

# INVERSE SPECTRAL PROBLEMS FOR SINGULAR NON-SELFADJOINT DIFFERENTIAL OPERATORS WITH DISCONTINUITIES IN AN INTERIOR POINT

G. Freiling and V. Yurko

**Abstract.** Non-selfadjoint second-order differential operators on the half-line having a discontinuity in an interior point are studied. We establish properties of the spectrum and investigate the inverse problem of recovering the operator from given spectral characteristics. For this inverse problem we prove the uniqueness theorem, obtain a procedure for constructing the solution and provide necessary and sufficient conditions for the solvability of the inverse problem.

Key words: differential equations, boundary value problems, spectral characteristics, inverse problems

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## 1. INTRODUCTION

This paper deals with the non-selfadjoint boundary value problem  $\mathcal{L}$  for the differential equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0 \quad (1)$$

on the half-line with the boundary condition

$$U(y) := y'(0) - hy(0) = 0 \quad (2)$$

and with the jump condition

$$\begin{bmatrix} y \\ y' \end{bmatrix} (a+0) = A \begin{bmatrix} y \\ y' \end{bmatrix} (a-0) \quad (3)$$

in an interior point  $a > 0$ . Here  $A = [a_{jk}]_{j,k=1,2}$  is a transition matrix with  $\det A \neq 0$ ,  $h$  and  $a_{jk}$  are complex numbers, and the potential  $q(x)$  is a complex-valued function satisfying

$$(1+x)q(x) \in L(0, \infty). \quad (4)$$

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. As a rule, such problems are connected with discontinuous material properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [1]-[2]. After reducing the corresponding mathematical model we obtain a boundary value problem with discontinuities in interior points where the potential must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium [3]-[4]. Boundary value problems with discontinuities in an interior point also

appear in geophysical models for oscillations of the earth [5]-[6]. Here the main discontinuity is caused by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behavior of solutions for such nonlinear equations. We also note that inverse problems considered here appear in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

For classical Sturm-Liouville operators without discontinuities inverse problems of spectral analysis have been studied fairly completely (see [7]-[11] and references therein). Some aspects of direct and inverse problems for discontinuous boundary value problems in various formulations have been considered in [4], [12]-[16] and other works. In particular, in [12] for the case of a finite interval  $x \in [0, T]$ , it was shown that if  $q(x)$  is known a priori on  $[0, T/2]$ , then  $q(x)$  is uniquely determined on  $[T/2, T]$  by the eigenvalues. Boundary value problems with singularities inside the interval have been studied in [17]-[18]; for further discussion see the references therein.

The presence of discontinuities produces essential qualitative modifications in the investigation of the operators, especially for singular non-selfadjoint operators. For the boundary value problem with discontinuities, the behavior of the spectrum is more complicated than for classical Sturm-Liouville operators. In particular, the discrete spectrum can be unbounded and partially lie on the continuous spectrum. In Section 2 we study properties of the spectrum, introduce and investigate the so-called spectral data which reflect behavior of the spectrum. Sections 3-4 are devoted to the inverse problem of recovering  $\mathcal{L}$  from the given spectral data. In Section 3 the uniqueness theorem is proved, and in Section 4 we provide a constructive procedure for the solution of the inverse problem and establish necessary and sufficient conditions for its solvability. For this purpose we use the method of spectral mappings (see [19], [20]) in which ideas of the contour integral method are used and developed.

We note that similar results can be obtained for the case of the Dirichlet condition  $y(0) = 0$  (instead of the boundary condition (2)). We also note that condition (4) on the potential  $q(x)$  is the classical one (see, for example, the monographs [8] and [9]). In particular, this yields a limit point type case at infinity and generates specific spectral properties of the boundary value problem  $\mathcal{L}$  (see Section 2). Other classes of potentials require different investigations.

## 2. PROPERTIES OF THE SPECTRUM

**2.1.** Let  $\lambda = \rho^2$ , and let for definiteness  $\text{Im } \rho \geq 0$ . Denote by  $\overline{\Pi}_+$  the  $\lambda$ -plane with the two-sided cut  $\Pi_0$  along the arc  $\Lambda_+ := \{\lambda : \lambda \geq 0\}$ , and  $\Pi := \overline{\Pi}_+ \setminus \{0\}$ ; notice that here  $\Pi_+, \Pi_0$  and  $\Pi$  must be considered as images of subsets of the Riemann surface of the square-root-function. Then, under the map  $\rho \rightarrow \rho^2 = \lambda$ ,  $\Pi_+, \Pi_0$  and  $\Pi$  correspond to the domains  $\Omega_+ = \{\rho : \text{Im } \rho > 0\}$ ,  $\Omega_0 = \{\rho : \text{Im } \rho = 0\}$  and  $\Omega = \{\rho : \text{Im } \rho \geq 0, \rho \neq 0\}$ , respectively. Denote

$$b_{\pm} = \frac{1}{2}(a_{11} \pm a_{22}).$$

The behavior of the spectrum of the boundary value problem  $\mathcal{L}$  depends on  $b_{\pm}$ . For definiteness, we confine ourselves to the particular case when  $|b_-| > |b_+| > 0$ ,  $a_{12} = 0$ . In this case, in contrast to classical Sturm-Liouville operators, the discrete spectrum is unbounded and there are new qualitative effects in the investigation direct and inverse problems of spectral analysis.

Let  $\varphi_j(x, \lambda)$ ,  $j = 1, 2$ , be the discontinuous solutions of equation (1) under the initial

conditions  $\varphi_j^{(m-1)}(0, \lambda) = \delta_{jm}$ ,  $m = 1, 2$  ( $\delta_{jm}$  is the Kronecker delta), and under the jump condition (3). For each fixed  $x$ , the functions  $\varphi_j^{(m-1)}(x, \lambda)$  are entire in  $\lambda$  and, by virtue of Liouville's formula for the Wronskian ([21, p.83]),

$$\langle \varphi_1(x, \lambda), \varphi_2(x, \lambda) \rangle = \begin{cases} 1 & \text{for } x \in J_-, \\ \det A & \text{for } x \in J_+, \end{cases} \quad (5)$$

where  $J_- := [0, a)$ ,  $J_+ := (a, \infty)$ , and  $\langle y, z \rangle := yz' - y'z$ .

**Lemma 1.** *For  $j = 1, 2$ ,  $m = 0, 1$ ,  $|\rho| \rightarrow \infty$ , uniformly in  $x$ , the following asymptotic formulae are valid*

$$\varphi_j^{(m)}(x, \lambda) = \frac{(i\rho)^{m-j+1}}{2} \left( \exp(i\rho x)[1] + (-1)^{m-j+1} \exp(-i\rho x)[1] \right), \quad x \in J_-, \quad (6)$$

$$\begin{aligned} \varphi_j^{(m)}(x, \lambda) &= \frac{(i\rho)^{m-j+1} b_+}{2} \left( \exp(i\rho x)[1] + (-1)^{m-j+1} \exp(-i\rho x)[1] \right) \\ &+ \frac{(i\rho)^{m-j+1} b_-}{2} \left( (-1)^{j-1} \exp(i\rho(x-2a))[1] + (-1)^m \exp(-i\rho(x-2a))[1] \right), \quad x \in J_+, \end{aligned} \quad (7)$$

where  $[1] = 1 + O(\rho^{-1})$ .

*Proof.* Let  $\{y_k(x, \rho)\}_{k=1,2}$  be the Birkhoff-type smooth fundamental system of solutions for equation (1) with the asymptotics

$$y_k^{(m-1)}(x, \rho) = (\rho R_k)^{m-1} \exp(\rho R_k x)[1], \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega, \quad m = 1, 2, \quad (8)$$

where  $R_k = (-1)^{k-1}i$  (see [22, Ch.1]). Then

$$\varphi_j(x, \lambda) = A_{j1}^\pm(\rho) y_1(x, \rho) + A_{j2}^\pm(\rho) y_2(x, \rho), \quad x \in J_\pm, \quad (9)$$

separately for  $J_+$  and  $J_-$ . Using the jump conditions (3) we calculate

$$\begin{bmatrix} A_{j1}^+(\rho) \\ A_{j2}^+(\rho) \end{bmatrix} = \begin{bmatrix} \gamma_{11}^+(\rho) & \gamma_{12}^+(\rho) \\ \gamma_{21}^+(\rho) & \gamma_{22}^+(\rho) \end{bmatrix} \begin{bmatrix} A_{j1}^-(\rho) \\ A_{j2}^-(\rho) \end{bmatrix}, \quad (10)$$

where

$$\begin{bmatrix} \gamma_{11}^+(\rho) & \gamma_{12}^+(\rho) \\ \gamma_{21}^+(\rho) & \gamma_{22}^+(\rho) \end{bmatrix} = \frac{1}{w(\rho)} \begin{bmatrix} y_2'(a, \rho) & -y_2(a, \rho) \\ -y_1'(a, \rho) & y_1(a, \rho) \end{bmatrix} A \begin{bmatrix} y_1(a, \rho) & y_2(a, \rho) \\ y_1'(a, \rho) & y_2'(a, \rho) \end{bmatrix}, \quad (11)$$

and  $w(\rho) := \langle y_1(x, \rho), y_2(x, \rho) \rangle$ . Clearly,  $\det[\gamma_{jk}^+(\rho)]_{j,k=1,2} \equiv \det A$ . By virtue of (8) and (11),

$$\begin{bmatrix} \gamma_{11}^+(\rho) & \gamma_{12}^+(\rho) \\ \gamma_{21}^+(\rho) & \gamma_{22}^+(\rho) \end{bmatrix} = \begin{bmatrix} [b_+] & [b_-] \exp(-2i\rho a) \\ [b_-] \exp(2i\rho a) & [b_+] \end{bmatrix}, \quad |\rho| \rightarrow \infty, \quad (12)$$

where  $[b_\pm] = b_\pm + O(\rho^{-1})$ . Since, according to the initial conditions on  $\varphi_j(x, \lambda)$ ,

$$A_{jk}^-(\rho) = \frac{(-1)^{k-j}}{w(\rho)} y_{3-k}^{(2-j)}(0, \rho), \quad j, k = 1, 2,$$

it follows from (8), (10) and (12) that for  $|\rho| \rightarrow \infty$ ,  $\text{Im } \rho \geq 0$ ,

$$A_{1k}^-(\rho) = \frac{1}{2}[1], \quad A_{2k}^-(\rho) = \frac{(-1)^{k-1}}{2i\rho}[1],$$

$$A_{1k}^+(\rho) = \frac{1}{2} \left( [b_+] + [b_-] \exp((-1)^k 2i\rho a) \right),$$

$$A_{2k}^+(\rho) = \frac{(-1)^{k-1}}{2i\rho} \left( [b_+] - [b_-] \exp((-1)^k 2i\rho a) \right).$$

Together with (8) and (9) this yields (6) and (7). Lemma 1 is proved.  $\square$

We note that estimates for  $\frac{\partial}{\partial \lambda} \varphi_j(x, \lambda)$  can be derived analogously to Lemma 1.

Let us now introduce the so-called discontinuous Jost solution  $e(x, \rho)$ ,  $x \geq 0$ , for equation (1) which satisfies the jump condition (3) and the condition

$$\lim_{x \rightarrow \infty} e(x, \rho) \exp(-i\rho x) = 1. \quad (13)$$

**Lemma 2.** *The function  $e(x, \rho)$  has the following properties:*

(i<sub>1</sub>) For each fixed  $x$ , the functions  $e^{(m)}(x, \rho)$ ,  $m = 0, 1$ , are analytic for  $\rho \in \Omega_+$ , are continuous for  $\text{Im } \rho \geq 0$ , and are continuously differentiable for  $\rho \in \Omega$ .

(i<sub>2</sub>) For  $|\rho| \rightarrow \infty$ ,  $\text{Im } \rho \geq 0$ , uniformly in  $x$ ,

$$e^{(m)}(x, \rho) = (i\rho)^m \exp(i\rho x) [1], \quad x \in J_+, \quad (14)$$

$$e^{(m)}(x, \rho) = \frac{(i\rho)^m}{\det A} \left( [b_+] \exp(i\rho x) + (-1)^{m+1} [b_-] \exp(i\rho(2a - x)) \right), \quad x \in J_-. \quad (15)$$

(i<sub>3</sub>) For real  $\rho \neq 0$ ,  $x \in J_{\pm}$ , the functions  $e(x, \rho)$  and  $e(x, -\rho)$  form a fundamental system of solutions for equation (1), and

$$\langle e(x, \rho), e(x, -\rho) \rangle = \begin{cases} -2i\rho(\det A)^{-1} & \text{for } x \in J_-, \\ -2i\rho & \text{for } x \in J_+. \end{cases}$$

*Proof.* Denote by  $e_0(x, \rho)$  the classical Jost solution which is smooth for  $x > 0$ , and  $\lim_{x \rightarrow \infty} e_0(x, \rho) \exp(-i\rho x) = 1$  (see [8]). Clearly,

$$e(x, \rho) = e_0(x, \rho) \quad \text{for } x > a, \quad (16)$$

and consequently, (14) holds. Representing  $e(x, \rho)$  by the fundamental system of solutions  $\{y_k(x, \rho)\}$ , we get

$$e(x, \rho) = B_1^{\pm}(\rho)y_1(x, \rho) + B_2^{\pm}(\rho)y_2(x, \rho), \quad x \in J_{\pm}.$$

Taking the jump condition (3) into account we infer similarly to (10):

$$\begin{bmatrix} B_1^+(\rho) \\ B_2^+(\rho) \end{bmatrix} = \begin{bmatrix} \gamma_{11}^+(\rho) & \gamma_{12}^+(\rho) \\ \gamma_{21}^+(\rho) & \gamma_{22}^+(\rho) \end{bmatrix} \begin{bmatrix} B_1^-(\rho) \\ B_2^-(\rho) \end{bmatrix}, \quad (17)$$

where the functions  $\gamma_{jk}^+(\rho)$  are defined by (11). By virtue of (14) and (8),  $B_1^+(\rho) = [1]$ ,  $B_2^+(\rho) = 0$ . Then, (12) and (17) yield

$$B_1^-(\rho) = \frac{[b_+]}{\det A}, \quad B_2^-(\rho) = -\frac{[b_-]}{\det A} \exp(2i\rho a), \quad |\rho| \rightarrow \infty,$$

and we arrive at (15).

Furthermore, let  $S_j(x, \lambda)$ ,  $j = 1, 2$ , be smooth solutions of equation (1) under the initial conditions  $S_j^{(m-1)}(0, \lambda) = \delta_{jm}$ ,  $m = 1, 2$ . Then

$$e(x, \rho) = D_1^{\pm}(\rho)S_1(x, \lambda) + D_2^{\pm}(\rho)S_2(x, \lambda), \quad x \in J_{\pm}. \quad (18)$$

According to the jump condition (3) we have

$$\begin{bmatrix} D_1^+(\rho) \\ D_2^+(\rho) \end{bmatrix} = \begin{bmatrix} \gamma_{11}^0(\lambda) & \gamma_{12}^0(\lambda) \\ \gamma_{21}^0(\lambda) & \gamma_{22}^0(\lambda) \end{bmatrix} \begin{bmatrix} D_1^-(\rho) \\ D_2^-(\rho) \end{bmatrix},$$

where

$$\begin{bmatrix} \gamma_{11}^0(\lambda) & \gamma_{12}^0(\lambda) \\ \gamma_{21}^0(\lambda) & \gamma_{22}^0(\lambda) \end{bmatrix} = \begin{bmatrix} S_2'(a, \lambda) & -S_2(a, \lambda) \\ -S_1'(a, \lambda) & S_1(a, \lambda) \end{bmatrix} A \begin{bmatrix} S_1(a, \lambda) & S_2(a, \lambda) \\ S_1'(a, \lambda) & S_2'(a, \lambda) \end{bmatrix}.$$

It follows from (18) and (16) that

$$e_0(x, \rho) = D_1^+(\rho)S_1(x, \lambda) + D_2^+(\rho)S_2(x, \lambda)$$

for  $x \in J_+$ , and consequently, for all  $x \geq 0$ . This yields  $D_j^+(\rho) = e_0^{(j-1)}(0, \rho)$ . Since the functions  $\gamma_{jk}^0(\lambda)$  are entire in  $\lambda$  and  $\det[\gamma_{jk}^0(\lambda)]_{j,k=1,2} \equiv \det A$ , it follows that the functions  $e^{(m)}(x, \rho)$  have the same smoothness with respect to  $\rho$  as the functions  $e_0^{(m)}(x, \rho)$ , and consequently,  $(i_1)$  is valid. Assertion  $(i_3)$  is obvious, hence Lemma 2 is proved.  $\square$

**2.2.** Denote

$$\Delta(\rho) = e'(0, \rho) - he(0, \rho). \quad (19)$$

The function  $\Delta(\rho)$  is called the characteristic function for  $\mathcal{L}$ . It follows from Lemma 2 that  $\Delta(\rho)$  is analytic for  $\rho \in \Omega_+$ , continuous for  $\text{Im } \rho \geq 0$ , and continuously differentiable for  $\rho \in \Omega$ . Moreover,

$$\Delta(\rho) = \frac{i\rho}{\det A} ([b_+] + [b_-] \exp(2i\rho a)), \quad |\rho| \rightarrow \infty, \text{Im } \rho \geq 0. \quad (20)$$

Similarly, one can calculate for  $|\rho| \rightarrow \infty, \text{Im } \rho \geq 0$ :

$$e(0, \rho) = \frac{1}{\det A} ([b_+] - [b_-] \exp(2i\rho a)), \quad (21)$$

$$\dot{\Delta}(\rho) = -\frac{a[b_-]}{\det A} \exp(2i\rho a), \quad (22)$$

where  $\dot{\Delta}(\rho) := \frac{d}{d\lambda}\Delta(\rho) = \frac{1}{2\rho} \frac{d}{d\rho}\Delta(\rho)$ . For sufficiently large  $|\rho|$ , the function  $\Delta(\rho)$  has simple zeros of the form

$$\rho_k = \rho_k^0 + O\left(\frac{1}{k}\right), \quad |k| \rightarrow \infty, \quad (23)$$

where  $\rho_k^0 = \frac{\pi}{a}(k + \theta)$  are the zeros of the function

$$\Delta_0(\rho) = b_+ + b_- \exp(2i\rho a), \quad (24)$$

and

$$\theta = -\frac{i}{2\pi} \ln \left| \frac{b_+}{b_-} \right| + \frac{1}{2\pi} \arg \left( -\frac{b_+}{b_-} \right). \quad (25)$$

Clearly,  $\text{Im } \theta > 0$ , since  $|b_-| > |b_+| > 0$ . For definiteness, let  $\arg \left( -\frac{b_+}{b_-} \right) \in [0, 2\pi)$ .

Denote

$$\begin{aligned} \Lambda &= \{\lambda = \rho^2 : \rho \in \Omega, \Delta(\rho) = 0\}, \\ \Lambda' &= \{\lambda = \rho^2 : \rho \in \Omega_+, \Delta(\rho) = 0\}, \\ \Lambda'' &= \{\lambda = \rho^2 : \rho \in \Omega_0, \rho \neq 0, \Delta(\rho) = 0\}. \end{aligned}$$

Obviously,  $\Lambda = \Lambda' \cup \Lambda''$ ,  $\Lambda'$  is a countable set, and  $\Lambda''$  is a bounded set. We put

$$\Phi(x, \lambda) = \frac{e(x, \rho)}{\Delta(\rho)}. \quad (26)$$

The function  $\Phi(x, \lambda)$  satisfies (1), (3) and, on account of (19) and Lemma 2, also the conditions  $U(\Phi) = 1$ ,  $\Phi(x, \lambda) = O(\exp(i\rho x))$ ,  $x \rightarrow \infty$  (while  $\Delta(\rho) \neq 0$ ), where  $U$  is defined by (2). Denote

$$M(\lambda) := \Phi(0, \lambda). \quad (27)$$

We will call  $M(\lambda)$  the *Weyl function* for  $\mathcal{L}$  since it is a generalization of the concept of the Weyl function for the classical Sturm-Liouville operator (see[7]). It follows from (26) and (27) that

$$M(\lambda) = \frac{e(0, \rho)}{\Delta(\rho)}. \quad (28)$$

Using the initial conditions in  $x = 0$  we get

$$\Phi(x, \lambda) = \varphi_2(x, \lambda) + M(\lambda)\varphi(x, \lambda), \quad (29)$$

where

$$\varphi(x, \lambda) := \varphi_1(x, \lambda) + h\varphi_2(x, \lambda). \quad (30)$$

By virtue of (5), (29) and (30),

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle = \begin{cases} 1 & \text{for } x \in J_-, \\ \det A & \text{for } x \in J_+. \end{cases} \quad (31)$$

**Theorem 1.** *The Weyl function  $M(\lambda)$  is analytic in  $\Pi_+ \setminus \Lambda'$  and continuously differentiable in  $\Pi \setminus \Lambda$ . The set of singularities of  $M(\lambda)$  (as an analytic function) coincides with the set  $\Lambda_0 := \Lambda_+ \cup \Lambda$ .*

Theorem 1 follows from (19), (28) and Lemma 2.

**Definition 1.** The set of singularities of the Weyl function  $M(\lambda)$  is called the spectrum of  $\mathcal{L}$  (and is denoted by  $\sigma(\mathcal{L})$ ). The values of the parameter  $\lambda$ , for which equation (1) has nontrivial solutions satisfying (3) and the conditions  $U(y) = 0$ ,  $y(\infty) = 0$  (i.e.  $\lim_{x \rightarrow \infty} y(x) = 0$ ), are called eigenvalues of  $\mathcal{L}$ , and the corresponding solutions are called eigenfunctions of  $\mathcal{L}$ .

**Remark 1.** One can introduce the operator

$$\mathcal{L}' : D(\mathcal{L}') \rightarrow L_2(0, \infty), \quad y \rightarrow -y'' + q(x)y$$

with the domain of definition  $D(\mathcal{L}') = \{y : y \in L_2(0, \infty) \cap AC(J_-) \cap AC_{loc}(J_+), y' \in AC(J_-) \cap AC_{loc}(J_+), \mathcal{L}'y \in L_2(0, \infty), U(y) = 0, y(x) \text{ satisfies (3)}\}$ . It is easy to verify that the spectrum of  $\mathcal{L}'$  coincides with  $\sigma(\mathcal{L})$ . There is no difference between working either with the operator  $\mathcal{L}'$  or with the boundary value problem  $\mathcal{L}$ .

**Theorem 2.**  *$\mathcal{L}$  has no eigenvalues  $\lambda > 0$ . Moreover, if  $\lambda_0 = \rho_0^2 > 0$  and  $\Delta(\rho_0) = 0$ , then  $\Delta(-\rho_0) \neq 0$ .*

*Proof.* Suppose that  $\lambda_0 = \rho_0^2 > 0$  is an eigenvalue, and let  $y_0(x)$  be a corresponding eigenfunction. Since the functions  $\{e(x, \rho_0), e(x, -\rho_0)\}$  form a fundamental system of solutions for equation (1) in  $J_{\pm}$ , and satisfy (3), we have  $y_0(x) = C_1 e(x, \rho_0) + C_2 e(x, -\rho_0)$ .

As  $x \rightarrow \infty$ ,  $y_0(x) \sim 0$ ,  $e(x, \pm\rho_0) \sim \exp(\pm i\rho_0 x)$ . But this is possible only if  $C_1 = C_2 = 0$ . Furthermore, if  $\lambda_0 = \rho_0^2 > 0$  and  $\Delta(\rho_0) = 0$ , then it follows from (19) that

$$0 \neq \langle e(x, \rho_0), e(x, -\rho_0) \rangle|_{x=0} = e(0, \rho_0)\Delta(-\rho_0),$$

hence  $\Delta(-\rho_0) \neq 0$ . Theorem 2 is proved.  $\square$

**Theorem 3.** The set  $\Lambda'$  coincides with the set of non-zero eigenvalues of  $\mathcal{L}$ , and

$$e(x, \rho_k) = \beta_k \varphi(x, \lambda_k), \quad \beta_k \neq 0 \quad \text{for all } \lambda_k = \rho_k^2 \in \Lambda'. \quad (32)$$

*Proof.* Let  $\lambda_k \in \Lambda'$ . Then  $U(e(x, \rho_k)) = \Delta(\rho_k) = 0$  and, by virtue of (13),  $\lim_{x \rightarrow \infty} e(x, \rho_k) = 0$ . Thus,  $e(x, \rho_k)$  is an eigenfunction, and  $\lambda_k = \rho_k^2$  is an eigenvalue. Moreover, it follows from (26) and (31) that  $\langle \varphi(x, \lambda_k), e(x, \rho_k) \rangle = 0$ , and consequently (32) is valid.

Conversely, let  $\lambda_k = \rho_k^2$ ,  $\text{Im } \rho_k > 0$  be an eigenvalue, and let  $y_k(x)$  be a corresponding eigenfunction. Clearly,  $y_k(0) \neq 0$ . Without loss of generality we put  $y_k(0) = 1$ . Then  $y_k'(0) = h$ , and hence  $y_k(x) \equiv \varphi(x, \lambda_k)$ . Since  $\lim_{x \rightarrow \infty} y_k(x) = 0$ , one gets  $y_k(x) = \beta_k^0 e(x, \rho_k)$ ,  $\beta_k^0 \neq 0$ . This yields (32). Consequently,  $\Delta(\rho_k) = U(e(x, \rho_k)) = 0$ , and  $\varphi(x, \lambda_k)$  and  $e(x, \rho_k)$  are eigenfunctions. Theorem 3 is proved.  $\square$

For brevity, we confine ourselves to the case of a simple spectrum in the following sense.

**Definition 2.** We shall say that  $\mathcal{L}$  has simple spectrum if all zeros of  $\Delta(\rho)$  are simple, have no finite limit points, and  $\rho M(\lambda) = O(1)$  as  $\rho \rightarrow 0$ .

Let  $\mathcal{L}$  have simple spectrum. Then  $\Lambda''$  is a finite set, and  $\Lambda = \Lambda' \cup \Lambda''$  is a countable set:

$$\Lambda = \{\rho_k^2\}_{k \in \omega}.$$

Here  $\omega = \omega_0 \cup \omega^0$ , where  $\omega_0$  is a finite set,  $\omega^0 = \{k \in \mathbf{Z} : |k| > k_0\}$  for some  $k_0$ , and the numbers  $\rho_k$  have the form (23) for  $k \in \omega^0$ . Each element of  $\Lambda'$  is an eigenvalue of  $\mathcal{L}$ . According to Theorem 2, the points of  $\Lambda''$  are not eigenvalues of  $\mathcal{L}$ , they are called *spectral singularities* of  $\mathcal{L}$ . Denote

$$M_k = \frac{e(0, \rho_k)}{\dot{\Delta}(\rho_k)}, \quad \rho_k \in \Lambda. \quad (33)$$

Obviously,  $M_k \neq 0$ , and

$$\lim_{\lambda \rightarrow \lambda_k, \lambda \in \Pi} (\lambda - \lambda_k)M(\lambda) = M_k. \quad (34)$$

Let

$$\alpha_k := \begin{cases} M_k & \text{for } \rho_k \in \Lambda', \\ \frac{1}{2}M_k & \text{for } \rho_k \in \Lambda'', \end{cases} \quad (35)$$

$$V(\lambda) := \frac{1}{2\pi i} (M^-(\lambda) - M^+(\lambda)), \quad \lambda > 0, \quad (36)$$

where  $M^\pm(\lambda) = \lim_{z \rightarrow 0, \text{Re } z > 0} M(\lambda \pm iz)$ . Using (21)-(23), (33) and (35) we calculate

$$\alpha_k = \frac{2}{a} + O\left(\frac{1}{k}\right), \quad |k| \rightarrow \infty. \quad (37)$$

By virtue of (28) and (36),

$$V(\lambda) = \frac{1}{2\pi i} \left( \frac{e(0, -\rho)}{\Delta(-\rho)} - \frac{e(0, \rho)}{\Delta(\rho)} \right), \quad \rho > 0,$$

and consequently, we get with Lemma 2, ( $i_3$ ):

$$V(\lambda) = \frac{\rho}{\pi \det A} \cdot \frac{1}{\Delta(\rho)\Delta(-\rho)}, \quad \rho > 0. \quad (38)$$

**Definition 3.** The data  $S := (\{V(\lambda)\}_{\lambda>0}, \{\lambda_k, \alpha_k\}_{k \in \omega})$  are called the *spectral data* of  $\mathcal{L}$ .

The spectral data describe the behavior of the spectrum;  $\{V(\lambda)\}_{\lambda>0}$  is connected with the continuous spectrum, and  $\{\lambda_k, \alpha_k\}_{k \in \omega}$  describe the discrete spectrum. Using the results obtained above we arrive at the following statement.

**Theorem 4.** *The spectral data  $S := (\{V(\lambda)\}_{\lambda>0}, \{\lambda_k, \alpha_k\}_{k \in \omega})$ ,  $\lambda_k = \rho_k^2$ , have the following properties:*

- ( $i_1$ )  $\rho_k \neq \rho_s$  for  $k \neq s$ ; moreover, if  $\rho_k \in \Lambda''$ , then  $-\rho_k \notin \Lambda''$ ;
- ( $i_2$ ) as  $|k| \rightarrow \infty$ ,

$$\rho_k = \frac{\pi}{a}(k + \theta) + O\left(\frac{1}{k}\right), \quad (39)$$

where  $\theta$  is defined by (25);

- ( $i_3$ )  $\alpha_k \neq 0$ , and (37) is valid;

( $i_4$ ) the function  $V(\lambda)$  is continuously differentiable for  $\Lambda_+ \setminus \Lambda''$ ; for  $\lambda_k \in \Lambda''$  there exist finite limits  $V_k := \lim_{\lambda \rightarrow \lambda_k} (\lambda - \lambda_k)V(\lambda) \neq 0$ , and

$$V_k = i\pi^{-1}\alpha_k \text{sign } \rho_k; \quad (40)$$

- ( $i_5$ ) as  $\lambda \rightarrow 0$ ,

$$\rho V(\lambda) = O(1), \quad (41)$$

and as  $\lambda \rightarrow +\infty$ ,

$$V(\lambda) = V^0(\lambda) + O\left(\frac{1}{\lambda}\right), \quad (42)$$

where

$$V^0(\lambda) := \frac{\det A}{\pi \rho} \cdot \frac{1}{\Delta_0(\rho)\Delta_0(-\rho)}, \quad \rho > 0. \quad (43)$$

and  $\Delta_0(\rho)$  is defined by (24).

The asymptotics (42) follows from (38), (20) and (24). Notice that relation (40) gives us a connection between  $V(\lambda)$ , which describes the continuous spectrum, and  $\{\lambda_k, \alpha_k\}$ ,  $\lambda_k \in \Lambda''$ , which describe the spectral singularities.

We note that the boundary value problem considered corresponds to a problem for equation (1) over the interval  $x > a$  with a  $\lambda$ -dependent boundary condition at  $x = a$  generated by  $\varphi^\nu(a-0, \lambda)$ ,  $\nu = 0, 1$  (see, for example, [20, Sect.4.4]).

### 3. UNIQUENESS THEOREM

Let us go on to studying the inverse problem of recovering the boundary value problem  $\mathcal{L}$  from its spectral data  $S$ . In this section we prove the uniqueness theorem for the solution of this inverse problem. For this purpose we use ideas of the contour integral method.

For studying the inverse problem we agree that together with  $\mathcal{L}$  we consider a boundary value problem  $\tilde{\mathcal{L}}$  of the same form but with different coefficients  $\tilde{q}$  and  $\tilde{h}$ . If a certain

symbol  $\gamma$  denotes an object related to  $\mathcal{L}$ , then the corresponding symbol  $\tilde{\gamma}$  with tilde will denote the analogous object related to  $\tilde{\mathcal{L}}$ , and  $\hat{\gamma} := \gamma - \tilde{\gamma}$ .

**Theorem 5.** *If  $S = \tilde{S}$ , then  $q(x) = \tilde{q}(x)$  a.e. for  $x > 0$ , and  $h = \tilde{h}$ . Thus, the specification of the spectral data uniquely determines the boundary value problem  $\mathcal{L}$ .*

*Proof.* From the given spectral data  $S$  one can reconstruct not only  $q$  and  $h$  but also  $a$  and  $A$  (see, for example, [12]). However, for brevity we assume that  $a$  and  $A$  are known a priori. Denote

$$\kappa_\delta^+(\lambda_k) := \{\lambda : \lambda \in [\lambda_k - \delta, \lambda_k + \delta]\}, \quad \lambda_k \in \Lambda''; \quad \xi_\delta := \Lambda_+ \setminus \left( \bigcup_{\lambda_k \in \Lambda''} \kappa_\delta^+(\lambda_k) \right),$$

$$G_\delta := \{\rho : \operatorname{Im} \rho \geq 0, |\rho - \rho_k| \geq \delta, \rho_k \in \Lambda\}.$$

Let us show that the specification of the spectral data  $S$  uniquely determines the Weyl function  $M(\lambda)$  via the formula

$$M(\lambda) = \sum_{\lambda_k \in \Lambda} \frac{\alpha_k}{\lambda - \lambda_k} + \int_0^\infty \frac{V(\mu)}{\lambda - \mu} d\mu, \quad \lambda \notin \sigma(\mathcal{L}), \quad (44)$$

where the integral is understood in the principal value sense:  $\int_0^\infty := \lim_{\delta \rightarrow 0} \int_{\xi_\delta}$ .

Indeed, using (20), (21) and (28) one gets

$$|\Delta(\rho)| \geq C_\delta |\rho|, \quad |M(\lambda)| \leq C_\delta |\rho|^{-1}, \quad \rho \in G_\delta. \quad (45)$$

It follows from (42), (43) and (24) that

$$\rho V(\lambda) = O(1), \quad \rho \rightarrow +\infty. \quad (46)$$

In particular, (46) implies that the integral in (44) converges absolutely at infinity. Moreover, in view of (37) and (39), the series in (44) converges absolutely too.

Take positive numbers  $r_N = ((N + \chi)\pi/a)^2$  such that the circles  $\theta_N := \{\lambda : |\lambda| = r_N\}$  lie in  $G_\delta$  for sufficiently small  $\delta > 0$ , and consider the contour integral

$$I_N(\lambda) := \frac{1}{2\pi i} \int_{\theta_N} \frac{M(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in \operatorname{int} \theta_N, \quad (47)$$

with counterclockwise circuit. It follows from (45) that

$$\lim_{N \rightarrow \infty} I_N(\lambda) = 0. \quad (48)$$

For each  $\lambda_k \in \Lambda''$  on the upper (lower) side of the cut  $\Pi_0$  we take a semicircle  $\kappa_\delta(\lambda_k) := \{\lambda : |\lambda - \lambda_k| = \delta, \operatorname{Im} \lambda > 0 \text{ (Im } \lambda < 0 \text{ respectively)}\}$  and choose  $\delta > 0$  such that the sets  $\operatorname{int} \kappa_\delta(\lambda_k)$  do not intersect each other, and do not contain points of  $\Lambda'$ . Let  $\Pi_\delta$  be the two-sided cut  $\Pi_0$  without the  $\delta$ -neighbourhoods of the points of  $\Lambda''$ , and let  $\Gamma_\delta := \Pi_\delta \cup \left( \bigcup_{\lambda_k \in \Lambda''} \kappa_\delta(\lambda_k) \right)$  be the contour with counterclockwise circuit. Denote  $\Gamma_{\delta,N} := \Gamma_\delta \cap \theta_{N,0}$ , where  $\theta_{N,0} = \{\lambda : |\lambda| \leq r_N\}$ . Contracting the contour  $\theta_N$  in (47) to the real axis and using (28), (33), (35) and the residue theorem, we get

$$M(\lambda) = \sum_{\substack{\lambda_k \in \Lambda' \\ |\lambda_k| < r_N}} \frac{\alpha_k}{\lambda - \lambda_k} + \frac{1}{2\pi i} \int_{\Gamma_{\delta,N}} \frac{M(\mu)}{\lambda - \mu} d\mu - I_N(\lambda).$$

By virtue of (48) this yields as  $N \rightarrow \infty$  :

$$M(\lambda) = \sum_{\lambda_k \in \Lambda'} \frac{\alpha_k}{\lambda - \lambda_k} + \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{M(\mu)}{\lambda - \mu} d\mu. \quad (49)$$

Taking (34) and (35) into account we calculate for each  $\lambda_k \in \Lambda''$  :

$$\lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{\kappa_\delta(\lambda_k)} \frac{M(\mu)}{\lambda - \mu} d\mu = \frac{\alpha_k}{\lambda - \lambda_k}.$$

Moreover, in view of (36),

$$\frac{1}{2\pi i} \int_{\Pi_\delta} \frac{M(\mu)}{\lambda - \mu} d\mu = \int_{\xi_\delta} \frac{V(\mu)}{\lambda - \mu} d\mu.$$

Therefore, from (49) as  $\delta \rightarrow 0$  we arrive at (44).

Let us now define the matrix  $\mathcal{P}(x, \lambda) = [\mathcal{P}_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$\mathcal{P}(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}. \quad (50)$$

By virtue of (31), this yields

$$\left. \begin{aligned} \mathcal{P}_{j1}(x, \lambda) &= (\eta(x))^{-1} (\varphi^{(j-1)}(x, \lambda) \tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}'(x, \lambda)) \\ \mathcal{P}_{j2}(x, \lambda) &= (\eta(x))^{-1} (\Phi^{(j-1)}(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}(x, \lambda)) \end{aligned} \right\}, \quad (51)$$

$$\left. \begin{aligned} \varphi(x, \lambda) &= \mathcal{P}_{11}(x, \lambda) \tilde{\varphi}(x, \lambda) + \mathcal{P}_{12}(x, \lambda) \tilde{\varphi}'(x, \lambda) \\ \Phi(x, \lambda) &= \mathcal{P}_{21}(x, \lambda) \tilde{\Phi}(x, \lambda) + \mathcal{P}_{22}(x, \lambda) \tilde{\Phi}'(x, \lambda) \end{aligned} \right\}, \quad (52)$$

where  $\eta(x) = 1$  for  $x \in J_-$ , and  $\eta(x) = \det A$  for  $x \in J_+$ . It follows from (26), (5) and Lemmas 1 and 2 that for  $x \geq 0$ ,  $m = 0, 1$ ,

$$\left. \begin{aligned} |\varphi^{(m)}(x, \lambda)| &\leq C |\rho|^m |\exp(-i\rho x)| \\ |\Phi^{(m)}(x, \lambda)| &\leq C_\delta |\rho|^{m-1} |\exp(i\rho x)|, \quad \rho \in G_\delta \end{aligned} \right\}, \quad (53)$$

$$\left. \begin{aligned} |\varphi^{(m)}(x, \lambda) - \tilde{\varphi}^{(m)}(x, \lambda)| &\leq C |\rho|^{m-1} |\exp(-i\rho x)| \\ |\Phi^{(m)}(x, \lambda) - \tilde{\Phi}^{(m)}(x, \lambda)| &\leq C_\delta |\rho|^{m-2} |\exp(i\rho x)|, \quad \rho \in G_\delta \end{aligned} \right\}. \quad (54)$$

Using (31), (51), (53) and (54) we obtain that for  $x \geq 0$ ,  $\rho \in G_\delta$ ,  $|\rho| \rightarrow \infty$  :

$$\mathcal{P}_{jk}(x, \lambda) - \delta_{jk} = O(\rho^{-1}), \quad j \leq k; \quad \mathcal{P}_{21}(x, \lambda) = O(1). \quad (55)$$

According to the assumption of Theorem 5,  $S = \tilde{S}$ . By virtue of (44) this implies  $M(\lambda) = \tilde{M}(\lambda)$ . Then, in view of (51) and (29), we conclude that for each fixed  $x$ , the functions  $\mathcal{P}_{jk}(x, \lambda)$  are entire in  $\lambda$ . Taking (55) into account we get  $\mathcal{P}_{11}(x, \lambda) \equiv 1$ ,  $\mathcal{P}_{12}(x, \lambda) \equiv 0$ . Substituting this into (52) we infer  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ ,  $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ , and consequently,  $\mathcal{L} = \tilde{\mathcal{L}}$ . Theorem 5 is proved.  $\square$

#### 4. SOLUTION OF THE INVERSE PROBLEM

In this section we give a constructive procedure for the solution of the inverse problem and formulate necessary and sufficient conditions for its solvability. The central role here is played by the so-called main equation of the inverse problem which connects spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation which is a linear equation in a suitable Banach space. Moreover, we prove the unique solvability of the main equation. Using the solution of the main equation we provide explicit formulae for the solution of the inverse problem considered.

Let  $\tilde{\mathcal{L}}$  be a certain known model boundary value problem with the spectral data  $\tilde{S} = (\{\tilde{V}(\lambda)\}_{\lambda>0}, \{\tilde{\lambda}_k, \tilde{\alpha}_k\}_{k \in \tilde{\omega}}, \tilde{\lambda}_k = \tilde{\rho}_k^2$ . Denote

$$\begin{aligned} \lambda_{k0} &= \lambda_k, \quad \lambda_{k1} = \tilde{\lambda}_k, \quad \alpha_{k0} = \alpha_k, \quad \alpha_{k1} = \tilde{\alpha}_k, \\ \varphi_{kj}(x) &= \varphi(x, \lambda_{kj}), \quad \varphi_\lambda(x) = \varphi(x, \lambda), \\ D(x, \lambda, \mu) &= \frac{1}{\eta(x)} \frac{\langle \varphi_\lambda(x), \varphi_\mu(x) \rangle}{\lambda - \mu}, \quad D_{sp}(x, \lambda, \mu) = \frac{\partial^{s+p}}{\partial \lambda^s \partial \mu^p} D(x, \lambda, \mu). \end{aligned}$$

It follows from Lemma 1 that for  $\lambda = \rho^2$ ,  $\mu = \theta^2$ ,  $0 \leq \text{Im } \rho \leq C$ ,  $0 \leq \text{Im } \theta \leq C$ , the following estimates hold

$$|D(x, \lambda, \mu)| \leq \frac{C}{|\rho - \chi\theta| + 1}, \quad \chi := \text{sign}(\text{Re } \rho \text{Re } \theta), \quad (56)$$

$$|\varphi(x, \lambda)| \leq C. \quad (57)$$

Similarly one can get

$$\left| \frac{\partial}{\partial \lambda} \varphi(x, \lambda) \right| \leq C, \quad |D_{sp}(x, \lambda, \mu)| \leq \frac{C}{|\rho - \chi\theta| + 1}, \quad s, p = 0, 1. \quad (58)$$

Denote

$$\begin{aligned} P_{\lambda, \mu}(x) &= D(x, \lambda, \mu) \hat{V}(\mu), \quad P_{\lambda, kj}(x) = D(x, \lambda, \lambda_{kj}) \alpha_{kj}, \\ P_{ni, \mu}(x) &= D(x, \lambda_{ni}, \mu) \hat{V}(\mu), \quad P_{ni, kj}(x) = D(x, \lambda_{ni}, \lambda_{kj}) \alpha_{kj}. \end{aligned}$$

We define  $\tilde{\varphi}_{kj}$ ,  $\tilde{D}$ ,  $\tilde{P}_{\lambda, \mu}$ ,  $\tilde{P}_{\lambda, kj}$ ,  $\tilde{P}_{ni, \mu}$ ,  $\tilde{P}_{ni, kj}$  by the same formulas but with  $\tilde{\varphi}$ ,  $\tilde{D}$  instead of  $\varphi$ ,  $D$ . If  $\omega_0 \neq \tilde{\omega}_0$ , then we define the corresponding functions identically zero (for example, if  $n \in \omega_0 \setminus \tilde{\omega}_0$ , then  $\varphi_{n1} = P_{n1, \mu} = P_{n1, kj} = P_{\lambda, n1} = P_{kj, n1} = 0$ , and the same for functions with tilde). Let  $\omega' := \omega \cup \tilde{\omega}$ , and let  $\omega_1$  be a set of indices  $v = (k, j)$ , where  $k \in \omega'$ ,  $j = 0, 1$ . Denote

$$\xi'_\delta := \Lambda_+ \setminus \left( \left( \bigcup_{\lambda_k \in \Lambda''} \kappa_\delta^+(\lambda_k) \right) \cup \left( \bigcup_{\tilde{\lambda}_k \in \tilde{\Lambda}''} \kappa_\delta^+(\tilde{\lambda}_k) \right) \right).$$

**Lemma 3.** *The following relations hold*

$$\tilde{\varphi}_\lambda(x) = \varphi_\lambda(x) + \int_0^\infty \tilde{P}_{\lambda, \mu}(x) \varphi_\mu(x) d\mu + \sum_{k \in \omega'} \left( \tilde{P}_{\lambda, k0}(x) \varphi_{k0}(x) - \tilde{P}_{\lambda, k1}(x) \varphi_{k1}(x) \right), \quad (59)$$

$$\begin{aligned} &P_{\lambda, \mu}(x) - \tilde{P}_{\lambda, \mu}(x) + \int_0^\infty \tilde{P}_{\lambda, \xi}(x) P_{\xi, \mu}(x) d\xi \\ &+ \sum_{s \in \omega'} \left( \tilde{P}_{\lambda, s0}(x) P_{s0, \mu}(x) - \tilde{P}_{\lambda, s1}(x) P_{s1, \mu}(x) \right) = 0, \end{aligned} \quad (60)$$

where the integrals are understood in the principal value sense:  $\int_0^\infty := \lim_{\delta \rightarrow 0} \int_{\xi'_\delta}$ .

*Proof.* Consider the contour integral (with counterclockwise circuit)

$$J_{N,k}(x, \lambda) = \frac{1}{2\pi i} \int_{\theta_N} \frac{\mathcal{P}_{1k}(x, \mu) - \delta_{1k}}{\lambda - \mu} d\mu, \quad k = 1, 2, \quad \lambda \in \text{int } \theta_N, \quad (61)$$

where  $\mathcal{P}_{1k}$  are defined by (50). In view of (55),

$$\lim_{N \rightarrow \infty} J_{N,k}(x, \lambda) = 0. \quad (62)$$

Take  $h > 0$  such that  $\text{Im } \rho_k < h$ ,  $\text{Im } \tilde{\rho}_k < h$  for all  $\rho_k \in \Lambda$ ,  $\tilde{\rho}_k \in \tilde{\Lambda}$ . Let  $\gamma = \{\lambda = u + iv : u = (2h^2)^{-2}v^2 - h^2\}$  be the image of  $\text{Im } \rho = h$  under the map  $\rho \rightarrow \lambda = \rho^2$ . Denote  $\gamma_N = (\gamma \cap \theta_{N,0}) \cup \{\lambda : |\lambda| = r_N, \lambda \in \text{int } \gamma\}$  (with counterclockwise circuit). Moving the contour in (61) through the pole  $\mu = \lambda$ , we get

$$\mathcal{P}_{1k}(x, \lambda) = \delta_{1k} + \frac{1}{2\pi i} \int_{\gamma_N} \frac{\mathcal{P}_{1k}(x, \mu)}{(\lambda - \mu)} d\mu + J_{N,k}(x, \lambda), \quad \lambda \notin \text{int } \gamma_N.$$

From this and from (52) it follows that

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \left( \tilde{\varphi}(x, \lambda) \mathcal{P}_{11}(x, \mu) + \tilde{\varphi}'(x, \lambda) \mathcal{P}_{12}(x, \mu) \right) \frac{d\mu}{\lambda - \mu} + J_N(x, \lambda), \quad (63)$$

where  $J_N(x, \lambda) = J_{N,1}(x, \lambda) \tilde{\varphi}(x, \lambda) + J_{N,2}(x, \lambda) \tilde{\varphi}'(x, \lambda)$ . By virtue of (62),  $\lim_{N \rightarrow \infty} J_N(x, \lambda) = 0$ . Substituting (51) into (63) we calculate

$$\begin{aligned} \varphi(x, \lambda) &= \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} (\eta(x))^{-1} \left( \tilde{\varphi}(x, \lambda) (\varphi(x, \mu) \tilde{\Phi}'(x, \mu) - \Phi(x, \mu) \tilde{\varphi}'(x, \mu)) \right. \\ &\quad \left. + \tilde{\varphi}'(x, \lambda) (\Phi(x, \mu) \tilde{\varphi}(x, \mu) - \varphi(x, \mu) \tilde{\Phi}(x, \mu)) \right) \frac{d\mu}{\lambda - \mu} + J_N(x, \lambda). \end{aligned}$$

In view of (29) this yields

$$\tilde{\varphi}_\lambda(x) = \varphi_\lambda(x) + \frac{1}{2\pi i} \int_{\gamma_N} \tilde{D}(x, \lambda, \mu) \hat{M}(\mu) \varphi_\mu(x) d\mu - J_N(x, \lambda), \quad (64)$$

since the terms with  $\varphi_2(x, \mu)$  vanish by Cauchy's theorem.

Furthermore, we consider the contour integral

$$J_{N,jk}(x, \lambda, \mu) = \frac{1}{2\pi i} \int_{\theta_N} \frac{\mathcal{P}_{jk}(x, \xi) d\xi}{(\lambda - \xi)(\xi - \mu)}, \quad \lambda, \mu \in \text{int } \theta_N. \quad (65)$$

In view of (55),

$$\lim_{N \rightarrow \infty} J_{N,jk}(x, \lambda, \mu) = 0. \quad (66)$$

Since

$$\frac{1}{\lambda - \mu} \left( \frac{1}{\lambda - \xi} - \frac{1}{\mu - \xi} \right) = \frac{1}{(\lambda - \xi)(\xi - \mu)},$$

we have from (65)

$$\frac{\mathcal{P}_{jk}(x, \lambda) - \mathcal{P}_{jk}(x, \mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_{\gamma_N} \frac{\mathcal{P}_{jk}(x, \xi) d\xi}{(\lambda - \xi)(\xi - \mu)} + J_{N,jk}(x, \lambda, \mu), \quad \lambda, \mu \notin \text{int } \gamma_N. \quad (67)$$

It follows from (50) and (51) that  $\det \mathcal{P}(x, \lambda) \equiv 1$ , and

$$\left. \begin{aligned} \mathcal{P}_{11}(x, \lambda)\varphi'(x, \lambda) - \mathcal{P}_{21}(x, \lambda)\varphi(x, \lambda) &= \tilde{\varphi}'(x, \lambda), \\ \mathcal{P}_{22}(x, \lambda)\varphi(x, \lambda) - \mathcal{P}_{12}(x, \lambda)\varphi'(x, \lambda) &= \tilde{\varphi}(x, \lambda), \end{aligned} \right\} \quad (68)$$

$$\mathcal{P}(x, \lambda) \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = (\eta(x))^{-1} \left( \langle y(x), \tilde{\Phi}(x, \lambda) \rangle \begin{bmatrix} \varphi(x, \lambda) \\ \varphi'(x, \lambda) \end{bmatrix} - \langle y(x), \tilde{\varphi}(x, \lambda) \rangle \begin{bmatrix} \Phi(x, \lambda) \\ \Phi'(x, \lambda) \end{bmatrix} \right) \quad (69)$$

for any smooth  $y(x)$ . Taking (66), (67) and (69) into account we calculate

$$\begin{aligned} \frac{\mathcal{P}(x, \lambda) - \mathcal{P}(x, \mu)}{\lambda - \mu} \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} &= \frac{(\eta(x))^{-1}}{2\pi i} \int_{\gamma_N} \left( \langle y(x), \tilde{\Phi}(x, \xi) \rangle \begin{bmatrix} \varphi(x, \xi) \\ \varphi'(x, \xi) \end{bmatrix} \right. \\ &\quad \left. - \langle y(x), \tilde{\varphi}(x, \xi) \rangle \begin{bmatrix} \Phi(x, \xi) \\ \Phi'(x, \xi) \end{bmatrix} \right) \frac{d\xi}{(\lambda - \xi)(\xi - \mu)} + \varepsilon_N(x, \lambda, \mu), \end{aligned} \quad (70)$$

where  $\lim_{N \rightarrow \infty} \varepsilon_N(x, \lambda, \mu) = 0$ . Using (50) and (68) we get

$$\begin{aligned} &\det \left( (\mathcal{P}(x, \lambda) - \mathcal{P}(x, \mu)) \begin{bmatrix} \tilde{\varphi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi(x, \lambda) \\ \varphi'(x, \lambda) \end{bmatrix} \right) \\ &= \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle - \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle. \end{aligned}$$

Consequently, (70) for  $y(x) = \tilde{\varphi}(x, \lambda)$  yields

$$\begin{aligned} \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} &= \frac{(\eta(x))^{-1}}{2\pi i} \int_{\gamma_N} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\Phi}(x, \xi) \rangle \langle \varphi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right. \\ &\quad \left. - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle \langle \Phi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right) d\xi + \varepsilon_N^0(x, \lambda, \mu), \end{aligned}$$

where  $\lim_{N \rightarrow \infty} \varepsilon_N^0(x, \lambda, \mu) = 0$ . In view of (29) this yields

$$D(x, \lambda, \mu) - \tilde{D}(x, \lambda, \mu) + \frac{1}{2\pi i} \int_{\gamma_N} \tilde{D}(x, \lambda, \xi) \hat{M}(\xi) D(x, \xi, \mu) d\xi + \varepsilon_N^1(x, \lambda, \mu), \quad (71)$$

where  $\lim_{N \rightarrow \infty} \varepsilon_N^1(x, \lambda, \mu) = 0$ .

Let  $\Pi'_\delta$  be the two-sided cut  $\Pi_0$  without the  $\delta$ -neighbourhoods of the points of  $\Lambda'' \cup \tilde{\Lambda}''$ , and let  $\Gamma'_\delta := \Pi'_\delta \cup \left( \bigcup_{\lambda_k \in \Lambda''} \kappa_\delta(\lambda_k) \right) \cup \left( \bigcup_{\tilde{\lambda}_k \in \tilde{\Lambda}''} \kappa_\delta(\tilde{\lambda}_k) \right)$  be the contour with counterclockwise circuit. Denote  $\Gamma'_{\delta, N} := \Gamma'_\delta \cap \theta_{N, 0}$ . Contracting the contour  $\gamma_N$  in (64) to the real axis through the poles of  $\Lambda' \cup \tilde{\Lambda}'$  and using the residue theorem, we get

$$\begin{aligned} \tilde{\varphi}_\lambda(x) &= \varphi_\lambda(x) + \frac{1}{2\pi i} \int_{\Gamma'_{\delta, N}} \tilde{D}(x, \lambda, \mu) \hat{M}(\mu) \varphi_\mu(x) d\mu \\ &+ \sum_{\substack{\lambda_k \in \Lambda' \\ |\lambda_k| < r_N}} \tilde{D}(x, \lambda, \lambda_{k0}) \alpha_{k0} \varphi_{k0}(x) - \sum_{\substack{\tilde{\lambda}_k \in \tilde{\Lambda}' \\ |\tilde{\lambda}_k| < r_N}} \tilde{D}(x, \lambda, \lambda_{k1}) \alpha_{k1} \varphi_{k1}(x) - J_N(x, \lambda). \end{aligned}$$

As  $N \rightarrow \infty$  this yields

$$\tilde{\varphi}_\lambda(x) = \varphi_\lambda(x) + \frac{1}{2\pi i} \int_{\Gamma'_\delta} \tilde{D}(x, \lambda, \mu) \hat{M}(\mu) \varphi_\mu(x) d\mu$$

$$+ \sum_{\lambda_k \in \Lambda'} \tilde{D}(x, \lambda, \lambda_{k0}) \alpha_{k0} \varphi_{k0}(x) - \sum_{\tilde{\lambda}_k \in \tilde{\Lambda}'} \tilde{D}(x, \lambda, \lambda_{k1}) \alpha_{k1} \varphi_{k1}(x). \quad (72)$$

Since

$$\lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{\kappa_\delta(\lambda_{ki})} \tilde{D}(x, \lambda, \mu) \hat{M}(\mu) \varphi_\mu(x) d\mu = (-1)^i \tilde{D}(x, \lambda, \lambda_{ki}) \alpha_{ki} \varphi_{ki}(x), \quad \lambda_{ki} \in \Lambda'' \cup \tilde{\Lambda}'',$$

$$\frac{1}{2\pi i} \int_{\Pi'_\delta} \tilde{D}(x, \lambda, \mu) \hat{M}(\mu) \varphi_\mu(x) d\mu = \int_{\xi'_\delta} \tilde{P}_{\lambda, \mu}(x) \varphi_\mu(x) d\mu,$$

from (72) as  $\delta \rightarrow 0$  we arrive at (59). Similarly, leaning on (71) we deduce (60). Lemma 3 is proved.  $\square$

**Remark 2.** For each fixed  $x \geq 0$ , relation (59) can be considered as a linear equation with respect to  $\varphi_\lambda(x)$  for  $\lambda \in \Lambda_+ \cup \Lambda \cup \tilde{\Lambda}$ . But the series in (59) converges only "with brackets," and the integral is understood in the principal value sense. Therefore, it is not convenient to use (59) as a main equation of the inverse problem. Below we will transfer (59) to a linear equation in a suitable Banach space (see (77)).

Denote

$$\xi_k = |\rho_k - \tilde{\rho}_k| + |\alpha_k - \tilde{\alpha}_k|, \quad \chi_k = \begin{cases} \xi_k^{-1} & \text{for } \xi_k \neq 0, \\ 0 & \text{for } \xi_k = 0, \end{cases}$$

$$\psi_{k0}(x) = (\varphi_{k0}(x) - \varphi_{k1}(x)) \chi_k, \quad \psi_{k1}(x) = \varphi_{k1}(x),$$

$$H_{\lambda, \mu}(x) = P_{\lambda, \mu}(x), \quad H_{\lambda, k0}(x) = P_{\lambda, k0}(x) \xi_k, \quad H_{\lambda, k1}(x) = P_{\lambda, k0}(x) - P_{\lambda, k1}(x),$$

$$H_{n0, \mu}(x) = (P_{n0, \mu}(x) - P_{n1, \mu}(x)) \chi_n, \quad H_{n1, \mu}(x) = P_{n1, \mu}(x),$$

$$H_{n0, k0}(x) = (P_{n0, k0}(x) - P_{n1, k0}(x)) \chi_n \xi_k, \quad H_{n1, k1}(x) = P_{n1, k0}(x) - P_{n1, k1}(x),$$

$$H_{n0, k1}(x) = (P_{n0, k0}(x) - P_{n1, k0}(x) - P_{n0, k1}(x) + P_{n1, k1}(x)) \chi_n, \quad H_{n1, k0}(x) = P_{n1, k0}(x) \xi_k.$$

Analogously we define  $\tilde{\psi}_{kj}(x)$ ,  $\tilde{H}_{\lambda, \mu}(x)$ ,  $\tilde{H}_{\lambda, kj}(x)$ ,  $\tilde{H}_{ni, \mu}(x)$  and  $\tilde{H}_{ni, kj}(x)$ . It follows from (56)-(58), (6)-(7) and Schwarz's lemma that for  $\lambda = \rho^2$ ,  $\mu = \theta^2$ ,  $\rho > 0$ ,  $\theta > 0$ ,  $i, j = 0, 1$ ,  $n, k \in \omega'$ , the following estimates hold

$$\left. \begin{aligned} & \left| \frac{\partial^j}{\partial \lambda^j} \varphi_\lambda(x) \right| \leq C, \quad |\psi_{kj}(x)| \leq C, \\ & \left| \frac{\partial^j}{\partial \lambda^j} H_{\lambda, \mu}(x) \right| \leq \frac{C |\hat{V}(\mu)|}{(|\rho - \theta| + 1)}, \quad \left| \frac{\partial^j}{\partial \lambda^j} H_{\lambda, kj}(x) \right| \leq \frac{C \xi_k}{(|\rho - |k| \pi/a| + 1)}, \\ & |H_{ni, \mu}(x)| \leq \frac{C |\hat{V}(\mu)|}{|\theta - |n| \pi/a| + 1}, \quad |H_{ni, kj}(x)| \leq \frac{C \xi_k}{||n| - |k|| + 1}. \end{aligned} \right\} \quad (73)$$

The same estimates are also valid for  $\tilde{\varphi}_\lambda(x)$ ,  $\tilde{\psi}_{kj}(x)$ ,  $\tilde{H}_{\lambda, \mu}(x)$ ,  $\tilde{H}_{\lambda, kj}(x)$ ,  $\tilde{H}_{ni, \mu}(x)$  and  $\tilde{H}_{ni, kj}(x)$ .

Consider the Banach space  $m$  of bounded sequences  $\alpha = [\alpha_v]_{v \in \omega_1}$  with the norm  $\|\alpha\|_m = \sup_{v \in \omega_1} |\alpha_v|$ . Define the vectors

$$\psi(x) = [\psi_v(x)]_{v \in \omega_1} = \left[ \begin{array}{c} \psi_{k0}(x) \\ \psi_{k1}(x) \end{array} \right]_{k \in \omega'}, \quad \tilde{\psi}(x) = [\tilde{\psi}_v(x)]_{v \in \omega_1} = \left[ \begin{array}{c} \tilde{\psi}_{k0}(x) \\ \tilde{\psi}_{k1}(x) \end{array} \right]_{k \in \omega'}.$$

It follows from (73) that for each fixed  $x$ ,  $\psi(x), \tilde{\psi}(x) \in m$ . Let  $B := C_1[0, \infty)$  be the Banach space of continuously differentiable functions  $\lambda \rightarrow f(\lambda)$  on the half-line  $\lambda \geq 0$  such that  $f(\lambda)$  and  $\frac{\partial}{\partial \lambda} f(\lambda)$  are bounded, with the norm  $\|f\|_B = \max_{j=0,1} \sup_{\lambda \geq 0} |\frac{\partial^j}{\partial \lambda^j} f(\lambda)|$ . It

follows from (73) that for each fixed  $x$ ,  $\varphi_\lambda(x), \tilde{\varphi}_\lambda(x) \in B$ . Consider the Banach space  $\mathcal{B}$  of vectors

$$F = \begin{bmatrix} f \\ \alpha \end{bmatrix},$$

where  $f \in B$ ,  $\alpha = [\alpha_v]_{v \in \omega_1} \in m$ , with the norm  $\|F\|_{\mathcal{B}} = \max(\|f\|_B, \|\alpha\|_m)$ . Denote

$$\Psi(x) = \begin{bmatrix} \varphi_\lambda(x) \\ \psi(x) \end{bmatrix}, \quad \tilde{\Psi}(x) = \begin{bmatrix} \tilde{\varphi}_\lambda(x) \\ \tilde{\psi}(x) \end{bmatrix}.$$

Clearly,  $\Psi(x), \tilde{\Psi}(x) \in \mathcal{B}$  for each fixed  $x$ . For fixed  $x$ , we define the operator  $\tilde{H} = \tilde{H}(x) : \mathcal{B} \rightarrow \mathcal{B}$  by the formulas

$$\begin{aligned} \tilde{F} &= \tilde{H}F, \quad F = \begin{bmatrix} f \\ \alpha \end{bmatrix} \in \mathcal{B}, \quad \tilde{F} = \begin{bmatrix} \tilde{f} \\ \tilde{\alpha} \end{bmatrix} \in \mathcal{B}, \\ \left. \begin{aligned} \tilde{f}(\lambda) &= \int_0^\infty \tilde{H}_{\lambda,\mu}(x) f(\mu) d\mu + \sum_{v \in \omega_1} \tilde{H}_{\lambda,v}(x) \alpha_v, \\ \tilde{\alpha}_u &= \int_0^\infty \tilde{H}_{u,\mu}(x) f(\mu) d\mu + \sum_{v \in \omega_1} \tilde{H}_{u,v}(x) \alpha_v, \end{aligned} \right\} & (74) \\ \lambda, \mu &\geq 0; \quad u = (n, i), \quad v = (k, j); \quad n, k \in \omega'; \quad i, j = 0, 1. \end{aligned}$$

Analogously we define the operator  $H = H(x)$ .

**Lemma 4.** *For each fixed  $x$ , the operators  $H(x)$  and  $\tilde{H}(x)$  are linear bounded operators acting from  $\mathcal{B}$  to  $\mathcal{B}$ .*

*Proof.* For definiteness, we consider here only one of the blocks in (74), the remaining blocks are studied similarly. Let  $f(\lambda) \in B$ , and let

$$f^*(\lambda) := \int_0^\infty \tilde{H}_{\lambda,\mu}(x) f(\mu) d\mu.$$

We will show that

$$f^*(\lambda) \in B, \quad \|f^*\|_B \leq C\|f\|_B. \quad (75)$$

Since  $\Lambda'' \cup \tilde{\Lambda}'' := \{\lambda_k^*\}_{k=\overline{1,p}}$  is a finite set, there exist numbers  $\{d_j\}_{j=\overline{-1,p+1}}$  such that  $0 = d_{-1} < d_0 < \lambda_1^* < d_1 < \dots < \lambda_p^* < d_p < d_{p+1} = \infty$ . Then

$$f^*(\lambda) = \sum_{k=0}^{p+1} f_k(\lambda),$$

where

$$f_k(\lambda) := \int_{d_{k-1}}^{d_k} \tilde{H}_{\lambda,\mu}(x) f(\mu) d\mu.$$

Since

$$\frac{\partial^j}{\partial \lambda^j} f_0(\lambda) = 2 \int_0^{\sqrt{d_0}} \tilde{D}_{j0}(x, \lambda, \mu) \theta \hat{V}(\mu) f(\mu) d\theta, \quad j = 0, 1, \quad \mu = \theta^2,$$

it follows from (41) and (58) that  $f_0(\lambda) \in B$  and  $\|f_0\|_B \leq C\|f\|_B$ . By virtue of (58),

$$\left| \frac{\partial^j}{\partial \lambda^j} f_{p+1}(\lambda) \right| \leq C\|f\|_B \int_{d_p}^\infty \frac{|\hat{V}(\mu)| d\mu}{|\rho - \theta| + 1}, \quad \lambda = \rho^2, \quad \mu = \theta^2, \quad \rho > 0, \quad \theta > 0.$$

According to (42),  $\hat{V}(\lambda) = O(\lambda^{-1})$ ,  $\lambda \rightarrow \infty$ , and consequently,  $f_{p+1}(\lambda) \in B$  and  $\|f_{p+1}\|_B \leq C\|f\|_B$ . For  $k = \overline{1, p}$ , we denote  $w_k(\mu) := \hat{V}(\mu)(\mu - \lambda_k^*)$ . Clearly,  $w_k(\mu) \in C[d_{k-1}, d_k]$ . Then

$$f_k(\lambda) = \int_{d_{k-1}}^{d_k} \tilde{D}(x, \lambda, \mu) w_k(\mu) \frac{f(\mu) - f(\lambda_k^*)}{\mu - \lambda_k^*} d\mu + f(\lambda_k^*) \int_{d_{k-1}}^{d_k} \frac{\tilde{D}(x, \lambda, \mu) - \tilde{D}(x, \lambda, \lambda_k^*)}{\mu - \lambda_k^*} w_k(\mu) d\mu + f(\lambda_k^*) \tilde{D}(x, \lambda, \lambda_k^*) \int_{d_{k-1}}^{d_k} \hat{V}(\mu) d\mu, \quad (76)$$

where the last integral is understood in the principal value sense. Using (76) and (58) it is easy to verify that  $f_k(\lambda) \in B$  and  $\|f_k\|_B \leq C\|f\|_B$  for  $k = \overline{1, p}$ . Thus, (75) holds. Lemma 4 is proved.  $\square$

**Theorem 6.** *For each fixed  $x$ , the vector  $\Psi(x) \in \mathcal{B}$  is the solution of the equation*

$$\tilde{\Psi}(x) = (E + \tilde{H}(x))\Psi(x) \quad (77)$$

in the Banach space  $\mathcal{B}$  ( $E$  is the identity operator). The operator  $E + \tilde{H}(x)$  has a bounded inverse operator, i.e. equation (77) is uniquely solvable.

*Proof.* Taking into account our notations we get from (59):

$$\tilde{\varphi}_\lambda(x) = \varphi_\lambda(x) + \int_0^\infty \tilde{H}_{\lambda, \mu}(x) \varphi_\mu(x) d\mu + \sum_{(k, j) \in \omega_1} \tilde{H}_{\lambda, kj}(x) \psi_{kj}(x), \quad \lambda > 0,$$

$$\tilde{\psi}_{ni}(x) = \psi_{ni}(x) + \int_0^\infty \tilde{H}_{ni, \mu}(x) \varphi_\mu(x) d\mu + \sum_{(k, j) \in \omega_1} \tilde{H}_{ni, kj}(x) \psi_{kj}(x), \quad (n, i) \in \omega_1,$$

which is equivalent to (77). Similarly, (60) takes the form

$$H(x) - \tilde{H}(x) + \tilde{H}(x)H(x) = 0$$

or

$$(E + \tilde{H}(x))(E - H(x)) = E.$$

Interchanging places for  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  we obtain in the same way

$$\Psi(x) = (E - H(x))\tilde{\Psi}(x), \quad (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence the operator  $(E + \tilde{H}(x))^{-1}$  exists, and it is a linear bounded operator. Theorem 6 is proved.  $\square$

Equation (77) is called the *main equation* of the inverse problem. Solving (77) we find the vector  $\Psi(x)$ , and consequently, the functions  $\varphi_\lambda(x)$  and  $\varphi_{ni}(x)$ . Since  $\varphi_\lambda(x)$  and  $\varphi_{ni}(x)$  are solutions of equation (1), we can construct the function  $q(x)$  and the coefficient  $h$ . Below Lemma 5 gives us explicit formulas for calculating  $q(x)$  and  $h$  from  $\varphi_\lambda(x)$  and  $\varphi_{ni}(x)$ . For simplicity, we assume that the model boundary value problem  $\tilde{\mathcal{L}}$  is chosen such that

$$\sum_k (k\xi_k)^2 < \infty, \quad \int_{\rho^*}^\infty \rho^4 |\hat{V}(\lambda)|^2 d\rho < \infty \quad (78)$$

for sufficiently large  $\rho^* > 0$ . Condition (78) is needed for technical reasons but such a choice is always possible, and for square integrable potentials it is trivial. Denote

$$\varepsilon_0(x) = (\eta(x))^{-1} \left( \int_0^\infty \varphi_\mu(x) \tilde{\varphi}_\mu(x) \hat{V}(\mu) d\mu + \sum_{k \in \omega'} (\varphi_{k0}(x) \tilde{\varphi}_{k0}(x) \alpha_{k0} - \varphi_{k1}(x) \tilde{\varphi}_{k1}(x) \alpha_{k1}) \right),$$

$$\varepsilon(x) = -2\varepsilon'_0(x). \quad (79)$$

**Lemma 5.** *The following relations hold*

$$q(x) = \tilde{q}(x) + \varepsilon(x), \quad (80)$$

$$h = \tilde{h} - \varepsilon_0(0). \quad (81)$$

*Proof.* Since the proof of Lemma 5 is similar to the proof of [20, Lemma 1.6.5], we outline here only the main steps of the proof. Differentiating (59) twice with respect to  $x \in J_{\pm}$  and using (79) we get

$$\begin{aligned} \tilde{\varphi}'_{\lambda}(x) - \varepsilon_0(x)\tilde{\varphi}_{\lambda}(x) &= \varphi'_{\lambda}(x) + \int_0^{\infty} \tilde{P}_{\lambda,\mu}(x)\varphi'_{\mu}(x) d\mu \\ &+ \sum_{k \in \omega'} \left( \tilde{P}_{\lambda,k_0}(x)\varphi'_{k_0}(x) - \tilde{P}_{\lambda,k_1}(x)\varphi'_{k_1}(x) \right), \quad (82) \\ \tilde{\varphi}''_{\lambda}(x) &= \varphi''_{\lambda}(x) + \int_0^{\infty} \tilde{P}_{\lambda,\mu}(x)\varphi''_{\mu}(x) d\mu + \sum_{k \in \omega'} \left( \tilde{P}_{\lambda,k_0}(x)\varphi''_{k_0}(x) - \tilde{P}_{\lambda,k_1}(x)\varphi''_{k_1}(x) \right) \\ + 2(\eta(x))^{-1}\tilde{\varphi}_{\lambda}(x) &\left( \int_0^{\infty} \tilde{\varphi}_{\mu}(x)\varphi'_{\mu}(x)\hat{V}(\mu) d\mu + \sum_{k \in \omega'} \left( \alpha_{k_0}\tilde{\varphi}_{k_0}(x)\varphi'_{k_0}(x) - \alpha_{k_1}\tilde{\varphi}_{k_1}(x)\varphi'_{k_1}(x) \right) \right) \\ &+ (\eta(x))^{-1} \left( \int_0^{\infty} (\tilde{\varphi}_{\lambda}(x)\tilde{\varphi}_{\mu}(x))'\hat{V}(\mu)\varphi_{\mu}(x) d\mu \right. \\ &\left. + \sum_{k \in \omega'} \left( (\tilde{\varphi}_{\lambda}(x)\tilde{\varphi}_{k_0}(x))'\alpha_{k_0}\varphi_{k_0}(x) - (\tilde{\varphi}_{\lambda}(x)\tilde{\varphi}_{k_1}(x))'\alpha_{k_1}\varphi_{k_1}(x) \right) \right). \quad (83) \end{aligned}$$

In (83) we replace the second derivatives using equation (1), and then we replace  $\varphi_{\lambda}(x)$  using (59). After cancelling terms with  $\tilde{\varphi}'_{\lambda}(x)$  we arrive at (80). Taking  $x = 0$  in (82) we get (81), since  $\tilde{D}(0, \lambda, \mu) = 0$ . Lemma 5 is proved.  $\square$

Denote by  $W_N$  the set of functions  $q(x)$ ,  $x \geq 0$  such that the functions  $q^{(j)}(x)$ ,  $j = \overline{0, N-1}$  are absolutely continuous on  $[0, T]$  for each fixed  $T > 0$ , and  $q^{(j)}(x) \in L(0, \infty)$ ,  $j = \overline{0, N}$ . Let  $Z_N$  be the set of functions  $q(x) \in W_N$  satisfying (4). We shall say that  $\mathcal{L} \in \mathcal{V}_N$  if  $q(x) \in Z_N$  and  $\mathcal{L}$  has simple spectrum. We shall solve the inverse problem in the classes  $\mathcal{V}_N$  for each fixed  $N \geq 0$ .

Using the results obtained above we arrive at the following statement.

**Theorem 7.** *Let  $S$  be the spectral data of  $\mathcal{L} \in \mathcal{V}_N$ . Then*

- 1)  $S$  has properties  $(i_1) - (i_5)$  stated in Theorem 4;
- 2) (asymptotics) there exists  $\tilde{\mathcal{L}} \in \mathcal{V}_N$  such that (78) holds;
- 3) for each fixed  $x$ , equation (77) has a unique solution  $\Psi(x) \in \mathcal{B}$ ;
- 4)  $\varepsilon(x) \in Z_N$  where the function  $\varepsilon(x)$  is defined by (79).

*The boundary value problem  $\mathcal{L}$  can be constructed by the following algorithm.*

**Algorithm 1.** *Let the spectral data  $S$  be given. Then*

- 1) Find  $\varphi_{\lambda}(x)$  and  $\varphi_{ni}(x)$  by solving the main equation (77).
- 2) Construct  $q(x)$  and  $h$  via (80)-(81).

**Remark 3.** Using the method of spectral mappings (see [19], [20]), one can prove that the conditions 1)-4) of Theorem 7 are not only necessary but also sufficient for the solvability of the inverse problem. In other words, for the data  $S$  to be the spectral data for a certain  $\mathcal{L} \in \mathcal{V}_N$ , it is necessary and sufficient that conditions 1)-4) of Theorem 7 are fulfilled.

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Freiling, Gerhard  
Fachbereich Mathematik, Universität Duisburg,  
D-47048 Duisburg, Germany  
freiling@math.uni-duisburg.de

Yurko, Vjacheslav Anatoljevich  
Department of Mathematics, Saratov State University,  
Astrakhanskaya 83, Saratov 410026, Russia  
yurkova@info.sgu.ru