Convergence and existence results for continuous- and discrete-time Riccati equations

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1 Introduction

This note is mainly concerned with the convergence properties of the solutions of two types of Riccati equations. The first one is the periodic Riccati differential equation (PRDE)

$$\dot{X} = -A^*(t)X - XA(t) - Q(t) + XS(t)X, \quad X(t_0) = X_0 \tag{PRDE}$$

where A, Q and S are $n \times n$ piecewise continuous, bounded matrix functions of period ω with Q and S hermitian. In the time-invariant case, where A, Q and S are constant matrices, we denote this equation by (RDE).

The second equation to be studied is the Riccati difference equation

$$X(t) = A^*X(t+1)(I + SX(t+1))^{-1}A + Q, \quad X(t_0) = X_0; \tag{DRE}$$

here A, Q and S are constant matrices of dimension $n \times n$ with Q and S hermitian; the results obtained are also valid in the case of periodic coefficients.

In the last three decades the matrix Riccati equations (RDE) and (DRE) and the corresponding algebraic Riccati equations (ARE) and (DARE) have received a great deal of attention (see [4] and [23]). One of the main topics in the analysis of Riccati differential and difference equations is the study of the attractiveness properties of its constant or – in the case of periodic coefficients – ω-periodic solutions as $t \to -\infty$; this topic has been studied among others in [1], [5], [7], [8], [9], [12], [17], [24], [25], [28] and [29]. In particular it is known from [8], [9], [12] and [17] that the solution X of (RDE) converges towards the strong solution X_+ of (ARE) if $S = BB^*$, (A, B) is stabilizable, $Q \geq 0$ and $X_0 > 0$. Shayman ([28]) has provided a complete phase portrait of the (LQ-type) periodic Riccati differential equation under controllability and the assumption that the characteristic multipliers of the corresponding Hamiltonian matrix (see Section 2) do not lie on the unit circle. An analogous analysis is available in the discrete-time and time-varying case ([15]). The convergence towards the strong solution of (DRE) has also been studied in [1], [5], [17] and [24].

The most general convergence results for (RDE) are due to Park and Kailath (see [26], where also a short survey of convergence results in the time-invariant case can be found). In particular Park and Kailath extended the work of Callier et al. ([7], [8] and [9]) and studied the situation where (A, B) is stabilizable but not necessarily controllable and where the basin of attraction of X_+ as $t \to -\infty$ may be unbounded from above and below.

The main object of this paper is twofold:

1. We show that the analogous results are also valid for (PRDE) and (DRE) since these equations have many applications in optimal control and filtering. The results obtained are best possible.
2. We present a new method which is mainly based on general comparison theorems and a representation formula for the solutions of Riccati differential respectively difference equations. The proofs of the main convergence results (see Theorems 4.5 and 5.10) are elementary and selfcontained; moreover the idea of the proof is the same for (PRDE) and (DRE).

Furthermore, in the Theorems 3.1 and 5.13 we provide necessary and sufficient conditions for the existence of a periodic equilibrium of (PRDE).

¿From the proofs of our main convergence results it becomes transparent how the comparison theorems ensure that the solutions considered do not have finite escape time $t < 0$ and how, subsequently, the representation formula (4.4) (resp. (5.2)) yields the desired convergence results; a key role is played here by Corollary 4.2 (resp. Corollary 5.7) which displays niceley the influence of the terminal value $X_0 = X(0)$ on the solution of (PRDE) (resp. (DRE)).

For the discussion of the basic structural properties of (PRDE) and (DRE) and the corresponding definitions (controllability, observability etc.) we refer to [2], [23] and [25].

2 Preliminary results and notations

For convenience of the reader we cite here without proof some results concerning (PRDE) which we use in Sections 3 and 4 of this paper.

With (PRDE) we associate the corresponding Hamiltonian matrix

$$H(t) = \begin{pmatrix} A(t) & -S(t) \\ -Q(t) & -A^*(t) \end{pmatrix},$$

and we introduce the Riccati function

$$\mathcal{R}(X; H(t)) := -A^*(t)X - XA(t) - Q(t) + XS(t)X.$$

Let $X(\cdot, X_0, t_0)$ denote the solution of (PRDE) with $X(t_0) = X_0$. Notice that $X(\cdot, X_0, t_0)$ is hermitian as long as it exists if and only if $X_0 = X_0^*$.

To formulate a comparison theorem, we define the $2n \times 2n$ matrices $\widetilde{H}$ and J by

$$\widetilde{H}(t) = \begin{pmatrix} \widetilde{A}(t) & -\widetilde{S}(t) \\ -\widetilde{Q}(t) & -\widetilde{A}^*(t) \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix.

2.1 Theorem (Comparison Theorem, [21]). *Let $\mathcal{I} \subset \mathbb{R}$ be some interval and $t_0 \in \mathcal{I}$. Assume that X resp. $\widetilde{X}$ is on $\mathcal{I}$ a solution of $\dot{X} = \mathcal{R}(X; H(t))$ resp. $\dot{X} = \mathcal{R}(X; \widetilde{H}(t))$. If*

$$JH(t) \leq J\widetilde{H}(t) \quad \text{for} \quad t \in \mathcal{I}$$

then $X(t_0) \leq \widetilde{X}(t_0)$ implies that $X(t) \leq \widetilde{X}(t)$ for $t \in \mathcal{I} \cap (-\infty, t_0]$.

Notice that this means in particular that the solution X depends monotonically on the initial value X_0 and on the coefficients Q and $-S$.

2.2 Lemma ([16]). *Let $X := X(\cdot, X_0, t_0)$ be a solution of (PRDE) which exists for $t_0 - \omega \leq t \leq t_0$. If*

$$X(t_0 - \omega) \overset{\geq}{\underset{\leq}{}} X(t_0)$$

then

$$X(t - \omega) \overset{\geq}{\underset{\leq}{}} X(t)$$

holds on the maximal common interval of existence of X and $X(\cdot - \omega)$.

2.3 Theorem ([10]). *Let $S(t) = B(t)B^*(t)$, (A, B) be a stabilizable pair, and assume that there exists an ω-periodic solution of the differential inequality $\dot{X} \geq \mathcal{R}(X; H(t))$. Then there exists an ω-periodic equilibrium X_+ of (PRDE) with the following properties:*

- *(i) $X_+ \geq X$ for every ω-periodic solution X of $\dot{X} \geq \mathcal{R}(X; H(t))$. In particular X_+ is the maximal ω-periodic equilibrium of (PRDE) and $X_+ \geq 0$ if $Q \geq 0$.*
- *(ii) All characteristic multipliers of the corresponding closed loop matrix $A_{X_+} := A - SX_+$ are contained in the closed unit disk of the complex plane.*

2.4 Lemma ([27], Section II, 2). *If X is a solution of (PRDE) on some interval $\mathcal{I}$ and Y is a fundamental matrix of*

$$\dot{Y} = [A(t) - S(t)X(t)]\, Y,$$

then Y, $Z = XY$ is a solution pair of

$$\dot{Y} = A(t)Y - S(t)Z, \quad \dot{Z} = -Q(t)Y - A^*(t)Z \tag{2.1}$$

on $\mathcal{I}$. Conversely, if Y, Z is a solution pair of (2.1) with Y non-singular on some interval $\mathcal{I}$, then $X = ZY^{-1}$ is a solution of (PRDE) on $\mathcal{I}$.

For any matrix function M, $\Phi_M(t, \tau)$ denotes its transition matrix; in particular $\Psi_H(t) := \Phi_H(t + \omega, t)$ is the monodromy matrix of H at t. It can be shown that its eigenvalues are independent of t; they are called the *characteristic multipliers* of H.

It is known (see [31]) that $\Phi_H(t, 0)$ may be expressed in the form $\Phi_H(t, 0) = P(t)e^{Rt}$, where P is a piecewise continuously differentiable matrix function of period ω with $P(0) = I$ and R is a non-singular $n \times n$ matrix. If $J_R = W^{-1}RW$ is a Jordan canonical form of R, then

$$\Phi_H(t, 0)W = V(t)e^{J_R t} \quad \text{with} \quad V(t) := P(t)W = \Big(v_1(t), \dots, v_{2n}(t)\Big) \tag{2.2}$$

is also a fundamental matrix of $\dot{x} = H(t)x$.

Let $\lambda_1, \lambda_2, \dots, \lambda_{2n}$ denote the eigenvalues of R with $\operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_2 \leq \dots \leq \operatorname{Re}\lambda_{2n}$. For definiteness we let the Jordan canonical form of R be given by

$$J_R = \operatorname{diag}(J_{r_1}(\mu_1), J_{r_2}(\mu_2), \dots, J_{r_p}(\mu_p))$$

with $1 \le r_i \le 2n$, $r_1 + \ldots + r_p = 2n$, where $J_{r_i}(\mu_i)$ is the $r_i \times r_i$ Jordan block with $\mu_i \in \{\lambda_1, \ldots, \lambda_{2n}\}$ on the main diagonal:

$$J_{r_i}(\mu_i) = \begin{pmatrix} \mu_i & 1 & 0 & \ldots & 0 \\ 0 & \mu_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & \mu_i \end{pmatrix}.$$

Then we infer from (2.2) that to each Jordan block $J_{r_i}(\mu_i)$, $1 \le i \le p$, there corresponds a set of r_i linearly independent solutions

$$\begin{aligned} y_{r_1+\ldots+r_{i-1}+1}(t) &= e^{\mu_i t} v_{r_1+\ldots+r_{i-1}+1}(t) =: e^{\mu_i t} p_{r_1+\ldots+r_{i-1}+1}(t), \\ &\vdots \\ y_{r_1+\ldots+r_{i-1}+r_i}(t) &= e^{\mu_i t} \sum_{j=1}^{r_i} \frac{t^{r_i-j}}{(r_i-j)!} v_{r_1+\ldots+r_{i-1}+j}(t) =: e^{\mu_i t} p_{r_1+\ldots+r_{i-1}+r_i}(t) \end{aligned} \tag{2.3}$$

of the differential equation $\dot{x} = H(t)x$, where v_i are ω-periodic vector functions.

If $p_k(t) = \Big(p_{k1}(t), \ldots, p_{k,2n}(t)\Big)^T$ are the polynomials with ω-periodic coefficients defined by (2.3), we set

$$\widetilde{p}_k := (p_{k1}, \ldots, p_{kn})^T \quad \text{and} \quad p_k^{(i,j)} := (p_{k1}, \ldots, p_{k,j-1}, p_{k,n+i}, p_{k,j+1}, \ldots, p_{kn})^T;$$

analogously we define $\widetilde{v}_k$ and $v_k^{(i,j)}$.

The next result follows by Lemma 2.4 and Cramer's rule, where the determinants are evaluated using the Cauchy-Binet formula for computing $n \times n$ minors of matrix products.

2.5 Theorem ([20]). *Let $X_0 \in \mathbb{C}^{n \times n}$ be given. If $d_1, \ldots, d_{2n}$ are the row vectors of $D = V^{-1}(0) \left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix} \right)$, then $X = (x_{ij})_{i,j=1,\ldots,n}$ with*

$$x_{ij}(t) = \frac{\displaystyle\sum_{1 \le k_1 < \ldots < k_n \le 2n} e^{(\lambda_{k_1} + \ldots + \lambda_{k_n})t} |p_{k_1}^{(i,j)}(t), \ldots, p_{k_n}^{(i,j)}(t)| \begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}}{\displaystyle\sum_{1 \le k_1 < \ldots < k_n \le 2n} e^{(\lambda_{k_1} + \ldots + \lambda_{k_n})t} |\widetilde{p}_{k_1}(t), \ldots, \widetilde{p}_{k_n}(t)| \begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}} \tag{2.4}$$

is the solution of (PRDE) with $X(0) = X_0$.

Notice that the determinants $\begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}$ in (2.4) are determined uniquely up to a common constant factor by X_0; they are called *Plücker coordinates* of this initial value and also of the subspace $\operatorname{Im} \left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix} \right)$.

2.6 Remark. From the definition of the polynomials $p_k(t)$ (see (2.3)) it follows that for each fixed n-tuple $(k_1, \ldots, k_n)$, $1 \le k_1 < k_2 < \ldots < k_n \le 2n$, there exist constants $M \in \mathbb{N}_0$ and

$N_0, N_1, \dots, N_M \in \mathbb{N}$ such that with $\alpha_{\mu\nu} \in \mathbb{R}$, $\ell_1(\mu,\nu), \dots, \ell_n(\mu,\nu) \in \{1, 2, \dots, 2n\}$, $0 \le \mu \le M$, $1 \le \nu \le N_\mu$, simultaneously

$$|\widetilde{p}_{k_1}(t), \dots, \widetilde{p}_{k_n}(t)| = \sum_{\mu=0}^{M} t^\mu \sum_{\nu=1}^{N_\mu} \alpha_{\mu\nu} |\widetilde{v}_{\ell_1(\mu,\nu)}(t), \dots, \widetilde{v}_{\ell_n(\mu,\nu)}(t)|$$

and for $1 \le i, j \le n$

$$|p_{k_1}^{(i,j)}(t), \dots, p_{k_n}^{(i,j)}(t)| = \sum_{\mu=0}^{M} t^\mu \sum_{\nu=1}^{N_\mu} \alpha_{\mu\nu} |v_{\ell_1(\mu,\nu)}^{(i,j)}(t), \dots, v_{\ell_n(\mu,\nu)}^{(i,j)}(t)|,$$

where in particular $N_M = 1$ and $v_{\ell_1(M,1)}(0), \dots, v_{\ell_n(M,1)}(0)$ are generalized eigenvectors of $\Phi_H(\omega, 0)$ corresponding to characteristic multipliers $e^{\lambda_{k_1}\omega}, \dots, e^{\lambda_{k_n}\omega}$ of H, spanning an n-dimensional $\Phi_H(\omega, 0)$-invariant subspace.

It follows now that

$$|\widetilde{p}_{k_1}(t), \dots, \widetilde{p}_{k_n}(t)| = t^M \left\{ |\widetilde{v}_{\ell_1(M,1)}(t), \dots, \widetilde{v}_{\ell_n(M,1)}(t)| + O(1/t) \right\}$$

and

$$|p_{k_1}^{(i,j)}(t), \dots, p_{k_n}^{(i,j)}(t)| = t^M \left\{ |v_{\ell_1(M,1)}^{(i,j)}(t), \dots, v_{\ell_n(M,1)}^{(i,j)}(t)| + O(1/t) \right\}.$$

If in particular $v_{k_1}(0), \dots, v_{k_n}(0)$ are spanning a $\Phi_H(\omega, 0)$-invariant subspace then

$$|\widetilde{p}_{k_1}(t), \dots, \widetilde{p}_{k_n}(t)| = |\widetilde{v}_{k_1}(t), \dots, \widetilde{v}_{k_n}(t)|$$

and

$$|p_{k_1}^{(i,j)}(t), \dots, p_{k_n}^{(i,j)}(t)| = |v_{k_1}^{(i,j)}(t), \dots, v_{k_n}^{(i,j)}(t)|;$$

notice that in this case $|\widetilde{v}_{k_1}(t), \dots, \widetilde{v}_{k_n}(t)| \ne 0$ for $t \in \mathbb{R}$ if and only if

$$\operatorname{Im}\{v_{k_1}(0), \dots, v_{k_n}(0)\} \cap \begin{bmatrix} 0 \\ \mathcal{C}_{A,B}^{\perp} \end{bmatrix} = \{0\}$$

where

$$\mathcal{C}_{A,B} := \sum_{r=0}^{n-1} \operatorname{Im}[\Phi_A^r(0,\omega) G_C(0,\omega)]$$

with

$$G_C(\tau, t) = \int_\tau^t \Phi(\tau, s) B(s) B^*(s) \Phi_A^*(\tau, s)\, ds.$$

3 An existence result

In general it is very difficult to prove the existence of a periodic solution of a nonlinear differential equation. For (PRDE) the following theorem provides a tool for checking the existence of an ω-periodic solution.

3.1 Theorem. *The following statements are equivalent:*

(i) (PRDE) has an ω-periodic equilibrium.

(ii) There exist two hermitian ω-periodic functions X_ℓ, X_u (which may coincide) with

a) $X_\ell(t) \leq X_u(t)$ for $t \in \mathbb{R}$,
b) $\dot{X}_\ell(t) \geq \mathcal{R}(X_\ell(t); H(t))$ for $t \in \mathbb{R}$,
c) $\dot{X}_u(t) \leq \mathcal{R}(X_u(t); H(t))$ for $t \in \mathbb{R}$.

(iii) There exist hermitian matrices X_1, X_2 with

α) $X_1 \leq X_2$,
β) $X_1 \leq X(t_0 - \omega, X_1, t_0)$,
γ) $X_2 \geq X(t_0 - \omega, X_2, t_0)$.

Proof. (i) $\Longrightarrow$ (ii) is trivial since a), b), c) are fulfilled with $X_\ell = X_u = \hat{X}$ for any ω-periodic equilibrium $\hat{X}$ of (PRDE).

(ii) implies that there are matrices Q_ℓ, Q_u with $Q_\ell(t) \leq 0 \leq Q_u(t)$, $t \in \mathbb{R}$, such that for $t \in \mathbb{R}$

$$\begin{aligned}\dot{X}_\ell(t) &= \mathcal{R}(X_\ell(t); H(t)) - Q_\ell(t),\\ \dot{X}_u(t) &= \mathcal{R}(X_u(t); H(t)) - Q_u(t).\end{aligned}$$

¿From a) and Theorem 2.1 we infer that for $t \leq t_0$

$$X_\ell(t) \leq X(t, X_\ell(t_0), t_0) \leq X(t, X_u(t_0), t_0) \leq X_u(t),$$

in particular it follows that

$$\begin{aligned}X_\ell(t_0 - \omega) = X_\ell(t_0) &=: X_1 \leq X(t_0 - \omega, X_1, t_0),\\ X_u(t_0 - \omega) = X_u(t_0) &=: X_2 \geq X(t_0 - \omega, X_2, t_0).\end{aligned}$$

Hence (ii) implies (iii).

Assume (iii) is valid. Then it follows from Lemma 2.2 and Theorem 2.1 that $X(\cdot, X_1, t_0)$ and $X(\cdot, X_2, t_0)$ exist for $t \leq t_0$ and fulfill

$$X(t, X_1, t_0) \leq X(t - \omega, X_1, t_0) \leq X(t - \omega, X_2, t_0) \leq X(t, X_2, t_0) \quad \text{for} \quad t \leq t_0.$$

Hence the limits

$$X_{j\infty}(t) = \lim_{k\to\infty} X(t - k\omega, X_j, t_0) = \lim_{k\to\infty} X(t - (k \mp 1)\omega, X_j, t_0) = X_{j\infty}(t \pm \omega)$$

exist for $j = 1, 2$ and $t \leq t_0$. Consequently (i) holds, since $X_{1\infty}$ and $X_{2\infty}$ are obviously periodic equilibria of (PRDE) (which may coincide). □

3.2 Remark. Notice that up to this point we did not make any assumptions on the definiteness of S or Q. In this case the main problem in the application of Theorem 3.1 arises when we have to check either the conditions a), b) and c) of (ii) (or, equivalently, the conditions α), β) and γ) of (iii)). Fortunately in most applications at least one of the matrix functions Q and S is positive semidefinite and/or the existence of an ω-periodic solution of at least one of the Riccati differential inequalities is guaranteed in advance.

If for example $Q \geq 0$ then obviously (ii), b) is fulfilled with $X_\ell(t) \equiv 0$; moreover, $S = BB^*$ and (A, B) stabilizable guarantee, according to Theorem 2.3, that there is an ω-periodic function $X_u \geq 0$ satisfying condition c) of Theorem 3.1, (ii) (even with "=" instead of "$\leq$").

Conditions (iii), β) and γ) could be tested numerically using standard software for the numerical solution of (PRDE) and by trying adequate initial values.

4 Convergence to X_+

In the first part of this section we assume that

$$S = BB^*,\ Q = C^*C \text{ and } (A, B) \text{ is controllable.} \tag{4.1}$$

For structural properties and basic definitions we refer the reader to [2] and [25]. According to Theorem 2.3 there exists a maximal periodic equilibrium X_+ and (similarly) a minimal periodic equilibrium X_- of (PRDE). They are unique up to a shift $t - t_0$ of the argument; we shall define these solutions uniquely by prescribing an initial value for $t = 0$. Moreover X_+ is positive semidefinite and the characteristic multipliers of A_{X_+} are contained in $\overline{\mathbb{D}}$; similarly X_- is negative semidefinite and the characteristic multipliers of A_{X_-} are contained in $\mathbb{C} \setminus \mathbb{D}$.

For $\diamond \in \{<, \leq, =, \geq, >\}$ $\mathcal{R}^d_\diamond(M)$ denotes the sum of all root subspaces $\mathcal{R}_\lambda(M) = \mathrm{Ker}(M - \lambda I)^n$ of $M \in \mathbb{C}^{n\times n}$ corresponding to eigenvalues λ of M with $|\lambda| \diamond 1$. Further,

$$G_O(\tau, t) = \int_\tau^t \Phi_A^*(s, \tau) C^*(s) C(s) \Phi_A(s, \tau)\, ds$$

denotes the observability Gramian, and the unobservable subspace is defined by

$$\mathcal{U}_{C,A} = \bigcap_{r=0}^{n-1} \mathrm{Ker}(G_O(0, \omega)\Phi_A^r(\omega, 0)).$$

¿From [25] we know:

(i) There is a one-to-one correspondence between the set

$$\left\{ \mathcal{S} = \mathrm{Im}\begin{pmatrix} Y \\ Z \end{pmatrix} \,\middle|\, \begin{array}{c} \mathcal{S} \text{ is } \Phi_H(\omega, 0)\text{-invariant,} \\ \det Y \neq 0 \text{ and } Y^*Z = Z^*Y \end{array} \right\}$$

and the set of all hermitian ω-periodic equilibria; this correspondence is defined by

$$\mathcal{S} = \mathrm{Im}\begin{pmatrix} Y \\ Z \end{pmatrix} \longleftrightarrow X(\cdot, ZY^{-1}, 0).$$

(ii) $\{y_1, \ldots, y_k\} \subset \mathrm{Ker}\, G_O(0, \omega)$ is a Jordan chain of $\Phi_A(\omega, 0)$ corresponding to the (C, A)-unobservable eigenvalue λ if and only if $\left\{ \binom{y_1}{0}, \ldots, \binom{y_k}{0} \right\}$ is a Jordan chain of $\Phi_H(\omega, 0)$ corresponding to the eigenvalue λ.

(iii) If $\Phi_H(\omega, 0)$ has eigenvalues of modulus 1 then

$$\dim \mathcal{R}^d_=(\Phi_H(\omega, 0)) = 2 \dim(\mathcal{U}_{C,A} \cap \mathcal{R}^d_=(\Phi_A(\omega, 0))) =: 2\kappa,$$

and the first half of each Jordan chain corresponding to each such eigenvalue has the form $\binom{y_1}{0}, \ldots, \binom{y_k}{0}$ with a Jordan chain $\{y_1, \ldots, y_k\} \subset \mathrm{Ker}\, G_O(0, \omega)$ of $\Phi_A(\omega, 0)$.

With the notations of Section 2 it follows from (2.2) that

$$\Phi_H(t,0) = V(t)e^{J_R t}V^{-1}(0).$$

A Jordan basis of $\Phi_H(\omega,0)$ is given by $v_1(0),\ldots,v_{2n}(0)$ where $v_1,\cdots,v_{n-\kappa}$ (and $v_{n+\kappa+1},\ldots,v_{2n}$) correspond to characteristic multipliers $e^{\lambda\omega}\in\mathbb{D}$ (and $e^{\lambda\omega}\in\mathbb{C}\setminus\overline{\mathbb{D}}$, respectively). In the case $\kappa>0$ we divide the index set $\{n-\kappa+1,\ldots,n+\kappa\}$ into the subsets $\{f_1,\ldots,f_\kappa\}$ and $\{s_1,\ldots,s_\kappa\}$ so that $\{v_j(0)\mid j=f_1,\ldots,f_\kappa\}$ contains the eigenvectors of the form $\binom{y_j}{0}$, where $y_j\in\operatorname{Ker}G_O(0,\omega)\cap\mathcal{R}^d_=(\Phi_A(\omega,0))$ (see the preceding remarks). With this notation the $\Phi_H(\omega,0)$-invariant subspaces corresponding to the maximal solution X_+ and the minimal solution X_- are

$$\mathcal{S}_+ = \operatorname{Im}\{v_j(0)\mid j=1,\ldots,n-\kappa,f_1,\ldots,f_\kappa\} \tag{4.2}$$

and

$$\mathcal{S}_- = \operatorname{Im}\{v_j(0)\mid j=n+\kappa+1,\ldots,2n,f_1,\ldots,f_\kappa\}, \tag{4.3}$$

respectively.

If H has no characteristic multipliers of modulus one we say that (PRDE) is *exponentially dichotomic*. In this case we have

$$\operatorname{Re}(\lambda_{k_1}+\ldots+\lambda_{k_n}) > \operatorname{Re}(\lambda_1+\ldots+\lambda_n) \quad \text{for} \quad (k_1,\ldots,k_n)\neq(1,\ldots,n).$$

If in addition $\begin{vmatrix} d_1\\ \vdots\\ d_n\end{vmatrix}\neq 0$ and X has no finite negative escape time we obtain from Remark 2.6 that for $t\le 0$

$$\begin{aligned} x_{ij}(t) &= \frac{\displaystyle\sum_{1\le k_1<\ldots<k_n\le 2n} e^{[(\lambda_{k_1}+\ldots+\lambda_{k_n})-(\lambda_1+\ldots+\lambda_n)]t}\,|p^{(i,j)}_{k_1}(t),\ldots,p^{(i,j)}_{k_n}(t)|\begin{vmatrix} d_{k_1}\\ \vdots\\ d_{k_n}\end{vmatrix}}{\displaystyle\sum_{1\le k_1<\ldots<k_n\le 2n} e^{[(\lambda_{k_1}+\ldots+\lambda_{k_n})-(\lambda_1+\ldots+\lambda_n)]t}\,|\widetilde{p}_{k_1}(t),\ldots,\widetilde{p}_{k_n}(t)|\begin{vmatrix} d_{k_1}\\ \vdots\\ d_{k_n}\end{vmatrix}} \\ &= \frac{|v^{(i,j)}_1(t),\ldots,v^{(i,j)}_n(t)|}{|\widetilde{v}_1(t),\ldots,\widetilde{v}_n(t)|} + O\left(t^\nu e^{\operatorname{Re}(\lambda_{n+1}-\lambda_n)t}\right) \end{aligned} \tag{4.4}$$

for some $\nu\in\{0,1,\ldots,n-1\}$. This means that $X(t)-X_+(t)\to 0$ as $t\to-\infty$ at an exponential rate.

In the case where H has characteristic multipliers of modulus one there are several summands in formula (2.4) having an exponential term $e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t}$ growing as fast as $e^{(\lambda_1+\ldots+\lambda_{n-\kappa})t}$. It is easy to see that under our assumptions among these terms the term with the Plücker coordinate

$$P_{\mathrm{dom}} := \begin{vmatrix} d_1\\ \vdots\\ d_{n-\kappa}\\ d_{s_1}\\ \vdots\\ d_{s_\kappa}\end{vmatrix} \tag{4.5}$$

(this definition makes sense also for $\kappa = 0$) is the dominant term as $t \to -\infty$ since – as a consequence of (2.3) – the degree of the polynomial (with periodic coefficients) appearing in this term is maximal. In this case (PRDE) is called *polynomially dichotomic.* Using the special structure of the polynomials $\widetilde{p}_k$ and $p_k^{(i,j)}$ (see Remark 2.6) it follows in an elementary way that if $P_{\mathrm{dom}} \neq 0$ and $X(t)$ exists for $t \leq 0$ then we have

$$x_{ij}(t) = \frac{|v_1^{(i,j)}(t), \ldots, v_{n-\kappa}^{(i,j)}(t), v_{f_1}^{(i,j)}(t), \ldots, v_{f_\kappa}^{(i,j)}(t)|}{|\widetilde{v}_1(t), \ldots, \widetilde{v}_{n-\kappa}(t), \widetilde{v}_{f_1}(t), \ldots, \widetilde{v}_{f_\kappa}(t)|} + O\left(\frac{1}{t}\right) \tag{4.6}$$

for $t \leq 0$, i.e. $X(t) - X_+(t) \to 0$ as $t \to -\infty$ at a polynomial rate.

For the subsequent convergence results it is important to know conditions under which the Plücker coordinate P_{dom} is nonzero.

4.1 Lemma. *Let $X_0 \in \mathbb{C}^{n\times n}$ be given and let $D = \begin{pmatrix} d_1 \\ \vdots \\ d_{2n} \end{pmatrix} := V^{-1}(0) \begin{pmatrix} I \\ X_0 \end{pmatrix}$. If $\{i_1, \ldots, i_n\}$ and $\{j_1, \ldots, j_n\}$ are two disjoint index sets with $\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_n\} = \{1, 2, \ldots, 2n\}$ then the following statements hold:*

(i)

$$p \in \operatorname{Ker} \begin{pmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{pmatrix} \iff \begin{pmatrix} I \\ X_0 \end{pmatrix} p \in \operatorname{Im}(v_{j_1}, \ldots, v_{j_n}).$$

(ii) If $\mathcal{S} = \operatorname{Im}(v_{j_1}, \ldots, v_{j_n}) = \operatorname{Im}\begin{pmatrix} Y \\ Z \end{pmatrix}$, then

$$\begin{vmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{vmatrix} \neq 0 \iff X_0Y - Z \text{ is invertible.}$$

If in addition $\det Y \neq 0$ and $X := ZY^{-1}$, this means

$$\begin{vmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{vmatrix} \neq 0 \iff X_0 - X \text{ is invertible.}$$

(iii) If $0 \neq p \in \operatorname{Ker} X_0$ and $\begin{pmatrix} p \\ 0 \end{pmatrix} \in \operatorname{Im}(v_{j_1}, \ldots, v_{j_n})$, then $\begin{vmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{vmatrix} = 0$.

Proof. (i) We set

$$V^{-1} =: \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix}.$$

Then

$$w_i v_j = \delta_{ij}, \quad 1 \leq i, j \leq 2n,$$

and it follows that

$$\begin{aligned} p \in \operatorname{Ker} \begin{pmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{pmatrix} \quad &\iff \quad \begin{pmatrix} w_{i_1} \\ \vdots \\ w_{i_n} \end{pmatrix} \begin{pmatrix} I \\ X_0 \end{pmatrix} p = 0 \\ &\iff \quad \begin{pmatrix} I \\ X_0 \end{pmatrix} p \in \operatorname{Im}(v_{j_1}, \ldots, v_{j_n}). \end{aligned}$$

(ii) From (i) we infer that $\begin{vmatrix} d_{i_1} \\ \vdots \\ d_{i_n} \end{vmatrix} = 0$ if and only if there exist $p, q \in \mathbb{C}^n \setminus \{0\}$ with

$$\begin{pmatrix} I \\ X_0 \end{pmatrix} p = \begin{pmatrix} Y \\ Z \end{pmatrix} q;$$

this is possible if and only if $X_0 Y - Z$ is not invertible.

Under the assumption $\det Y \neq 0$, $X = ZY^{-1}$, the second assertion of (ii) follows from $X_0 Y - Z = (X_0 - X)Y$, and (iii) is a consequence from (i). ☐

4.2 Corollary. *Let* $X_0 \in \mathbb{C}^{n \times n}$ *be given and let* $D := V^{-1}(0) \left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix} \right)$.

(i) If (4.1) holds then

$$P_{\text{dom}} \neq 0 \iff X_0 - X_-(0) \textit{ is invertible.}$$

(ii) If (4.1) holds and $\Phi_H(\omega, 0)$ *has no eigenvalues on* $\partial\mathbb{D}$ *then, assuming that* $X(t, X_0, 0)$ *exists for* $t \leq 0$, *we have*

$$\lim_{t \to -\infty} (X(t, X_0, 0) - X_+(t)) = 0 \iff X_0 - X_-(0) \textit{ is invertible.}$$

(iii) If

$$\lim_{t \to -\infty} (X(t, X_0, 0) - X_+(t)) = 0,$$

then

$$\operatorname{Ker} X_0 \cap \mathcal{U}_{C,A} \cap \mathcal{R}^d_>(\Phi_A(\omega, 0)) = \{0\}.$$

Proof. (i) follows immediately from (4.3), (4.5) and Lemma 4.1, (ii). Together with (4.4) one gets (ii).

To proof (iii) suppose for a contradiction that there is a nonzero $p \in \operatorname{Ker} X_0 \cap \mathcal{U}_{C,A} \cap \mathcal{R}^d_>(\Phi_A(\omega, 0))$. Then $\left(\begin{smallmatrix} p \\ 0 \end{smallmatrix} \right)$ is a generalized eigenvector of $\Phi_H(\omega, 0)$ corresponding to an eigenvalue λ with $\operatorname{Re} \lambda > 0$, i.e., $\left(\begin{smallmatrix} p \\ 0 \end{smallmatrix} \right) \in \operatorname{Im}\{v_j(0) \mid j = n + \kappa + 1, \ldots, 2n\}$. Therefore as a consequence of Lemma 4.1, (iii), all Plücker coordinates $\begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}$ with $1 \leq k_1 < \ldots < k_n \leq n + \kappa$ are zero. ☐

Using the preceding corollary and the representation formula we obtain a short proof of the next theorem, which extends [28], Theorem 8, c).

4.3 Theorem. *Let (4.1) hold. If* $X_0 = X_0^* \geq X_-(0)$ *then*

$$X_-(t) \leq X(t, X_0, 0) \leq \widetilde{X}(t) \quad \text{for} \quad t \leq 0, \tag{4.7}$$

where $\widetilde{X}$ *is the solution of*

$$\dot{\widetilde{X}} = -A^*(t)\widetilde{X} - \widetilde{X}A(t) - Q(t), \quad \widetilde{X}(0) = X_0, \tag{4.8}$$

and for $X_0 > X_-(0)$

$$\lim_{t \to -\infty} (X(t, X_0, 0) - X_+(t)) = 0. \tag{4.9}$$

Moreover, if H *has no characteristic multipliers of modulus 1 the convergence in (4.9) takes place at an exponential rate, see (4.4); otherwise we have convergence at a polynomial rate, see (4.6).*

Proof. (4.7) follows from the Comparison Theorem 2.1, therefore $X(t, X_0, 0)$ exists for $t \leq 0$. $X_0 > X_-(0)$ yields (see Corollary 4.2) that the Plücker coordinate P_{dom} is nonzero. Hence (4.9) follows easily from (4.4) and (4.6). $\square$

4.4 Lemma. *Let (4.1) hold. If* $\Phi_H(\omega, 0)$ *has no eigenvalues on* $\partial\mathbb{D}$ *and* $X_0 \not\geq X_-(0)$ *then* $X(\cdot, X_0, 0)$ *has at least one pole on* $(-\infty, 0)$.

Proof. (i) Assume in addition to $X_0 \not\geq X_-(0)$ that $X(\cdot, X_0, 0)$ does not have a finite negative excape time and even converges to X_+, i.e.,

$$\lim_{t \to -\infty} (X(t, X_0, 0) - X_+(t)) = 0.$$

Let X_1, X_2 be two matrices with $X_0 \leq X_1 \leq X_2$, $\det(X_1 - X_-(0)) = 0$ and $X_2 > X_-(0)$. From the Comparison Theorem 2.1 we infer that for $t \leq 0$

$$X(t, X_0, 0) \leq X(t, X_1, 0) \leq X(t, X_2, 0).$$

Hence, according to Corollary 4.2, (ii),

$$\lim_{t \to -\infty} (X(t, X_j, 0) - X_+(t)) = 0 \quad \text{for} \quad j = 0, 1, 2,$$

which yields, again by making use of Corollary 4.2, (ii), a contradiction to $\det(X_1 - X_-(0)) = 0$.

(ii) Let now $X_0 = X_0^* \not\geq X_-(0)$ be arbitrary. Then there exists a matrix $X_3 = X_3^* \geq X_0$ with $\det(X_3 - X_-(0)) \neq 0$ and $X_3 \not\geq X_-(0)$. Assume that $X(\cdot, X_0, 0)$ has no finite excape in negative time. Then it follows, using again the Comparison Theorem 2.1 and Corollary 4.2, (ii), that $X(\cdot, X_3, 0)$ has no finite escape time and converges to X_+, in contradiction to part (i) of the proof.

This proves the assertion of the lemma. $\square$

In the remaining part of this section we assume instead of (4.1) only that

$$S = BB^*, \; Q = C^*C \text{ and } (A, B) \text{ is stabilizable.} \tag{4.10}$$

In this case Theorem 2.3 yields that X_+ exists but X_- is undefined if (A, B) is not controllable.

For the formulation of our main convergence theorem, which generalizes the results of Callier and Willems [7], Callier and Winkin [8] and Park and Kailath [26] to the periodic case we assume without loss of generality (otherwise we have to perform first an ω-periodic transformation of coordinates, see [3]) that

$$\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) =: \left(\begin{array}{ccc|c} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & B_2 \\ A_{31} & A_{32} & A_{33} & B_3 \\ \hline C_1 & C_2 & 0 & 0 \end{array}\right),$$

where $\left(\left(\begin{smallmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{smallmatrix}\right), \left(\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right)\right)$ is controllable, A_{11} is exponentially stable (i.e. its characteristic multipliers are in $\mathbb{D}$), (C_2, A_{22}) has no unobservable characteristic multipliers on $\partial\mathbb{D}$ and A_{33} has only multipliers on $\partial\mathbb{D}$.

Now we partition the solution $X(\cdot, X_0, 0)$ of (PRDE) as

$$X(\cdot, X_0, 0) = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & \hat{X} \end{pmatrix}$$

with $\hat{X} \in \mathbb{C}^{r\times r}$ where $r = \dim \mathcal{C}_{A,B}$. Then (PRDE) decomposes into the following three equations:

$$\begin{aligned} \dot{X}_{11} &= -A_{11}^* X_{11} - X_{11} A_{11} - C_1^* C_1 \\ &\quad - X_{12} \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} - (A_{21}^* \ A_{31}^*) X_{12}^* + X_{12} \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \ B_3^*) X_{12}, \\ \dot{X}_{12} &= -A_{11}^* X_{12} - X_{12} \left[\begin{pmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{pmatrix} - \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \ B_3^*) \hat{X} \right] - (A_{21}^* \ A_{31}^*) \hat{X} - (C_1^* C_2 \ 0), \\ \dot{\hat{X}} &= -\begin{pmatrix} A_{22}^* & A_{32}^* \\ 0 & A_{33}^* \end{pmatrix} \hat{X} - \hat{X} \begin{pmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{pmatrix} - \begin{pmatrix} C_2^* C_2 & 0 \\ 0 & 0 \end{pmatrix} + \hat{X} \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \ B_3^*) \hat{X}. \quad (4.11) \end{aligned}$$

If $\hat{X}$ exists for $t \le 0$ then the differential equations for X_{11} and X_{12} are linear with coefficients depending on $\hat{X}$ and therefore it is clear that $X(\cdot, X_0, 0)$ exists for $t \le 0$ if and only if $\hat{X}$ exists for $t \le 0$.

The reduced Riccati differential equation (4.11) has a maximal ω-periodic equilibrium of the form $\hat{X}_+ = \left(\begin{smallmatrix} X_{22+} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and a minimal ω-periodic equilibrium of the form $\hat{X}_- = \left(\begin{smallmatrix} X_{22-} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where X_{22+} and X_{22-} are the maximal and the minimal ω-periodic equilibrium of

$$\dot{X}_{22} = -A_{22}^*(t) X_{22} - X_{22} A_{22}(t) - C_2^*(t) C_2(t) + X_{22} B_2(t) B_2^*(t) X_{22}. \quad (4.12)$$

Notice that according to Lemma 4.4 the solution of this differential equation exists for $t \le 0$ if and only if $X_{22}(0) \ge X_{22-}(0)$.

With the preceding notations we get

4.5 Theorem. *Let (4.10) hold,*

$$X_0 = X_0^* = \begin{pmatrix} X_{011} & X_{012} & X_{013} \\ X_{012}^* & X_{022} & X_{023} \\ X_{013}^* & X_{023}^* & X_{033} \end{pmatrix}$$

with

$$\hat{X}_0 := \begin{pmatrix} X_{022} & X_{023} \\ X_{023}^* & X_{033} \end{pmatrix} \ge \begin{pmatrix} X_{22-}(0) + \varepsilon I & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{for some} \quad \varepsilon > 0.$$

Then $X(t, X_0, 0)$ exists for $t \le 0$ and

$$\lim_{t\to-\infty} (X(t, X_0, 0) - X_+(t)) = 0.$$

Proof. Let $\hat{X}$ denote the solution of (4.11) with $\hat{X}(0) = \hat{X}_0$. If $\hat{X}_\ell$ and $\hat{X}_u$ are solutions of (4.11) with

$$\hat{X}_\ell(0) := \left(\begin{array}{cc} X_{22-}(0) + \varepsilon I & 0 \\ 0 & 0 \end{array} \right) \le \hat{X}_0 < \hat{X}_u(0),$$

then

$$\hat{X}_\ell(t) = \begin{pmatrix} X_{\ell 22}(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad t \le 0$$

where $X_{\ell 22}$ satisfies (4.12). Moreover it follows from the Comparison Theorem and Corollary 4.2, (ii), applied to (4.11) and (4.12) that $\hat{X}_\ell(t)$ and $\hat{X}_u(t)$ exist for $t \le 0$ with

$$\hat{X}_\ell(t) \le \hat{X}(t) \le \hat{X}_u(t) \quad \text{for} \quad t \le 0$$

and

$$\lim_{t\to-\infty} \left[\hat{X}_\ell(t) - \hat{X}_+(t) \right] = 0 = \lim_{t\to-\infty} \left[\hat{X}_u(t) - \hat{X}_+(t) \right].$$

Hence

$$\lim_{t\to-\infty} \left[\hat{X}(t) - \hat{X}_+(t) \right] = 0.$$

Now we study the difference

$$D := \left(\begin{array}{cc} D_{11} & D_{12} \\ D_{12}^* & \hat{D} \end{array} \right) := X(\cdot, X_0, 0) - X_+$$

which solves the Riccati differential equation

$$\dot{D} = - \left(A_{X_+} \right)^* (t) D - D A_{X_+}(t) + D B(t) B^*(t) D, \quad D(0) = X_0 - X_+(0).$$

If we partition X_+ conformally with the partition of D, we see that A_{X_+} takes the form

$$A_{X_+} = \begin{pmatrix} A_{11} & 0 \\ A_{X_+}^{21} & A_{X_+}^{22} \end{pmatrix} \quad \text{with} \quad A_{X_+}^{22} = \begin{pmatrix} (A_{22})_{X_{22+}} & 0 \\ * & A_{33} \end{pmatrix}$$

and we have

$$\begin{aligned} \dot{D}_{11} &= -A_{11}^*(t) D_{11} - D_{11} A_{11}(t) - F(t), \\ \dot{D}_{12} &= -A_{11}^*(t) D_{12} - D_{12} G(t) - \left(A_{X_+}^{21} \right)^* (t) \hat{D}(t) \end{aligned}$$

where

$$\begin{aligned} F &:= D_{12} A_{X_+}^{21} + \left(A_{X_+}^{21} \right)^* D_{12}^* - D_{12} \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*) D_{12}^*, \\ G &:= A_{X_+}^{22} - \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*) \hat{D}. \end{aligned}$$

¿From the well-known representation formula for the solutions of Sylvester differential equations it follows that

$$\begin{aligned} D_{12}(t) &= \Phi_{-A_{11}^*}(t,0) D_{12}(0) \Phi_{-G^*}^*(t,0) \\ &\qquad + \int_t^0 \Phi_{-A_{11}^*}(t,\tau) \left(A_{X_+}^{21} \right)^* (\tau) \hat{D}(\tau) \Phi_{-G^*}^*(t,\tau) \, d\tau. \end{aligned} \tag{4.13}$$

Since $G(t) - A^{22}_{X_+}(t) \to 0$ as $t \to -\infty$ and since all the characteristic multipliers of A_{11} (resp. $A^{22}_{X_+}$) lie in the open (resp. closed) unit disk there are constants $K_1, K_2, \varepsilon > 0$, $0 < \delta < \varepsilon$ such that

$$\|\Phi_{-A^*_{11}}(t,\tau)\| \le K_1 e^{\varepsilon(t-\tau)}, \quad \|\Phi^*_{-G^*}(t,\tau)\| \le K_2 e^{-\delta(t-\tau)}$$

for $t \le \tau$. From (4.13) we get now

$$\|D_{12}(t)\| \le K_1 K_2 \left\{ e^{(\varepsilon-\delta)t} \|D_{12}(0)\| + \int_t^0 e^{(\varepsilon-\delta)(t-\tau)} \| \left(A^{21}_{X_+}\right)^* (\tau) \hat{D}(\tau)\| \, d\tau \right\}.$$

Since $A^{21}_{X_+}(\tau)$ is uniformly bounded for $\tau \le 0$ and $\hat{D}(\tau)$ tends to zero as $\tau \to -\infty$ this implies (see [6], proof of Theorem 7.2.75) that $\lim_{t\to-\infty} D_{12}(t) = 0$ and, consequently, $\lim_{t\to-\infty} F(t) = 0$ from which similarly follows that $\lim_{t\to-\infty} D_{11}(t) = 0$ for arbitrary initial values.

This proves the assertion of the theorem. □

It is easy to show that the solution $\hat{X}(\cdot, \hat{X}_0, 0)$ has at least one pole on $(-\infty, 0)$ if $\hat{X}_0 - \left(\begin{smallmatrix} X_{22-} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ has less then $\dim X_{22-}$ positive eigenvalues. On the other hand $\hat{X}(t, \hat{X}_0, 0)$ exists for all $t \le 0$ if $\hat{X}_0 \ge \left(\begin{smallmatrix} X_{22-} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Up to now it is not known if $\hat{X}(\cdot, \hat{X}_0, 0)$ always has finite escape time $t < 0$ if $\hat{X}_0 \not\ge \left(\begin{smallmatrix} X_{22-} & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Instead we get by combining Lemma 4.4 with Theorem 4.5

4.6 Corollary. *If* $\mathcal{U}_{C,A} \cap \mathcal{R}_=(\Phi_A(\omega, 0)) = \{0\}$ *(which is here fulfilled if and only if* $\Phi_H(\omega, 0)$ *has no eigenvalues on* $\partial\mathbb{D}$*) then* $X(t, X_0, 0)$ *exists for all* $t \le 0$ *if and only if* $X_{022} \ge X_{22-}(0)$*, and*

$$\lim_{t\to-\infty} (X(t, X_0, 0) - X_+(t)) = 0$$

if and only if $X_{022} > X_{22-}(0)$.

The necessity of the last condition follows by a reasoning similar to that of step (i) in the proof of Lemma 4.4.

5 The discrete-time case

In this section we derive the discrete-time versions of the convergence results of Section 4. In order to abbreviate the representation of the results we consider here only the *time-invariant* Riccati difference equation

$$X(t) = A^* X(t+1)(I + SX(t+1))^{-1} A + Q, \quad X(t_0) = X_0; \tag{DRE}$$

where A, $Q = C^*C$ and $S = BB^*$ are $n \times n$ matrices with A invertible – analogous results are also valid in the case of periodic coefficients.

Since the idea of the proof is essentially the same as in Section 4 we give here only a sketch of the proofs; moreover we list the main tools needed for a detailed proof.

With (DRE) we associate the corresponding Hamiltonian matrix

$$H = \begin{pmatrix} A & -S \\ -Q & -A^* \end{pmatrix}$$

and the symplectic matrices

$$T = \begin{pmatrix} A^{-1} & A^{-1}S \\ QA^{-1} & A^* + QA^{-1}S \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} A + SA^{-*}Q & -SA^{-*} \\ -A^{-*}Q & A^{-*} \end{pmatrix}.$$

For $X \in \mathbb{C}^{n\times n}$ such that $I + SX$ is invertible we define the discrete-time Riccati function

$$\mathcal{R}^d(X; H) := A^*X(I + SX)^{-1}A + Q.$$

Since $S = BB^*$ this can be written as

$$\mathcal{R}^d(X; H) = A^*XA + Q - A^*XB(I + B^*XB)^{-1}B^*XA,$$

see [23], Proposition 12.1.1. If

$$\widetilde{H} = \begin{pmatrix} \widetilde{A} & -\widetilde{S} \\ -\widetilde{Q} & -\widetilde{A}^* \end{pmatrix}$$

denotes another Hamiltonian matrix we can formulate the following comparison theorem:

5.1 Theorem ([30]). *Let terminal values* X_0 *and* $\widetilde{X}_0$ *be given with* $X_0 \le \widetilde{X}_0$. *Assume that there is a solution* X *of (DRE) with* $I + B^*X(t)B > 0$ *for* $t = t_-, t_- + 1, \ldots, t_0$. *If*

$$JH \le J\widetilde{H}$$

then the solution $\widetilde{X}$ *of* $X(t) = \mathcal{R}^d(X(t+1); \widetilde{H})$, $\widetilde{X}(t_0) = \widetilde{X}_0$ *exists and satisfies the inequalities* $I + B^*\widetilde{X}(t)B > 0$ *and* $X(t) \le \widetilde{X}(t)$ *for* $t = t_-, t_- + 1, \ldots, t_0 - 1$.

Notice that the preceding theorem is also valid for time-varying coefficient matrices. In [18] a generalized comparison theorem is proved where Q and S are not assumed to be positive semidefinite.

5.2 Theorem ([23], Theorem 13.1.1). *Let* (A, B) *be a d-stabilizable pair and assume that there is a solution* $\widetilde{X}$ *of the discrete Riccati inequality* $\mathcal{R}^d(X; H) \ge X$ *for which* $I + B^*\widetilde{X}B > 0$. *Then there exists a unique equilibrium* $X_+ = X_+^*$ *of (DRE) such that* $I + B^*X_+B > 0$ *and* $X_+ \ge X$ *for all hermitian solutions of* $\mathcal{R}^d(X; H) \ge X$. *Moreover, all the eigenvalues of the corresponding discrete-time closed loop matrix*

$$A_{X_+} := (I + SX_+)^{-1}A = A - B(I + B^*X_+B)^{-1}B^*X_+A$$

lie in the closed unit disk.

5.3 Remark. It is known (see e.g. [23], Proposition 12.2.2) that there is a one-to-one correspondence between the set of hermitian equilibria of (DRE) and the set

$$\left\{ \mathcal{S} = \operatorname{Im}\begin{pmatrix} Y \\ Z \end{pmatrix} \,\middle|\, \mathcal{S} \text{ is } T^{-1}\text{-invariant, } \det Y \ne 0 \text{ and } Y^*Z = Z^*Y \right\}.$$

But since a root subspace of T^{-1} corresponding to an eigenvalue λ is also a root subspace of T corresponding to an eigenvalue $1/\lambda$ it is also possible to represent the equilibria of (DRE) in terms of root subspaces of T. In particular the subspace $\mathcal{S}_+$ corresponding to the maximal solution X_+ is spanned by the generalized eigenvectors of T corresponding to eigenvalues $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and by the first halves of Jordan chains corresponding to eigenvalues of modulus 1 which take the form $\left(\begin{smallmatrix} y_1 \\ 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} y_k \\ 0 \end{smallmatrix}\right)$ (the special form of these Jordan chains is obtained as in [25]). If (A, B) is controllable then the minimal solution X_- also exists. Its corresponding subspace $\mathcal{S}_-$ is spanned by the generalized eigenvectors of T corresponding to eigenvalues $\lambda \in \mathbb{D}$ and by the first halves of Jordan chains corresponding to eigenvalues of modulus 1.

The following lemma relates the solutions of (DRE) to the solutions of a linear difference equation.

5.4 Lemma ([19]). *If X is a solution of (DRE) on some interval $\mathcal{I}$ of $\mathbb{Z}$ and Y is a fundamental matrix of the difference equation*

$$Y(t) = A^{-1}(I + SX(t+1))Y(t+1),$$

then Y, $Z := XY$ is a solution pair of the linear difference equation

$$\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} A^{-1} & A^{-1}S \\ QA^{-1} & A^* + QA^{-1}S \end{pmatrix} \begin{pmatrix} Y(t+1) \\ Z(t+1) \end{pmatrix}. \tag{5.1}$$

Conversely, if Y, Z is a solution pair of (5.1) with Y non-singular on some interval $\mathcal{I}$ of $\mathbb{Z}$, then $X := ZY^{-1}$ is a solution of (DRE) on $\mathcal{I}$.

With $\widetilde{Y}(t) = Y(-t)$, $\widetilde{Z}(t) = Z(-t)$ we can write (5.1) as a forward recursion. So we obtain the difference equation

$$\begin{pmatrix} \widetilde{Y}(t+1) \\ \widetilde{Z}(t+1) \end{pmatrix} = T \begin{pmatrix} \widetilde{Y}(t) \\ \widetilde{Z}(t) \end{pmatrix}$$

to which the standard theory of linear difference equations can be applied.

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ denote the eigenvalues of T with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{2n}|$. Let $V = (v_1, \ldots, v_{2n}) \in \mathbb{C}^{2n \times 2n}$ be the matrix defined by a Jordan basis of corresponding generalized eigenvectors of T. We consider a Jordan canonical form of T,

$$J_T := V^{-1}TV = \operatorname{diag}(J_{r_1}(\mu_1), J_{r_2}(\mu_2), \ldots, J_{r_p}(\mu_p))$$

with $1 \leq r_i \leq 2n$, $r_1 + \ldots + r_p = 2n$, where $J_{r_i}(\mu_i)$ is the $r_i \times r_i$ Jordan block with $\mu_i \in \{\lambda_1, \ldots, \lambda_{2n}\}$ on the main diagonal (see Section 2 in the continuous-time case). To each such Jordan block $J_{r_i}(\mu_i)$, $1 \leq i \leq p$, there corresponds a set of r_i linearly independent solutions

$$\begin{aligned} y_{r_1+\ldots+r_{i-1}+1}(t) &:= \mu_i^t v_{r_1+\ldots+r_{i-1}+1}, \\ &\vdots \\ y_{r_1+\ldots+r_{i-1}+r_i}(t) &:= \sum_{j=1}^{r_i} \binom{t}{r_i - j} \mu_i^{t-r_i+j} v_{r_1+\ldots+r_{i-1}+j} \end{aligned}$$

of the difference equation $x(t+1) = Tx(t)$.

For an arbitrary vector $v = (v_1, \ldots, v_{2n})^T \in \mathbb{C}^{2n}$ we define as in Section 2

$$\widetilde{v} := (v_1, \ldots, v_n)^T \quad \text{and} \quad v^{(i,j)} := (v_1, \ldots, v_{j-1}, v_{n+i}, v_{j+1}, \ldots, v_n)^T.$$

With these notations we can formulate the discrete-time version of Theorem 2.5:

5.5 Theorem ([19]). *Let $X_0 \in \mathbb{C}^{n \times n}$ be given. If $d_1, \ldots, d_{2n}$ are the row vectors of $D = V^{-1}(0) \left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix} \right)$, then $X = (x_{ij})_{i,j=1,\ldots,n}$ with*

$$x_{ij}(-t) = \frac{\displaystyle\sum_{1 \leq k_1 < \ldots < k_n \leq 2n} |y_{k_1}^{(i,j)}(t), \ldots, y_{k_n}^{(i,j)}(t)| \begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}}{\displaystyle\sum_{1 \leq k_1 < \ldots < k_n \leq 2n} |\widetilde{y}_{k_1}(t), \ldots, \widetilde{y}_{k_n}(t)| \begin{vmatrix} d_{k_1} \\ \vdots \\ d_{k_n} \end{vmatrix}}, \quad t \in \mathbb{N}_0, \tag{5.2}$$

is the solution of (DRE) with $X(0) = X_0$.

5.6 Remark. (i) Just like in the continuous-time case (see Remark 2.6) it is easy to see that for each fixed n-tupel $(k_1,\ldots,k_n)$, $1 \le k_1 < k_2 < \ldots < k_n \le 2n$, there exist constants $M \in \mathbb{N}_0$ and $N_0, N_1, \ldots, N_M \in \mathbb{N}$ such that with $\alpha_{\mu\nu} \in \mathbb{R}$, $\ell_1(\mu,\nu),\ldots,\ell_n(\mu,\nu) \in \{1,2,\ldots,2n\}$, $0 \le \mu \le M$, $1 \le \nu \le N_\mu$

$$|\widetilde{y}_{k_1}(t),\ldots,\widetilde{y}_{k_n}(t)| = (\lambda_{k_1}\cdot\ldots\cdot\lambda_{k_n})^t \sum_{\mu=0}^{M} t^{\mu} \sum_{\nu=1}^{N_\mu} \alpha_{\mu\nu} |\widetilde{v}_{\ell_1(\mu,\nu)},\ldots,\widetilde{v}_{\ell_n(\mu,\nu)}|$$

where in particular $N_M = 1$ and $v_{\ell_1(M,1)},\ldots,v_{\ell_n(M,1)}$ are generalized eigenvectors of T corresponding to eigenvalues $\lambda_{k_1},\ldots,\lambda_{k_n}$ of T, spanning an n-dimensional T-invariant subspace.

It follows now that

$$|\widetilde{y}_{k_1}(t),\ldots,\widetilde{y}_{k_n}(t)| = (\lambda_{k_1}\cdot\ldots\cdot\lambda_{k_n})^t t^M \left\{|\widetilde{v}_{\ell_1(M,1)}(t),\ldots,\widetilde{v}_{\ell_n(M,1)}(t)| + O(1/t)\right\}.$$

If in particular $v_{k_1},\ldots,v_{k_n}$ are spanning a T-invariant subspace then

$$|\widetilde{y}_{k_1}(t),\ldots,\widetilde{y}_{k_n}(t)| = (\lambda_{k_1}\cdot\ldots\cdot\lambda_{k_n})^t |\widetilde{v}_{k_1},\ldots,\widetilde{v}_{k_n}|;$$

notice that in this case $|\widetilde{v}_{k_1},\ldots,\widetilde{v}_{k_n}| \neq 0$ if and only if $\operatorname{Im}\{v_{k_1},\ldots,v_{k_n}\} \cap \begin{bmatrix} 0 \\ \mathcal{C}^{\perp}_{A,B} \end{bmatrix} = \{0\}$.

Of course analogous formulae are valid simulaneously for $|y^{(i,j)}_{k_1}(t),\ldots,y^{(i,j)}_{k_n}(t)|$.

(ii) If the coefficients of (DRE) are ω-periodic then a similar representation formula like (5.2) remains valid if we replace above the constant vectors $v_1,\ldots,v_{2n}$ by adequate ω-periodic vector functions.

If T has no eigenvalues of modulus one we say that (DRE) is *exponentially dichotomic*. In this case we have

$$|\lambda_{k_1}\cdot\ldots\cdot\lambda_{k_n}| < |\lambda_1\cdot\ldots\cdot\lambda_n| \quad \text{for} \quad (k_1,\ldots,k_n) \neq (1,\ldots,n).$$

If in addition $\begin{vmatrix} d_1 \\ \vdots \\ d_n \end{vmatrix} \neq 0$ and X has no finite negative escape time we obtain from (5.2) using the estimates of Remark 5.6 that for $t \in \mathbb{N}_0$

$$x_{ij}(-t) = \frac{|v^{(i,j)}_1,\ldots,v^{(i,j)}_n|}{|\widetilde{v}_1,\ldots,\widetilde{v}_n|} + O\left(t^{\nu}\left|\frac{\lambda_{n+1}}{\lambda_n}\right|^t\right) \tag{5.3}$$

for some $\nu \in \{0,1,\ldots,n-1\}$. This means that $X(t) \to X_+$ as $t \to -\infty$ at an exponential rate assumed that $X(t)$ exists for all $t \in \mathbb{Z}_0^-$ where $\mathbb{Z}_0^- := \{0,-1,-2,\ldots\}$.

If T has eigenvalues of modulus one and in addition P_{dom} (as defined in (4.5)) is different from zero we have

$$X(t) = X_+ + O(1/t) \quad \text{for} \quad t \in \mathbb{Z}_0^-,$$

i.e. $X(t) \to X_+$ as $t \to -\infty$ at a polynomial rate (assumed that $X(t)$ exists for $t \in \mathbb{Z}_0^-$).

As in the continuous-time case the asymptotic behaviour of $X(t)$ as $t \to -\infty$ essentially depends on P_{dom}. So the convergence results obtain in Section 4 are also valid for the discrete-time case.

Analogously to Corollary 4.2 we get

5.7 Corollary. *Let* $X_0 \in \mathbb{C}^{n\times n}$ *be given and let* $D := V^{-1}(0)\left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix}\right)$.

(i) If (A, B) *is controllable then*

$$P_{\text{dom}} \neq 0 \iff X_0 - X_- \text{ is invertible.}$$

(ii) If (A, B) *is controllable and* T *has no eigenvalues on* $\partial\mathbb{D}$ *then, assuming that* $X(t, X_0, 0)$ *exists for* $t \in \mathbb{Z}_0^-$, *we have*

$$\lim_{t\to-\infty} X(t, X_0, 0) = X_+ \iff X_0 - X_- \text{ is invertible.}$$

(iii) If

$$\lim_{t\to-\infty} X(t, X_0, 0) = X_+,$$

then

$$\operatorname{Ker} X_0 \cap \mathcal{U}_{C,A} \cap \mathcal{R}^d_>(T) = \{0\}.$$

The next theorem which generalizes the main result of [24] is an immediate consequence of the comparison theorem 5.1, the representation formula (5.2), Remark 5.6 and Corollary 5.7.

5.8 Theorem. *Assume that* (A, B) *is controllable. If* $X_0 = X_0^* \geq X_-$ *then*

$$X_- \leq X(t, X_0, 0) \leq \widetilde{X}(t) \quad \text{for} \quad t \in \mathbb{Z}_0^-,$$

where $\widetilde{X}$ *is the solution of*

$$\widetilde{X}(t) = A^*\widetilde{X}(t+1)A + Q, \quad \widetilde{X}(0) = X_0, \tag{5.4}$$

and for $X_0 > X_-$

$$\lim_{t\to-\infty} X(t, X_0, 0) = X_+.$$

Moreover, if T *has no eigenvalues of modulus 1 the convergence takes place at an exponential rate, see (5.3); otherwise we have convergence at a polynomial rate.*

The following remark concerns the discrete version of Lemma 4.4.

5.9 Remark. Assume that (A, B) is controllable. If T has no eigenvalues on $\partial\mathbb{D}$ and $X_0 \not\geq X_-$ then the sequence $\{X(t, X_0, 0)\}_{t\in\mathbb{Z}_0^-}$ may have finite negative escape time since for such terminal values $I + SX(t)$ may become singular for some $t \leq 0$.

For the formulation of the discrete-time version of our convergence theorem 4.5 we assume again that

$$\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) =: \left(\begin{array}{ccc|c} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & B_2 \\ A_{31} & A_{32} & A_{33} & B_3 \\ \hline C_1 & C_2 & 0 & 0 \end{array}\right),$$

where $\left(\left(\begin{smallmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{smallmatrix}\right), \left(\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right)\right)$ is controllable, A_{11} is exponentially stable (i.e. here its eigenvalues are in $\mathbb{D}$), (C_2, A_{22}) has no unobservable eigenvalues on $\partial\mathbb{D}$ and A_{33} has only eigenvalues on $\partial\mathbb{D}$.

As in the continuous-time case we partition the solution $X(\cdot, X_0, 0)$ of (DRE) as

$$X(\cdot, X_0, 0) = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & \hat{X} \end{pmatrix}$$

with $\hat{X} \in \mathbb{C}^{r\times r}$ where $r = \dim \mathcal{C}_{A,B}$. Then (DRE) decomposes into the following three equations:

$$\begin{aligned} X_{11}(t) &= A_{11}^* X_{11}(t+1)A_{11} + C_1^*C_1 + (A_{21}^* \; A_{31}^*)X_{12}^*(t+1)A_{11} \\ &\quad + \left[A_{11}^*X_{12}(t+1) + (A_{21}^* \; A_{31}^*)\hat{X}(t+1)\right]\left[I + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}(B_2^* \; B_3^*)\hat{X}(t+1)\right]^{-1} \\ &\quad \times \left[\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} - \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}(B_2^* \; B_3^*)X_{12}^*(t+1)A_{11}\right], \\ X_{12}(t) &= \left[A_{11}^*X_{12}(t+1) + (A_{21}^* \; A_{31}^*)\hat{X}(t+1)\right] \times \\ &\quad \left[I + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}(B_2^* \; B_3^*)\hat{X}(t+1)\right]^{-1}\begin{pmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{pmatrix} + (C_1^*C_2 \; 0), \\ \hat{X}(t) &= \begin{pmatrix} A_{22}^* & A_{32}^* \\ 0 & A_{33}^* \end{pmatrix}\hat{X}(t+1)\left(I + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}(B_2^* \; B_3^*)\hat{X}(t+1)\right)^{-1}\begin{pmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{pmatrix} \\ &\quad + \begin{pmatrix} C_2^*C_2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.5}$$

As in the continuous-time case it follows from the preceding decomposition that X_{11} and X_{12} exist for $t \in \mathbb{Z}^-$ iff $\hat{X}$ exists for $t \in \mathbb{Z}^-$.

The reduced Riccati difference equation (5.5) has a maximal equilibrium of the form $\hat{X}_+ = \left(\begin{smallmatrix} X_{22+} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and a minimal equilibrium of the form $\hat{X}_- = \left(\begin{smallmatrix} X_{22-} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where X_{22+} and X_{22-} are the maximal and the minimal equilibrium of

$$X_{22}(t) = A_{22}^*X_{22}(t+1)(I + B_2B_2^*X_{22}(t+1))^{-1}A_{22} + C_2^*C_2. \tag{5.6}$$

With these notations we have

5.10 Theorem. *Let (A, B) be a d-stabilizable pair,*

$$X_0 = X_0^* = \begin{pmatrix} X_{011} & X_{012} & X_{013} \\ X_{012}^* & X_{022} & X_{023} \\ X_{013}^* & X_{023}^* & X_{033} \end{pmatrix}$$

with

$$\hat{X}_0 := \begin{pmatrix} X_{022} & X_{023} \\ X_{023}^* & X_{033} \end{pmatrix} \geq \begin{pmatrix} X_{22-} + \varepsilon I & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{for some} \quad \varepsilon > 0.$$

Then $X(t, X_0, 0)$ exists for $t \in \mathbb{Z}_0^-$ and

$$\lim_{t\to-\infty} X(t, X_0, 0) = X_+.$$

In the proof of Theorem 5.10 we will essentially use the fact that the difference of two solutions of a Riccati difference equation satisfies an associated homogeneous Stein difference equation; for convenience of the reader we provide here some details.

5.11 Lemma ([11]). *Let X and Y be hermitian $n \times n$ matrices such that $I + SX$ and $I + SY$ are invertible. Then*

$$\mathcal{R}^d(Y;H) - \mathcal{R}^d(X;H) = A_X^*(Y-X)A_Y = A_Y^*(Y-X)A_X.$$

The next lemma gives an explicit formula for the solution of a Stein difference equation; it can be verified easily by induction.

5.12 Lemma. *The solution of the Stein difference equation*

$$S(t) = M(t)S(t+1)N(t) + \Gamma(t), \quad S(t_0) = S_0,$$

is given by

$$\begin{aligned} S(t_0 - t) &= \left(\Phi^d_{M^*}\right)^* (t_0, t_0 - t)\, S_0\, \Phi^d_N(t_0, t_0 - t) \\ &\quad + \sum_{j=1}^{t} \left(\Phi^d_{M^*}\right)^* (t_0 - j, t_0 - t)\, \Gamma(t_0 - j)\, \Phi^d_N(t_0 - j, t_0 - t), \quad t \in \mathbb{N}_0, \end{aligned}$$

where

$$\Phi^d_M(t,\tau) = \begin{cases} M(t-1)M(t-2)\cdot \ldots \cdot M(\tau) & \text{for} \quad t > \tau, \\ I & \text{for} \quad t = \tau. \end{cases}$$

Proof of Theorem 5.10. Let $\hat{X}$ denote the solution of (5.5) with $\hat{X}(0) = \hat{X}_0$. If $\hat{X}_\ell$ and $\hat{X}_u$ are solutions of (5.5) with

$$\hat{X}_\ell(0) := \begin{pmatrix} X_{22-} + \varepsilon I & 0 \\ 0 & 0 \end{pmatrix} \le \hat{X}_0 < \hat{X}_u(0),$$

then it follows from the Comparison Theorem 5.1 and Corollary 5.7, (ii), applied to (5.5) and (5.6) that $\hat{X}_\ell(t)$ and $\hat{X}_u(t)$ exist for $t \in \mathbb{Z}_0^-$ with

$$\hat{X}_\ell(t) \le \hat{X}(t) \le \hat{X}_u(t) \quad \text{for} \quad t \in \mathbb{Z}_0^-$$

and

$$\lim_{t \to -\infty} \left[\hat{X}_\ell(t) - \hat{X}_+(t)\right] = 0 = \lim_{t \to -\infty} \left[\hat{X}_u(t) - \hat{X}_+(t)\right].$$

Hence

$$\lim_{t \to -\infty} \left[\hat{X}(t) - \hat{X}_+(t)\right] = 0.$$

According to Lemma 5.11 the difference

$$D := \begin{pmatrix} D_{11} & D_{12} \\ D_{12}^* & \hat{D} \end{pmatrix} := X(\cdot, X_0, 0) - X_+$$

satisfies the Stein difference equation

$$D(t) = (A_X)^*(t) D(t+1) A_{X_+}, \quad D(0) = X_0 - X_+,$$

with

$$A_X(t) = (I + SX(t+1))^{-1}A = \begin{pmatrix} A_{11} & 0 \\ A_X^{21}(t) & A_X^{22}(t) \end{pmatrix}$$

where

$$A_X^{22} = \begin{pmatrix} (A_{22})_{X_{22}} & 0 \\ * & A_{33} \end{pmatrix}$$

and

$$A_X^{21}(t) = \left(I + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*)\hat{X}(t+1)\right)^{-1} \left[\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} - \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*) X_{12}^*(t+1)A_{11}\right].$$

It follows now that

$$\begin{aligned} D_{11}(t) &= A_{11}^* D_{11}(t+1)A_{11} + A_{11}^* D_{12}(t+1)A_{X_+}^{21} \\ &\quad + \left(A_X^{21}\right)^*(t)\left(D_{12}^*(t+1)A_{11} + \hat{D}(t+1)A_{X_+}^{21}\right), \\ D_{12}(t) &= A_{11}^* D_{12}(t+1)A_{X_+}^{22} + \left(A_X^{21}\right)^*(t)\hat{D}(t+1)A_{X_+}^{22} \\ &= A_{11}^* D_{12}(t+1)F(t) + G(t)\hat{D}(t+1)A_{X_+}^{22} \end{aligned}$$

where

$$\begin{aligned} F(t) &= \left[I - \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*)\left(I + \hat{X}(t+1)\begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*)\right)^{-1} \hat{D}(t+1)\right] A_{X_+}^{22}, \\ G(t) &= \left[(A_{21}^* \; A_{31}^*) - A_{11}^*\,(X_+)_{12}\begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*)\right]\left(I + \hat{X}(t+1)\begin{pmatrix} B_2 \\ B_3 \end{pmatrix} (B_2^* \; B_3^*)\right)^{-1}. \end{aligned}$$

Since $F(t) \to A_{X_+}^{22}$ as $t \to -\infty$ and since all the eigenvalues of A_{11} (resp. $A_{X_+}^{22}$) lie in the open (resp. closed) unit disk there are constants $K_1, K_2, \varepsilon > 0$, $0 < \delta < \varepsilon$ such that

$$\|\left(\Phi_{A_{11}}^d\right)^*(t,\tau)\| \leq K_1 e^{-\varepsilon(t-\tau)}, \quad \|\Phi_F^d(t,\tau)\| \leq K_2 e^{\delta(t-\tau)}$$

for $t \geq \tau$. From Lemma 5.12 we get now

$$\|D_{12}(-t)\| \leq K_1 K_2 \left\{ e^{-(\varepsilon-\delta)t}\|D_{12}(0)\| + \sum_{j=1}^{t} e^{-(\varepsilon-\delta)(t-j)}\|G(-j)\hat{D}(1-j)A_{X+}^{22}\| \right\}.$$

Since $G(t)$ is uniformly bounded for $t \in \mathbb{Z}_0^-$ and $\hat{D}(t)$ tends to zero as $t \to -\infty$ this implies that $\lim\limits_{t \to -\infty} D_{12}(t) = 0$ and, consequently, $\lim\limits_{t \to -\infty} D_{11}(t) = 0$ for arbitrary initial values.

This proves the assertion of the theorem. □

The formulation of the periodic version of Theorem 5.1 is left to the reader.

Using the comparison theorem for time-varying discrete-time Riccati equations (see [18]) and the discrete-time version of Lemma 2.2 (see [14]) we obtain analogously to the proof of Theorem 3.1 the following existence theorem for Riccati difference equations with periodic coefficients:

5.13 Theorem. *The following statements are equivalent:*

(i) The periodic Riccati difference equation

$$X(t) = \mathcal{R}^d(X(t+1)) := A^*(t)X(t+1)(I + S(t)X(t+1))^{-1}A(t) + Q(t), \tag{5.7}$$

where A, Q and S are $n \times n$ bounded matrix functions of period ω with Q and S hermitian, has an ω-periodic equilibrium X such that $I + S(t)X(t+1) > 0$ for all $t_0 - \omega + 1 \leq t \leq t_0$.

(ii) There exist two hermitian ω-periodic functions X_ℓ, X_u with

  a) $X_\ell(t) \leq X_u(t)$ for $t \in \mathbb{Z}$ and $I + S(t)X_\ell(t+1) > 0$ for all $t_0 - \omega + 1 \leq t \leq t_0$,

  b) $X_\ell(t) \geq \mathcal{R}^d(X_\ell(t+1))$ for $t \in \mathbb{Z}$,

  c) $X_u(t) \leq \mathcal{R}^d(X_u(t+1))$ for $t \in \mathbb{Z}$.

(iii) There exist hermitian matrices X_1, X_2 with

  α) $X_1 \leq X_2$,

  β) $X_1 \leq X(t_0 - \omega, X_1, t_0)$ and $I + S(t)X(t, X_1, t_0) > 0$ for all $t_0 - \omega + 1 \leq t \leq t_0$,

  γ) $X_2 \geq X(t_0 - \omega, X_2, t_0)$.

Here we denoted by $X(\cdot, X_0, t_0)$ the solution of (5.7) with $X(t_0) = X_0$.

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