(t-\gamma_j)^2} S_{kj}(t,\lambda)dt,\ x \in \omega_j \cup \omega_{j-1},$$

where

$$g_j(x,t,\lambda) = C_{1j}(t,\lambda)C_{2j}(x,\lambda) - C_{2j}(t,\lambda)C_{1j}(x,\lambda).$$

The functions $S_{kj}(x,\lambda)$ are entire in λ of order $\frac{1}{2}$, and they form a FSS of equation (8). Moreover,

$$\det[S_{kj}^{(m-1)}(x,\lambda)]_{k,m=1,2} \equiv 1,$$

$$|S_{kj}^{(m)}(x,\lambda)| \le C|(x-\gamma_j)^{\mu_{kj}-m}|,\ |S_{kj}(x,\lambda) - C_{kj}(x,\lambda)| \le C|(x-\gamma_j)^{2\nu_j+\mu_{kj}}|,\ |\rho(x-\gamma_j)| \le 1.$$

Here and below one and the same symbol C denotes various positive constants in estimates not depending on x and ρ.

In [16] asymptotic properties of $S_{kj}(x,\lambda)$ and corresponding Stokes multipliers are investigated. In particular, for $x \in \omega_j \cup \omega_{j-1},\ (\rho,x) \in \Omega := \{(\rho,x) :\quad |\rho(x-\gamma_j)| \ge 1,$ $j=\overline{1,N}\}$, the following asymptotic formula is valid

$$\begin{aligned} S_{kj}^{(m)}(x,\lambda) = \beta_{kj}\rho^{-\mu_{kj}}((-i\rho)^m \exp(-i\rho(x-\gamma_j))[1]_j + \\ (i\rho)^m \exp(i\pi\mu_{kj}\ \mathrm{sign}\ (\gamma_j - x))\exp(i\rho(x-\gamma_j))[1]_j), \end{aligned} \tag{12}$$

where

$$[1]_j = 1 + O((\rho(x-\gamma_j))^{-1}),\ \beta_{1j}\beta_{2j} = (-4i\sin\pi\nu_j)^{-1}.$$

We note that the FSS's $\{S_{kj}(x,\lambda)\}$ will be used for matching together solutions at the singular points. In particular, if $q_0(x)$ is an analytic function, it corresponds to matching solutions by the analytic continuation in the upper half-plane $\mathrm{Im}\ x > 0$.

Let us extend the functions $S_{kj}(x,\lambda)$ to the whole segment $[0,T]$ by the formula

$$S_{kj}(x,\lambda) = A_{kj}^{1s}(\lambda)S_{1s}(x,\lambda) + A_{kj}^{2s}(\lambda)S_{2s}(x,\lambda),\ x \in \omega_s \cup \omega_{s-1}. \tag{13}$$

Lemma 1. *For $x \in \omega_s,\ (\rho,x) \in \Omega$,*

$$\begin{aligned} S_{kj}^{(m)}(x,\lambda) &= \beta_{kj}\rho^{-\mu_{kj}}((-i\rho)^m \exp(-i\rho(x-\gamma_j))[1]_\gamma \\ &+ (i\rho)^m \exp(-i\pi\mu_{kj})\exp(i\rho(x-\gamma_j))[1]_\gamma \\ &- 2i(i\rho)^m \sum_{p=j+1}^{s} \cos\pi\nu_p \cdot \exp(i\rho(x+\gamma_j-2\gamma_p))[1]_\gamma),\ s \ge j, \end{aligned} \tag{14}$$

$$\begin{aligned} S_{kj}^{(m)}(x,\lambda) &= \beta_{kj}\rho^{-\mu_{kj}}((-i\rho)^m \exp(-i\rho(x-\gamma_j))[1]_\gamma \\ &+ (i\rho)^m \exp(i\pi\mu_{kj})\exp(i\rho(x-\gamma_j))[1]_\gamma \\ &+ 2i(i\rho)^m \sum_{p=s+1}^{j-1} \cos\pi\nu_p \cdot \exp(i\rho(x+\gamma_j-2\gamma_p))[1]_\gamma),\ s < j, \end{aligned} \tag{15}$$

$$[1]_\gamma = 1 + \sum_{j=1}^{N} O(\rho(x-\gamma_j))^{-1}).$$

Proof. We prove the lemma by induction. For definiteness we confine ourselves to the case when $s \geq j$. For $s = j$ (i.e. for $x \in \omega_j$), (14) holds by virtue of (12). Fix $s > j$ and assume that for $x \in w_{s-1}$,

$$\begin{aligned} lS_{kj}^{(m)}(x,\lambda) &= \beta_{kj}\rho^{-\mu_{kj}}((-i\rho)^m \exp(-i\rho(x-\gamma_j))[1]_\gamma \\ &\quad + (i\rho)^m \exp(-i\pi\mu_{kj})\exp(i\rho(x-\gamma_j))[1]\gamma \\ &\quad - 2i(i\rho)^m \sum_{p=j+1}^{s-1} \cos\pi\nu_p \exp(i\rho(x+\gamma_j-2\gamma_p))[1]_\gamma). \end{aligned} \tag{16}$$

It follows from (12) that for $x \in \omega_{s-1}$,

$$\begin{aligned} &S_{ks}^{(m)}(x,\lambda) = \beta_{ks}\rho^{-\mu_{ks}}((-i\rho)^m \exp(-i\rho(x-\gamma_s))[1]_\gamma + \\ &(i\rho)^m \exp(i\pi\mu_{kj})\exp(i\rho(x-\gamma_s))[1]_\gamma. \end{aligned} \tag{17}$$

Substituting (16) and (17) into (13) we obtain the following algebraic system with respect to $A_{kj}^{rs}(\lambda)$:

$$\left.\begin{aligned} &A_{kj}^{1s}(\lambda)\beta_{1s}\rho^{-\mu_{1s}}[1] + A_{kj}^{2s}(\lambda)\beta_{2s}\rho^{-\mu_{2s}} = \beta_{kj}\rho^{-\mu_{kj}}\exp i\rho(\gamma_j-\gamma_s), \\ &A_{kj}^{1s}(\lambda)\beta_{1s}\rho^{-\mu_{1s}}\exp(i\pi\mu_{1s})[1] + A_{kj}^{2s}(\lambda)\beta_{2s}\exp(i\pi\mu_{2s})[1] = \beta_{kj}\rho^{-\mu_{kj}}B_{kjs}(\lambda), \end{aligned}\right\} \tag{18}$$

where

$$[1] = 1 + O(\rho^{-1}),$$

$$B_{kjs}(\lambda) = \exp(-i\pi\mu_{kj})\exp(i\rho(\gamma_s-\gamma_j)) - 2i\sum_{p=j+1}^{s-1}\cos\pi\nu_p\exp(i\rho(\gamma_s+\gamma_j-2\gamma_p))[1].$$

Since

$$\mu_{1s}+\mu_{2s}=1,\ \beta_{1s}\beta_{2s}=(-4i\sin\pi\nu_s)^{-1},\ \exp(i\pi\mu_{1s})-\exp(i\pi\mu_{2s})=2\sin\pi\nu_s,$$

the determinant of (18) is equal to $(2i\rho)^{-1}[1]$. Solving (18) we calculate

$$\left.\begin{aligned} &A_{kj}^{1s}(\lambda) = 2i\rho^{1-\mu_{ki}-\mu_{2s}}\beta_{kj}\beta_{2s}(\exp(i\pi\mu_{2s}\exp(i\rho(\gamma_j-\gamma_s))[1]B_{kjs}(\lambda)[1]), \\ &A_{kj}^{2s}(\lambda) = 2i\rho^{1-\mu_{kj}-\mu_{1s}}\beta_{kj}\beta_{1s}(-\exp(i\pi\mu_{1s})\exp(i\rho(\gamma_j-\gamma_s))[1] + B_{kjs}(\lambda)[1]). \end{aligned}\right\} \tag{19}$$

Let now $x \in \omega_s$. Then, by virtue of (12),

$$\begin{aligned} &S_{kj}^{(m)}(x,\lambda) = \beta_{ks}\rho^{-\mu_{ks}}((-i\rho)^m \exp(-i\rho(x-\gamma_\gamma))[1]_\gamma \\ &+ (i\rho)^m \exp(-i\pi\mu_{ks})\exp(i\rho(x-\gamma_s))[1]_\gamma). \end{aligned} \tag{20}$$

Substituting (20) and (19) into (13) we arrive at (14). Relation (15) is proved similarly.

□

2.3 Denote

$$\varphi_k(x,\lambda) = (-1)^{k-1}(S_{2j}^{(2-k)}(0,\lambda)S_{1j}(x,\lambda) - S_{1j}^{(2-k)}(0,\lambda)S_{2j}(x,\lambda)),\ k=1,2.$$

The functions $\varphi_k(x,\lambda)$ are solutions of (8) and $\varphi_k^{(m-1)}(0,\lambda) = \delta_{km},\ k,m=1,2$ (δ_{km} is the Kronecker delta). Clearly,

$$\langle \varphi_1(x,\lambda), \varphi_2(x,\lambda)\rangle \equiv 1, \tag{21}$$

where $\langle y,z\rangle := yz' - y'z$. Using (14) - (15) we get for $x\in\omega_s,\ (\rho,x)\in\Omega$,

$$\begin{aligned}\varphi_k^{(m-1)}(x,\lambda) &= \tfrac{1}{2}(i\rho)^{m-k}(\exp(i\rho x)[1]_\gamma + (-1)^{m-k}\exp(-i\rho x)[1]_\gamma\\ &+ (-1)^k 2i\sum_{j=1}^{s}\cos\pi\nu_j\exp(i\rho(x-2\gamma_j))[1]_\gamma),\quad |\rho|\to\infty,\quad k,m=1,2.\end{aligned} \tag{22}$$

Analogously, the functions

$$\psi_k(x,\lambda) := (-1)^{k-1}(S_{2j}^{(2-k)}(T,\lambda)S_{1j}(x,\lambda) - S_{1j}^{(2-k)}(T,\lambda)S_{2j}(x,\lambda)),\ k=1,2$$

satisfy (8) and the conditions $\psi_k^{(m-1)}(T,\lambda) = \delta_{k,m},\quad k,m=1,2.$
Moreover,

$$\langle \psi_1(x,\lambda), \psi_2(x,\lambda)\rangle \equiv 1, \tag{23}$$

and for $x\in\omega_s,\ (\rho,x)\in\Omega$,

$$\begin{aligned}\psi_k^{(m-1)}(x,\lambda) &= \tfrac{1}{2}(i\rho)^{m-k}((-1)^{m-k}\exp(i\rho(T-x))[1]_\gamma + \exp(-i\rho(T-x))[1]_\gamma\\ &+ (-1)^{k-1}2i\sum_{j=s+1}^{N}\cos\pi\nu_j\exp(i\rho(x+T-2\gamma_j))[1]_\gamma,\quad |\rho|\to\infty,\ k,m=1,2.\end{aligned} \tag{24}$$

In view of (9), we have

$$u(x,\rho) = \varphi_1(x,\lambda) - i\rho\varphi_2(x,\lambda). \tag{25}$$

It follows from (25), (22) and (11) that for $|\rho|\to\infty,\ \ \mathrm{Im}\,\rho\ge 0$,

$$\begin{aligned}&u^{(m)}(x,\rho) = (-i\rho)^m\exp(-i\rho x)[1]_\gamma - 2i(ip)^m\sum_{j=1}^{s}\cos\pi\nu_j\exp(i\rho(x-2\gamma_j))[1]_\gamma,\\ &x\in\omega_s,\ (\rho,x)\in\Omega,\ m=0,1,\end{aligned} \tag{26}$$

$$\Delta(\rho) = \exp(-i\rho T)[1] - 2i\sum_{j=1}^{N}\cos\pi\nu_j\exp(i\rho)(T-2\gamma_j))[1]. \tag{27}$$

Lemma 2. *The following relation holds*

$$|\Delta(\rho)| + |\Delta(-\rho)| \ne 0. \tag{28}$$

Proof. It follows from (5) for $\rho=0$ that

$$R(x)|u(x,0)|^2 = |y_1(x,0)|^2. \tag{29}$$

On the other hand, using (1) for $\rho = 0$, we get $y_1'(x, 0) \equiv 0$, and consequently

$$|y_1(x, 0)|^2 = A_j, \ x \in \omega_j, \ j = \overline{0, N}, \tag{30}$$

where A_j are constants, and $A_0 = 1$. Let us show that $A_j \neq 0$ for all $j = \overline{0, N}$. Suppose on the contrary that for a certain s $A_s = 0, \ A_j \neq 0 \ (j = \overline{0, s-1})$. Using the FSS $\{S_{ks}(x, \lambda)\}$ on can write

$$u(x, 0) = \alpha_{1s} S_{1s}(x, 0) + \alpha_{2s} S_{2s}(x, 0), \ x \in \omega_s \cup \omega_{s-1}.$$

Since $A_s = 0$, we have in view of (29) and (30) that $u(x, 0) \equiv 0$ for $x \in \omega_s$, and consequently $\alpha_{1s} = \alpha_{2s} = 0$. Hence $u(x, 0) \equiv 0$ for $x \in \omega_{s-1}$, i.e. $A_{s-1} = 0$. This contradicts our assumption. Since $A_0 \neq 0$, we conclude that $A_j \neq 0$ for all $j = \overline{0, N}$. In particular, this yields

$$\Delta(0) \neq 0. \tag{31}$$

Furthermore, it follows from (21) and (25) that

$$\langle u(x, \rho), \ u(x, -\rho) \rangle \equiv 2i\rho. \tag{32}$$

Hence for $\rho \neq 0$ (28) is valid. Together with (31) this gives (28) for all ρ. □

Denote $\Pi_+ = \{\rho : \ \text{Im}\ \rho > 0\}$,

$$\omega(\rho) := \Delta(\rho) \exp(i\rho T). \tag{33}$$

Let $\{\rho_n\}_{n \geq 1}$ be zeros of $\omega(\rho)$ in $\overline{\Pi}_+$. We note that if R has no singularities and turning points (i.e. $N = 0$), then $\omega(\rho)$ has no zeros in $\overline{\Pi}_+$ (see [4], [12]). However in the general case $\omega(\rho)$ has a finite or countable set of zeros in $\overline{\Pi}_+$.

Using (27) and (33), by the well-known methods (see, for example [17]) one can obtain the following properties of $\{\rho_n\}$.

1) There exist $h > 0, C_h > 0$ such that $|\omega(\rho)| \geq C_h$ for $\text{Im}\ \rho \geq h$. Hence in $\overline{\Pi}_+$ the zeros of $\omega(\rho)$ lie in the strip $\Pi_h := \{\rho : 0 \leq \ \text{Im}\ \rho \leq h\}$.

2) The number N_a of zeros of $\omega(\rho)$ in the rectangle $R_a = \{\rho : \text{Re}\ \rho \in [a, a+1], \ \ \text{Im}\ \rho \in [0, h]\}$ is bounded with respect to a.

3) Denote $G_\delta := \{\rho : |\rho - \rho_n| \geq \delta\} \cap \overline{\Pi}_+$. Then

$$|\omega(\rho)| \geq C_s, \ \rho \in G_\delta. \tag{34}$$

4) There exist numbers $R_n \to \infty$ such that for sufficiently small $\delta > 0$, the semicircles $|\rho| = R_n, \ \ \text{Im}\ \rho \geq 0$ lie in G_δ for all n.

For simplicity we confine ourselves to the case when all zeros of $\omega(\rho)$ in $\overline{\Pi}_+$ are simple.

3. The uniqueness theorem

3.1 Denote

$$p(x)=\begin{cases} q(T-x), & 0\le x\le T,\\ 0, & x>T,\end{cases}$$

and consider the equation

$$-y''+p(x)y=\lambda y,\quad x>0,\ \lambda=\rho^2 \tag{35}$$

on the half-line. Clearly,

$$p(x)=\sum_{j=1}^{N}\frac{a_{j1}}{(x-x_j)^2}+p_0(x),\quad p_0(x)\cdot\prod_{j=1}^{N}|x-x_j|^{1-2\nu_{j1}}\in L(0,T),$$

where $x_j = T-\gamma_{N-j+1},\ a_{j1} = a_{N-j+1}, \nu_{j1} = \nu_{N-j+1}$. We introduce the function $e(x,\rho), x>0$, by the formula

$$e(x,\rho)=\begin{cases} u(T-x,\rho)\exp(i\rho T), & 0\le x\le T,\\ \exp(i\rho x), & x>T.\end{cases}$$

Then $e(x,\rho)$ is a solution of equation (35), and $e(0,\rho)=\omega(\rho)$. Denote

$$\Phi(x,\lambda)=\frac{e(x,\rho)}{\omega(\rho)},\ M(\lambda)=\frac{e'(0,\rho)}{\omega(\rho)}. \tag{36}$$

The function $M(\lambda)$ is called the Weyl function.

Put $C(x,\lambda):=\psi_1(T-x,\lambda),\ S(x,\lambda):=-\psi_2(T-x,\lambda)$. It is easy to verify that $C(x,\lambda)$ and $S(x,\lambda)$ satisfy (35), and $C(0,\lambda)=S'(0,\lambda)=1,\ S(0,\lambda)=C'(0,\lambda)=0$. In view of (23), $\langle C(x,\lambda),\ S(x,\lambda)\rangle\equiv 1$. Since $\Phi(0,\lambda)=1,\ \Phi'(0,\lambda)=M(\lambda)$, we have

$$\Phi(x,\lambda)=C(x,\lambda)+M(\lambda)S(x,\lambda), \tag{37}$$

$$\langle\Phi(x,\lambda),S(x,\lambda)\rangle\equiv 1. \tag{38}$$

Denote $\Omega_\varepsilon:=\{x: x\in[0,T], |x-x_j|\ge\varepsilon, j=\overline{1,N}\}$ with a fixed $\varepsilon>0$. It follows from (24), (26) and (27) that for $|\rho|\to\infty,\ \rho\in\overline{\Pi}_+,\ x\in\Omega_\varepsilon\cap\omega_s,\ m=0,1$,

$$e^{(m)}(x,\rho)=(i\rho)^m\exp(i\rho x)[1]-2i(-i\rho)^m\sum_{j=s+1}^{N}\cos\pi\nu_{j1}\exp(i\rho(2x_j-x))[1], \tag{39}$$

$$\omega(\rho)=[1]-2i\sum_{j=1}^{N}\cos\pi\nu_{j1}\exp(2i\rho x_j)[1], \tag{40}$$

$$\begin{aligned} S^{(m)}(x,\lambda)&=\tfrac{1}{2}(i\rho)^{m-1}(\exp(i\rho x)[1]-(-1)^m\exp(-i\rho x)[1]\\ &+2i(-1)^m\sum_{j=1}^{s}\cos\pi\nu_{j1}\exp(i\rho)(2x_j-x))[1]).\end{aligned} \tag{41}$$

Denote by Π the λ-plane with the cut $\Gamma:=\{\lambda:\lambda\ge 0\}$, and $\Lambda:=\{\lambda_k\}_{k\ge1},\ \lambda_k=\rho_k^2,\ \operatorname{Im}\rho_k\ge 0$; notice that here Π and $\overline{\Pi}$ must be considered as subsets of the Riemann surface of the squareroot-function.

Theorem 1. *The Weyl function $M(\lambda)$ is analytic in $\Pi \setminus \Lambda$. In the points $\lambda = \lambda_k$ the Weyl functions $M(\lambda)$ has simple poles, and*

$$\operatorname*{Res}_{\lambda=\lambda_k} M(\lambda) = \frac{4i\rho_k^2}{\dot\omega(\rho_k)\omega(-\rho_k)}, \tag{42}$$

where $\dot\omega(\rho) = \dfrac{d}{d\rho}\omega(\rho)$. For $\lambda \in \Gamma \setminus \Lambda$ there exist the finite limits

$$M_\pm(\lambda) := \lim M(\lambda \pm iz), \quad z \to 0, \; Re\, z > 0,$$

and

$$V(\lambda) := \frac{1}{2\pi i}(M^+(\lambda) - M^-(\lambda)) = \frac{\rho}{\pi\omega(\rho)\omega(-\rho)}, \quad \rho > 0, \lambda = \rho^2. \tag{43}$$

Moreover,

$$|M(\lambda)| \le C_\delta|\rho|, \; \rho \in G_\delta; \quad |V(\lambda)| \le C_\delta|\rho|, \; \rho \in G_\delta \cap \{\rho : \rho > 0\}, \tag{44}$$

$$M(\lambda) = i\rho[1], \; |\rho| \to \infty, \; \arg\rho \in [\varepsilon_0, \pi - \varepsilon_0], \; \varepsilon_0 > 0. \tag{45}$$

Proof. The domain Π in the λ-plane corresponds to Π_+ in the ρ-plane. By virtue of (36), $M(\lambda)$ is analytic in $\Pi \setminus \Lambda$, continuous in $\overline{\Pi} \setminus \Lambda$, and we have

$$\operatorname*{Res}_{\lambda=\lambda_k} M(\lambda) = e'(0,\rho_k)\Big(\Big(\frac{d}{d\lambda}\omega(\rho)\Big)_{|\rho=\rho_k}\Big)^{-1} = \frac{2\rho_k e'(0,\rho_k)}{\dot\omega(\rho_k)}. \tag{46}$$

It follows from (32), (36), (37) and (38) that

$$\langle e(x,\rho), e(x,-\rho)\rangle \equiv -2i\rho, \quad \langle e(x,\rho), S(x,\lambda)\rangle \equiv \omega(\rho). \tag{47}$$

This implies

$$e(x,\rho_k) = e'(0,\rho_k)S(x,\lambda_k), \; \omega(-\rho_k)e'(0,\rho_k) = 2i\rho_k.$$

Together with (46) this yields (42).

Furthermore, using (36) and (47) we calculate for $\rho > 0$:

$$M^+(\lambda) - M^-(\lambda) = \frac{e'(0,-\rho)}{\omega(-\rho)} - \frac{e'(0,\rho)}{\omega(-\rho)} = \frac{2i\rho}{\omega(\rho)\omega(-\rho)},$$

i.e. (43) is valid. At last, the estimates (44) - (45) follow from (34), (36), (39), (40) and (43). □

Denote $\beta_k := -4i\rho_k^2(\dot\omega(\rho_k)\omega(-\rho_k))^{-1}$ if $\rho_k \in \Pi_+$, and $\beta_k := -2i\rho_k^2\dot\omega(\rho_k)\omega(-\rho))^{-1}$ if $\operatorname{Im}\rho_k = 0$. The set $S := (V(\lambda), \{\rho_k, \beta_k\})$ is called the spectral data.

3.2. Consider two wave resistances R and $\tilde R$. We agree that if a certain symbol α denotes an object related to R, then $\tilde\alpha$ denotes the analogous object related to $\tilde R$, and $\hat\alpha := \alpha - \tilde\alpha$.

Theorem 2. *If $r = \tilde{r}$, then $R = \tilde{R}$. Thus, the specification of the amplitude reflection coefficient uniquely determines the wave resistance.*

Proof. By virtue of (6) and (11),

$$f_j(\rho) = \frac{u(T,\rho)}{2} + \frac{(-1)^j}{2i\rho}\,(u'(T,\rho) + h(T)u(T,\rho)),\ j = 1, 2.$$

Togehter with (32) this implies

$$f_1(\rho)f_1(-\rho) - f_2(\rho)f_2(-\rho) \equiv 1. \tag{48}$$

It follows from (48) that the functions $f_1(\rho)$ and $f_2(\rho)$ has no common zeros. Hence, in view of (4) the functions $\Delta(\rho)$ and $f_1(\rho)$ has no common zeros. Furthermore, taking (3), (4) and (33) into account, we have

$$r(\rho) + 1 = \frac{\omega(\rho)\exp(-i\rho T)}{f_1(\rho)}.$$

The function $\omega(\rho)$ is entire in ρ, of exponential type, and according to (40),

$$\omega(\rho) = 1 + O(\rho^{-1}),\ |\rho| \to \infty,\ \arg\rho \in [\varepsilon_0, \pi - \varepsilon_0\},\ \varepsilon_0 > 0.$$

Let $\{\rho_k\}$ be the sequence of zeros of $r(\rho) + 1$; i.e. zeros of $\omega(\rho)$. Then

$$\omega(\rho) = a\exp(b\rho)G(\rho),$$

where

$$G(\rho) = \prod_k (1 - \frac{\rho}{\rho_k})\exp(\frac{\rho}{\rho_k}),$$
$$b = -\lim_{|\rho|\to\infty}\frac{G(\rho)}{\rho},\ a = (\lim_{|\rho|\to\infty}\exp(b\rho)G(\rho))^{-1},\ \ \arg\rho \in [\varepsilon_0, \pi - \varepsilon_0].$$

Thus, the specification of the amplitude reflection coefficient uniquely determines the function $\omega(\rho)$. In other words, under the hypothesis of Theorem 2 we have

$$\omega(\rho) = \tilde{\omega}(\rho). \tag{49}$$

Using (42) - (45) and (49) we infer that the function $\hat{M}(\lambda)$ is entire in λ.

Let us now define the matrix $P(x,\lambda) = [P_{kj}(x,\lambda)]_{k,j=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix} \tilde{\Phi}(x,\lambda) & \tilde{S}(x,\lambda) \\ \tilde{\Phi}'(x,\lambda) & \tilde{S}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \Phi(x,\lambda) & S(x,\lambda) \\ \Phi'(x,\lambda) & S'(x,\lambda) \end{bmatrix}.$$

In view of (38) we have

$$P_{jk}(x,\lambda) = (-1)^{k-1}(\Phi^{(j-1)}(x,\lambda)\tilde{S}^{(2-k)}(x,\lambda) - S^{(j-1)}(x,\lambda)\tilde{\Phi}^{2-k)}(x,\lambda)), \tag{50}$$

$$\Phi(x,\lambda) = P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda),\ S(x,\lambda) = P_{11}(x,\lambda)\tilde{S}(x,\lambda) + P_{12}(x,\lambda)\tilde{S}'(x,\lambda). \tag{51}$$

Fix $\varepsilon > 0$. It follows from (34), (36), (39) - (41) that for $x \in \Omega_\varepsilon,\ \rho \in \overline{\Pi}_+,\ m = 0, 1.$

$$\left.\begin{aligned} &|S^{(m)}(x,\lambda)| \le C(|\rho|+1)^{m-1}\exp(\tau x),\\ &|S^{(m)}(x,\lambda) - \tilde{S}^{(m)}(x,\lambda)| \le C(|\rho|+1)^{m-1}\exp(\tau x), \end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned} &|\Phi^{(m)}(x,\lambda)| \le C_\delta(|\rho|+1)^{m}\exp(-\tau x),\ \rho \in C_\delta,\\ &|\Phi^{(m)}(x,\lambda) - \tilde{\Phi}^{(m)} \le C_\delta(|\rho|+1)^{m-1}\exp(-\tau x),\ \rho \in G_\delta^+ := G_\delta \cap \tilde{G}_\delta. \end{aligned}\right\} \tag{53}$$

Using (50), (52) - (53) and (38) we obtain for $x \in \Omega_\varepsilon,\ \rho \in G_\delta^+$,

$$|P_{jk}(x,\lambda) - \delta_{jk}| \le \frac{C_\delta}{|\rho|+1}, \quad j \le k; \quad |P_{21}(x,\lambda)| \le C_\delta. \tag{54}$$

Furthermore, according to (37) and (50),

$$P_{11}(x,\lambda) = (C(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{C}'(x,\lambda)) + \hat{M}(\lambda)S(x,\lambda)\tilde{S}'(x,\lambda),$$

$$P_{12}(x,\lambda) = (S(x,\lambda)\tilde{C}(x,\lambda) - C(x,\lambda)\tilde{S}(x,\lambda)) + \hat{M}(\lambda)S(x,\lambda)\tilde{S}(x,\lambda).$$

Since $\hat{M}(\lambda)$ is entire in λ, we get that for each fixed $x \neq x_j$, the functions $P_{1k}(x,\lambda)$ are entire in λ. Together with (54) this yields $P_{11}(x,\lambda) \equiv 1,\ P_{12}(x,\lambda) \equiv 0$. Substituting this into (51) we derive

$$\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda),\ S(x,\lambda) \equiv \tilde{S}(x,\lambda)$$

for all x and λ. Consequently, $p(x) = \tilde{p}(x)$, i.e. $q(x) = \tilde{q}(x)$. In particular, this gives $u(x,\rho) \equiv \tilde{u}(x,\rho)$. Using (2), (29) and (30) we get $R_0(x) = C_j\tilde{R}_0(x),\ x \in \omega_j,\ j = \overline{0, N}$. Since $C_0 = 1$ and $R_0, \tilde{R}_0$ are continuous positive functions, we conclude that $C_j = 1$, $j = \overline{1, N}$, and $R_0 = \tilde{R}_0, R = \tilde{R}$. □

4. Solution of the inverse problem.

In this section we provide a constructive procedure for the solution of the inverse problem. The central role here is played by the so-called main equation of the inverse problem which is a linear equation in a corresponding Banach space. We give a derivation of the main equation, prove its unique solvability and show how to use the main equation for the solution of the inverse problem.

Let R and $\tilde{R}$ be such that $\gamma_j = \tilde{\gamma}_j,\ p_j = \tilde{p}_j, j = \overline{1, N}$, and

$$\xi_k = O(1),\ \int_0^\infty \frac{\hat{V}(\mu)}{|\theta|+1}\,d\mu + \sum_{k\ge 1}\frac{|\beta_k|\xi_k}{|\rho_k|+1} < \infty, \quad \mu = \theta^2, \tag{55}$$

where

$$\xi_k := |\rho_k - \tilde{\rho}_k| + |1 - \frac{\tilde{\beta}_k}{\beta_k}|.$$

Denote

$$\tilde{D}(x,\lambda,\mu) := \frac{\langle \tilde{S}(x,\lambda), \tilde{S}(x,\mu)\rangle}{\lambda - \mu}. \tag{56}$$

Since

$$-\tilde{S}''(x,\lambda)+\tilde{p}(x)\tilde{S}(x,\lambda)=\lambda\tilde{S}(x,\lambda),$$

we have

$$\tilde{D}'(x,\lambda,\mu)=\tilde{S}(x,\lambda)\tilde{S}(x,\mu). \tag{57}$$

According to (52),

$$|\tilde{S}^{(m)}(x,\lambda)|\le C(|\rho|+1)^{m-1},\ \rho\in\Pi_h,\ x\in\Omega_\varepsilon,\ m=0,1. \tag{58}$$

Using (56) - (58) and the maximum principle [19, p.128] we obtain for $x\in\Omega_\varepsilon,\ \rho,\theta\in\Pi_h$,

$$\left.\begin{aligned} |\tilde{D}(x,\lambda,\mu)| &\le \frac{C}{(|\rho|+1)(|\theta|+1)(||\rho|-|\theta||+1)},\\ |\tilde{D}^{(m+1)}(x,\lambda,\mu)| &\le \frac{C}{(|\rho|+1)(|\theta|+1)}\,(|\rho|+|\theta|+1)^m, \end{aligned}\right\} \tag{59}$$

where $\lambda=\rho^2,\mu=\theta^2$, and C depends on ε and h.

Theorem 3. *The following relations hold*

$$\begin{aligned} \tilde{S}(x,\lambda) &= S(x,\lambda)+\int_0^\infty \tilde{D}(x,\lambda,\mu)\hat{V}(\mu)S(x,\mu)d\mu\\ &+\sum_{k\ge1}(\tilde{D}(x,\lambda,\lambda_k)\beta_k S(x,\lambda_k)-\tilde{D}(x,\lambda,\tilde{\lambda}_k)\tilde{\beta}_k S(x,\tilde{\lambda}_k)), \end{aligned} \tag{60}$$

$$\begin{aligned} &D(x,\lambda,\mu)-\tilde{D}(x,\lambda,\mu)+\int_0^\infty \tilde{D}(x,\lambda,\xi)D(x,\xi,\mu)\hat{V}(\xi)d\xi\\ &+\sum_{k\ge1}(\tilde{D}(x,\lambda,\lambda_k)\beta_k D(x,\lambda_k,\mu)-\tilde{D}(x,\lambda,\tilde{\lambda}_k)\tilde{\beta}_k D(x,\tilde{\lambda}_k,\mu))=0. \end{aligned} \tag{61}$$

Proof. Let $h>0$ be such that for $\text{Im}\rho\ge h$ the functions $\omega(\rho)$ and $\tilde{\omega}(\rho)$ have no zeros, and choose $r_n=R_n^2$ such that for sufficiently small $\delta>0$, the semicircles $|\rho|=R_n, \text{Im}\rho\ge0$ lie in G_δ^+ for all n (see Section 2). Let $\Gamma=\{\lambda=u+iv : u=(2h^2)^{-2}v^2-h^2\}$ be the image of $\text{Im}\rho=h$ under the mapping $\lambda=\rho^2$. Denote $\Gamma_h=\Gamma\cap\{\lambda:|\lambda|\le r_n\}$ and consider the closed contours

$$\Gamma_{n0}=\Gamma_n\cup\{\lambda:|\lambda|=r_n,\lambda\notin\ \text{int}\ \Gamma\},\ \Gamma_{n1}=\Gamma_n\cup\{\lambda:|\lambda|=r_n,\lambda\in\ \text{int}\ \Gamma\}$$

(with counterclockwise circuit). According to (36) and (50), the functions $P_{jk}(x,\lambda)$ are analytic for $\lambda\in$ int Γ_{n0} for each fixed x. Then, by Cauchy's integral formula [19, p.84]

$$P_{1k}(x,\lambda)-\delta_{1k}=\frac{1}{2\pi i}\int_{\Gamma_{n0}}\frac{P_{1k}(x,\mu)-\delta_{1k}}{\mu-\lambda}d\mu,\quad \lambda\in\ \text{int}\ \Gamma_{n0}.$$

Using (54) we obtian

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{|\mu|=r_n}\frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}d\mu=0.$$

Since

$$\int_{\Gamma_{n0}}=\int_{|\mu|=r_n}-\int_{\Gamma_{n1}},$$

we have

$$P_{1k}(x,\lambda)=\delta_{1k}+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{P_{1k}(x,\mu)}{\lambda-\mu}d\mu,\ \lambda\notin \operatorname{int}\Gamma. \tag{62}$$

Similarly, one can derive

$$\frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu}=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}d\xi,\quad \lambda,\mu\notin \operatorname{int}\Gamma. \tag{63}$$

In view of (51) and (62) we infer

$$S(x,\lambda)=\tilde S(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}(\tilde S(x,\lambda)P_{11}(x,\mu)-\tilde S'(x,\lambda)P_{12}(x,\mu))\frac{d\mu}{\lambda-\mu}.$$

Substituting here $P_{1k}(x,\mu)$ from (50) and taking (37) into account we get

$$\tilde S(x,\lambda)=S(x,\lambda)-\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\tilde D(x,\lambda,\mu)\hat M(\mu)S(x,\mu)d\mu,\ \lambda\notin \operatorname{int}\Gamma.$$

From this, using Theorem 1 we arrive at (60).
Further, by virtue of (50) we have

$$P(x,\xi)\begin{bmatrix}\tilde S(x,\lambda)\\ \tilde S'(x,\lambda)\end{bmatrix}=\langle\tilde\Phi(x,\xi),\tilde S(x,\lambda)\rangle\begin{bmatrix}S(x,\xi)\\ S'(x,\xi)\end{bmatrix}-\langle\tilde S(x,\xi),\tilde S(x,\lambda)\rangle\begin{bmatrix}\Phi(x,\xi)\\ \Phi'(x,\xi)\end{bmatrix},$$

$$\det\Big((P(x,\lambda)-P(x,\mu))\begin{bmatrix}\tilde S(x,\lambda)\\ \tilde S'(x,\lambda)\end{bmatrix},\begin{bmatrix}S(x,\mu)\\ S'(x,\mu)\end{bmatrix}\Big)=$$

$$\langle S(x,\lambda),S(x,\mu)\rangle-\langle\tilde S(x,\lambda),\tilde S(x,\mu)\rangle.$$

Then, using (63) and (37) we calculate

$$D(x,\lambda,\mu)-\tilde D(x,\lambda,\mu)-\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\tilde D(x,\lambda,\xi)D(x,\xi,\mu)\hat M(\xi)d\xi=0,\quad \lambda,\mu\in \operatorname{int}\Gamma.$$

From this and Theorem 1 we arrive at (61). □

Theorem 4. *The following relation holds*

$$p(x)=\tilde p(x)-2\varepsilon'(x), \tag{64}$$

where

$$\varepsilon(x)=\int_0^\infty S(x,\lambda)\tilde S(x,\lambda)\hat V(\lambda)d\lambda+\sum_{k\ge1}(S(x,\lambda_k)\tilde S(x,\lambda_k)\beta_k-S(x,\tilde\lambda_k)\tilde S(x,\tilde\lambda_k)\tilde\beta_k). \tag{65}$$

Proof. Differentiating (60) twice with respect to x and using (57) and (35) we obtain

$$(\tilde p(x)-\lambda)\tilde S(x,\lambda)=(p(x)-\lambda)S(x,\lambda)+\int_0^\infty\tilde D(x,\lambda,\mu)\hat V(\mu)(p(x)-\mu)S(x,\mu)d\mu$$

$$+\sum_{k\ge1}(\tilde D(x,\lambda,\lambda_k)\beta_k(p(x)-\lambda_k)S(x,\lambda_k)-\tilde D(x,\lambda,\tilde\lambda_k)-\tilde D(x,\lambda,\tilde\lambda_k)\tilde\beta_k(p(x)-\tilde\lambda_k)S(x,\tilde\lambda_k))$$

$$+2\tilde S(x,\lambda)\Big(\int_0^\infty\tilde S(x,\mu)S'(x,\mu)\hat V(\mu)d\mu+\sum_{k\ge1}(\tilde S(x,\lambda_k)S'(x,\lambda_k)\beta_k-\tilde S(x,\lambda_k)S'(x,\tilde\lambda_k)\tilde\beta_k)\Big)$$

$$+\int_0^\infty(\tilde S(x,\lambda)\tilde S(x,\mu))'\hat V(\mu)S(x,\mu)d\mu+\sum_{k\ge1}((\tilde S(x,\lambda)\tilde S(x,\lambda_k))'\beta_kS(x,\lambda_k)-$$

$$(\tilde S(x,\lambda)\tilde S(x,\tilde\lambda_k))'\tilde\beta_kS(x,\tilde\lambda_k)).$$

We replace here $S(x,\lambda)$ by (60):

$$\tilde p(x)\tilde S(x,\lambda)=p(x)\tilde S(x,\lambda)+\int_0^\infty\langle\tilde S(x,\lambda),\tilde S(x,\mu)\rangle\hat V(\mu)S(x,\mu)d\mu$$

$$+\sum_{k\ge1}(\langle\tilde S(x,\lambda),\tilde S(x,\lambda_k)\rangle\beta_kS(x,\lambda_k)-\langle\tilde S(x,\lambda),\tilde S(x,\tilde\lambda_k)\rangle\tilde\beta_kS(x,\tilde\lambda_k))$$

$$+2\tilde S(x,\lambda)\Big(\int_0^\infty S(x,\mu)S'(x,\mu)\hat V(\mu)d\mu+\sum_{k\ge1}(\tilde S(x,\lambda_k)S'(x,\lambda_k)\beta_k-\tilde S(x,\tilde\lambda_k)S'(x,\tilde\lambda_k)\tilde\beta_k)\Big)$$

$$+\int_0^\infty(\tilde S(x,\lambda)\tilde S(x,\mu))'\hat V(\mu)S(x,\mu)d\mu+\sum_{k\ge1}((\tilde S(x,\lambda)\tilde S(x,\lambda_k))'\beta_kS(x,\lambda_k)-$$

$$(\tilde S(x,\lambda)\tilde S(x,\tilde\lambda_k))'\tilde\beta_kS(x,\tilde\lambda_k)).$$

After cancelation terms with $\tilde S'(x,\lambda)$ we arrive at (64). □

Denote

$$\lambda_{k0}=\lambda_k,\lambda_{k1}=\tilde\lambda_k,\beta_{k0}=\beta_k,\beta_{k1}=\tilde\beta_k,s_{kj}(x)=S(x,\lambda_{kj}),\tilde s_{kj}(x)=\tilde S(x,\lambda_{kj}),$$

$$\tilde P_{\lambda\mu}(x)=\tilde D(x,\lambda,\mu)\hat V(\mu),\ \tilde P_{\lambda,kj}(x)=\tilde D(x,\lambda,\lambda_{kj})\beta_{kj},$$

$$\tilde P_{ni,\mu}(x)=\tilde D(x,\lambda_{ni},\mu)\hat V(\mu),\ \tilde P_{ni,k_j}(x)=\tilde D(x,\lambda_{ni},\lambda_{kj})\beta_{kj},$$

$$\lambda>0,\ \mu<0,\ i,j=0,1.$$

The functions $P_{\lambda\mu}(x),P_{\lambda,kj}(x),P_{ni,\mu}(x)$ and $P_{ni,kj}(x)$ are defined analogously. Then, the relations (60) and (61) yield

$$\left.\begin{array}{rcl}\tilde S(x,\lambda)&=&S(x,\lambda)+\int_0^\infty\tilde P_{\lambda,\mu}(x)S(x,\mu)d\mu\\&+&\sum_{k\ge1}(\tilde P_{\lambda,k0}(x)s_{k0}(x)-\tilde P_{\lambda,k1}(x),s_{k1}(x)),\ \lambda>0,\\&&\tilde s_{ni}(x)=s_{ni}(x)+\int_0^\infty P_{ni,\mu}(x)S(x,\mu)d\mu\\&+&\sum_{k\ge1}(\tilde P_{ni,k0}(x)s_{k0}(x)-\tilde P_{ni,k1}(x)s_{k1}(x)),\ n\ge1,i=0,1,\end{array}\right\}\qquad(66)$$

$$\left.\begin{array}{rcl}P_{\lambda\mu}(x)-\tilde P_{\lambda\mu}(x)&+&\int_0^\infty\tilde P_{\lambda\xi}(x)P_{\xi\mu}(x)d\xi\\&+&\sum_{s\ge1}(\tilde P_{\lambda,s0}(x)P_{so,\mu}(x)-\tilde P_{\lambda,s1}(x)P_{s1,\mu}(x))=0,\\&&P_{\lambda,kj}(x)-\tilde P_{\lambda,kj}(x)+\int_0^\infty\tilde P_{\lambda\xi}(x)P_{\xi,kj}(x)d\xi\\&+&\sum_{s\ge1}(\tilde P_{\lambda,s0}(x)P_{s0,kj}(x)-\tilde P_{\lambda,s1}(x)P_{s1,kj}(x))=0,\end{array}\right\}\qquad(67)$$

$$\left.\begin{aligned} & P_{ni,\mu}(x) - \tilde{P}_{ni,\mu}(x) + \int_0^\infty \tilde{P}_{ni,\xi}(x) P_{\xi,\mu}(x) d\xi \\ + \;& \textstyle\sum_{s\geq 1} (\tilde{P}_{ni,s0}(x) P_{s0,\mu}(x) - \tilde{P}_{ni,s1}(x) P_{s1,\mu}(x)) = 0, \\ & P_{ni,kj}(x) - \tilde{P}_{ni,kj}(x) + \int_0^\infty \tilde{P}_{ni,\xi}(x) P_{\xi,kj}(x) d\xi \\ + \;& \textstyle\sum_{s\geq 1} (\tilde{P}_{ni,s0}(x) P_{s0,kj}(x) - \tilde{P}_{ni,s0}(x) P_{s1,kj}(x)) = 0. \end{aligned}\right\} \qquad (67)$$

Let

$$\begin{aligned} &\Omega_\lambda = (|\rho| + 1)^{-1}, \ \lambda = \rho^2 > 0, \\ &\Omega_n = (|\rho_n| + 1)^{-1}, n \geq 1, \end{aligned} \qquad \chi_n = \begin{cases} \xi_n^{-1}, & \xi_n \neq 0, \\ 0, & \xi_n = 0. \end{cases}$$

We introduce the functions

$$\tilde{\psi}(x,\lambda) = \Omega_\lambda^{-1} \tilde{S}(x,\lambda), \tilde{\psi}_{n0}(x) = \chi_n (\tilde{s}_{n0}(x) - \tilde{s}_{n1}(x)) \Omega_n^{-1},$$

$$\tilde{\psi}_{n1}(x) = s_{n1}(x) \Omega_n^{-1}, \ \tilde{H}_{\lambda\mu}(x) = \Omega_\lambda^{-1} \tilde{P}_{\lambda\mu}(x) \Omega_\mu,$$

$$\tilde{H}_{\lambda,k0}(x) = \Omega_\lambda^{-1} \tilde{P}_{\lambda,k0}(x) \xi_k \Omega_k, \ \tilde{H}_{\lambda,k1}(x) = \Omega_\lambda^{-1} (\tilde{P}_{\lambda,k0}(x) - \tilde{P}_{\lambda,k1}(x)) \Omega_k,$$

$$\tilde{H}_{n0,\mu}(x) = \Omega_n^{-1} \chi_n (\tilde{P}_{n0,\mu}(x) - \tilde{P}_{n1,\mu}(x)) \Omega_\mu, \ \tilde{H}_{ni,\mu}(x) = \Omega_n^{-1} \tilde{P}_{n1,\mu}(x) \Omega_\mu,$$

$$\begin{bmatrix} \tilde{H}_{n0,k0}(x) & \tilde{H}_{n0,k1}(x) \\ \tilde{H}_{n1,k0}(x) & \tilde{H}_{n1,k1}(x) \end{bmatrix} = \Omega_n^{-1} \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{P}_{n0,k0}(x) & \tilde{P}_{n0,k1}(x) \\ \tilde{P}_{n1,k0}(x) & \tilde{P}_{n1,k0}(x) \end{bmatrix} \begin{bmatrix} \xi_k & 1 \\ 0 & -1 \end{bmatrix} \Omega_k,$$

$\lambda, \mu > 0; \ n, k \geq 1.$

Similarly we define $\psi(x,\lambda), \psi_{ni}(x), H_{\lambda,\mu}(\lambda), H_{\lambda,kj}(x), H_{ni,\mu}(x)$ and $H_{ni,kj}(x)$. It follows from (52) and (59) that

$$\left.\begin{aligned} & |\psi(x,\lambda) \leq C, \ |\psi_{ni}(x)| \leq C, \ |\tilde{H}_{\lambda\mu}(x)|, \ |\tilde{H}_{ni,\mu}(x)| \leq \frac{C|\hat{V}(\mu)|}{(|\theta| + 1)^2}, \\ & |\tilde{H}_{\lambda,kj}(x)|, \ \tilde{H}_{ni,kj}(x)| \leq \frac{C\xi_k |\beta_k|}{(|\rho_k| + 1)^2}; \ \lambda, \mu > 0; \ n, k \geq 1; \ i, j = 0, 1. \end{aligned}\right\} \qquad (68)$$

The same estimates are valid for $\psi(x,\lambda), \psi_{kj}(x), H_{\lambda\mu}(x), H_{\lambda,kj}(x), H_{ni,\mu}(x)$ and $H_{ni,kj}(x)$. By virtue of (55) and (68) we have

$$\left.\begin{aligned} & \sup_{\lambda \geq 0} \Big(\int_0^\infty |\tilde{H}_{\lambda\mu}(x)| d\mu + \sum_{k,j} |\tilde{H}_{\lambda,kj}(x)| \Big) < \infty, \\ & \sup_{ni} \Big(\int_0^\infty |\tilde{H}_{ni,\mu}(x)| d\mu + \sum_{k,j} |\tilde{H}_{ni,kj}(x)| \Big) < \infty. \end{aligned}\right\} \qquad (69)$$

Let θ be the set of indices $v = (k,j), k \geq 1, j = 0, 1$. Consider the Banach space m of bounded sequences $\alpha = [\alpha_v]_{v\in\theta}$ with the norm $\|\alpha\|_m = \sup_{v\in\theta} |\alpha_v|$. Define the vectors

$$\psi(x) = [\psi_v(x)]_{v\in\theta} = \begin{bmatrix} \psi_{k0}(x) \\ \psi_{ki}(x) \end{bmatrix}_{k\geq 1}, \ \tilde{\psi}(x) = [\tilde{\psi}_v(x)]_{v\in\theta} = \begin{bmatrix} \tilde{\psi}_{k0}(x) \\ \tilde{\psi}_{k1}(x) \end{bmatrix}_{k\geq 1}.$$

It follows from (68) that for each fixed $x \in \Omega := \{x : x \in 0, T], \ x \neq x_j\}, \ \psi(x), \tilde{\psi}(x) \in m.$

Let $C = C[0,\infty)$ be the Banach space of continuous bounded functions $f : [0,\infty) \to \mathbb{C},\ \lambda \to f(\lambda)$ on the half-line $\lambda \ge 0$ with the norm $\|f\|_C = \sup_{\lambda\ge 0}|f(\lambda)|$. It follows from (68) that for each fixed $x \in \Omega,\ \psi(x,\cdot),\tilde\psi(x,\cdot) \in C$.

Consider the Banach space B of vectors

$$F = \begin{bmatrix} f \\ \alpha \end{bmatrix},\ f \in C,\ \alpha = [\alpha_v]_{v\in\theta} \in m$$

with the norm $\|F\|_B = \max(\|f\|_C,\ \|\alpha\|_m)$. Denote

$$\Psi(x) = \begin{bmatrix} \psi(x,\cdot) \\ \psi(x) \end{bmatrix}, \tilde\Psi(x) = \begin{bmatrix} \tilde\psi(x,\cdot) \\ \tilde\psi(x) \end{bmatrix}.$$

Then, $\Psi(x),\tilde\Psi(x) \in B$ for each fixed $x \in \Omega$. For a fixed $x \in \Omega$, let $\tilde Q = \tilde Q(x) : B \to B$ be the operator defined by

$$\tilde F = \tilde Q F,\ F = \begin{bmatrix} f \\ \alpha \end{bmatrix} \in B,\ \tilde F = \begin{bmatrix} \tilde f \\ \tilde\alpha \end{bmatrix} \in B,$$

$$\begin{aligned}
&\tilde f(\lambda) = \int_0^\infty \tilde H_{\lambda,\mu} f(\mu)d\mu + \sum_{v\in\theta} \tilde H_{\lambda v}\alpha_v,\\
&\tilde\alpha_u = \int_0^\infty \tilde H_{u,\mu} f(\mu)d\mu \sum_{v\in\theta} \tilde H_{uv}\alpha_v,\\
&\lambda,\mu \ge 0,\ u = (n,i) \in \theta,\ v = (k,j) \in \theta.
\end{aligned}$$

Analogously we define the operator $Q = Q(x)$. If follows from (69) that for each fixed $x \in \Omega$, the operators $E + \tilde Q(x)$ and $E - Q(x)$ (here E is the identity operator), acting from B to B, are linear bounded operators.

Theorem 5. *For each fixed $x \in \Omega$, the vector $\Psi(x) \in B$ is the solution of the equation*

$$\tilde\Psi(x) = (E + \tilde Q(x))\Psi(x) \tag{70}$$

in the Banach space B. The operator $(E + \tilde Q(x))$ has a bounded inverse operator, i.e. equation (70) is uniquely solvable. Equation (70) is called the main equation of the inverse problem.

Proof. Taking into account our notations we can rewrite (66) in the form

$$\begin{aligned}
\tilde\psi(x,\lambda) &= \psi(x,\lambda) + \int_0^\infty \tilde H_{\lambda\mu}(x)\psi(x,\mu)d\mu + \sum_{k,j} \tilde H_{\lambda,kj}(x)\psi_{kj}(x),\\
\tilde\psi_{ni}(x) &= \psi_{ni}(x) + \int_0^\infty \tilde H_{ni,\mu}(x)\psi(x,\mu)d\mu + \sum_{k,j} \tilde H_{ni,kj}(x)\psi_{kj}(x),
\end{aligned}$$

which gives (70). Similarly, (67) takes the form

$$(E + \tilde Q(x))(E - Q(x)) = E.$$

Interchanging places for R and $\tilde R$ we obtain analogously

$$\Psi(x) = (E - Q(x))\tilde\Psi(x),\ (E - Q(x))(E + \tilde Q(x)) = E.$$

Hence the operator $(E + \tilde{Q}(x))^{-1}$ exists, and it is a linear bounded operator. □

Using the main equation (70) we obtain the following algorithm for the solution of the inverse problem:

Given the amplitude reflection coefficient r. Then

1) Construct the function ω and the spectral data S (see Section 2).
2) Choose $\tilde{R}$ and caluculate $\tilde{Q}$ and $\tilde{\Psi}$.
3) Find Ψ by solving the main equation (70).
4) Calculate $p(x)$ by (64)-(65).
5) Construct $u(x) := u(x, 0)$ from the relations $u''(x) = p(T - x)u(x), u(0) = 1, u'(0) = 0$.
6) Calculate $R(x)$ using (29)-(30).

We note that this method also allows one to obtain necessary and sufficient condititons for the solvability of the inverse problem.

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