A Survey of Nonsymmetric Riccati Equations

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Abstract: We survey recent and also older results on nonsymmetric matrix Riccati differential equations and - in the time invariant case - on the corresponding algebraic Riccati equations. In particular we cite various applications connected with matrix Riccati equations.

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1 Introduction

The main purpose of this paper is to summarize the most important results on nonsymmetric matrix Riccati differential equations

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W \qquad (RDE)$$

with piecewise continuous (or locally integrable) coefficients $M_{11},\ M_{12},\ M_{21},\ M_{22}$ of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively, and -in the case of constant coefficients- on the corresponding algebraic matrix Riccati equation

$$0 = M_{21} + M_{22}W - WM_{11} - WM_{12}W. \qquad (ARE)$$

With (RDE) and (ARE) we associate the matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$.

In Section 2 we present several typical examples of problems leading to nonsymmetric (differential or algebraic) matrix Riccati equations. Section 3 contains a survey of the most important results derived up to now for nonsymmetric matrix Riccati equations; in particular we

- consider Radon's Lemma and the linear system of differential equations associated with (RDE),
- explain how the solutions of (ARE) can be obtained from the M-invariant graph-subspaces,
- describe some special properties of symmetric matrix Riccati equations,
- give a representation formula for the solutions of RDE) and describe the Riccati flow generated by (RDE) on the Graßmann-manifold $G(n, n+m)$,
- state existence results for the solutions of (RDE),
- survey further results for (RDE).

Moreover we give an extensive list of references which, as we hope, will help the interested reader to find further and more detailed results on matrix Riccati equations. We are aware that this survey is by no means complete, in particular we have omitted details on those topics which have already been surveyed previously in [Reid72], [Zakh73], [Shay91] and [LaRo95].

2 Some applications of Riccati equations

2.1 The Riccati- and the LK Transformation

For the investigation of linear differential equations, in particular in singular perturbation problems and also in control theory (see [OMal74], [KOS76], [Ande82], [SGS92], [AgGa93], [Waso95], [Frid95]) one frequently uses special linear transformations in order to reduce high-order systems to lower order ones or in order to (partially) decouple the system.
If the original system is partitioned as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u, \tag{2.1}$$

where $A_{11} \in \mathbb{R}^{n\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $A_{21} \in \mathbb{R}^{m\times n}$, $A_{22} \in \mathbb{R}^{m\times n}$, $B_1 \in \mathbb{R}^{n\times k}$ and $B_2 \in \mathbb{R}^{m\times k}$, it may be conveniently transformed into decoupled subsystems by a two step transformation of the form $y = T_2T_1x =: TX$ with

$$T_1 = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}, \; T_2 = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \; T = \begin{pmatrix} I+KL & K \\ L & I \end{pmatrix};$$

here $L \in \mathbb{R}^{m\times n}, K \in \mathbb{R}^{n\times m}$ and I denotes identity matrices of dimensions n and m, respectively.
Notice that T, T_1 T_2 are always nonsingular and have the explicit inverses

$$T^{-1} = \begin{pmatrix} I & -K \\ -L & I+LK \end{pmatrix}, \; T_1^{-1} = \begin{pmatrix} I & -I \\ -L & I \end{pmatrix}, \; T^{-1} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}.$$

Hence, if L is a solution of the nonsymmetric algebraic Riccati equation

$$F(L) := A_{21} - A_{22}L + LA_{11} - LA_{12}L = 0, \tag{2.2}$$

then the Riccati-transformation $x \mapsto T_1x$ transforms (2.1) to block-triangular form, since

$$T_1^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} - A_{12}L & A_{12} \\ F(L) & A_{22} + LA_{12} \end{pmatrix}; \tag{2.3}$$

i.e. (2.1) ist partially decoupled. If moreover K satisfies the algebraic Sylvester equation

$$K\tilde{A}_{22}(L) - \tilde{A}_{11}(L)K + A_{12} = 0, \tag{2.4}$$

where $\tilde{A}_{11}(L) = A_{11} - A_{12}L$, $\tilde{A}_{22}(L) = A_{22} + LA_{12}$, then (2.1) ist transformed by $x \mapsto T_2T_1x =: y$ to the block-diagonal form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11}(L) & 0 \\ 0 & \tilde{A}_{22}(L) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u, \tag{2.5}$$

where $\tilde{B}_1 = (I+KL)B_1 + KB_2$, $\tilde{B}_2 = LB_1 + B_2$.
Therefore - if (2.2) and (2.4) are solvable - the transformation $x \mapsto Tx$ completely decouples the system

(2.1) and is sometimes called LK-transformation (see e.g. [Lyas55], [Ande82]).

The LK-transformation has for example been used in [SGS92] in order to exploit the reduced-order slow and fast subsystems of a singular perturbed linear control system.
In [SuGa91] the LK-transformation is applied as an essential tool in the study of continuous and discrete weakly coupled systems (see also [GPSh90] for further details).
A slightly different approach has been used in [HoTi88] and [AgGa93] for the investigation of weakly coupled bilinear control systems of the form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} B_1 & \varepsilon B_2 \\ \varepsilon B_3 & B_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$+ \left\{ \sum_{i=1}^{n_1} y_{1i} \begin{pmatrix} N_{ai} & N_{bi} \\ N_{ci} & N_{di} \end{pmatrix} + \sum_{i=n_1+1}^{n_1+n_2} y_{2,i-n_1} \begin{pmatrix} N_{aj} & N_{bj} \\ N_{cj} & N_{dj} \end{pmatrix} \right\} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Bigg\}.$$

with a quadratic cost functional. Using recursively LK-transformations with time varying matrices L_k and K_k it is shown in [AgGa93] that the bilinear system can be recursively decoupled if L_k and K_k are solutions of nonsymmetric Riccati differential equations of the form

$$\dot{L}_k = T_{1k} + T_{2k} L_k + L_k T_{2k} + L_k T_{3k} L_k \tag{2.6}$$

and of Sylvester differential equations

$$\dot{K}_k = K_k M_{1k}(L_k) + M_{2k}(L_k) K_k + M_{2k}, \tag{2.7}$$

respectively. The main, partially unsolved, problem that arises here is to find conditions that guarantee that the solutions of (2.6), satisfying some initial conditions, do not blow up (see Subsection 3.5).

2.2 Spectral factorization

Each solution L of the algebraic Riccati equation (2.2) corresponds to a certain spectral factorization of the matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ since the eigenvalues of A are (counting multiplicity) exactly the eigenvalues of the matrices $\tilde{A}_{11}(L) = A_{11} - A_{12}L$ and $\tilde{A}_{22}(L) = A_{22} + LA_{12}$; this follows from (2.3). In particular it is possible to chose n, m and the solution L of (2.3) such that the spectrum of $\tilde{A}_{11}(L)$ and $\tilde{A}_{22}(L)$ consist of the desired parts of the spectrum of A (see Subsection 3.2).
A similar important role is played by Riccati equations in more general factorization problems; Chapter 5 of the book of Bart, Gohberg and Kaashoek [BGK79] contains further information on the connection between factorization problems and algebraic Riccati equations, the work of Fuhrmann [Fuhr95] describes factorization theory from a polynomial point of view and [Fuhr95] is concerned with the characterization and parametrization of minimal spectral factors.
The first remarkable results on nonsymmetric algebraic Riccati equations have been published in 1976 by Meyer [Meye76] (who generalized earlier basic results of Potter [Pott66] and Martensson [Mart71] on the representation and on the properties of the solutions of symmetric algebraic Riccati equations) and nearly simultaneously by Clements and Anderson [ClAn76], who studied a spectral factorization problem leading to the nonsymmetric algebraic Riccati equation

$$P(A_+ - b_+ c_+^T) + (A_-^T - c_- b_-^T)P - P b_+ b_-^T P - c_- c_+^T = 0. \tag{2.8}$$

A part of the results of [Meye76] and [ClAn76] are summarized in Subsections 3.1-3.3.
Discrete-time spectral factorization problems leading to nonsymmetric algebraic Riccati equations have been studied by Fairman et al. [FDLZ92].

2.3 Riccati differential equations

Initial and/or terminal value problems for symmetric and nonsymmetric Riccati differential equations and the corresponding operator equations appear in many branches of applied mathematics, notably in variational theory, optimal control and filtering, H^∞-control, invariant embedding and scattering processes, dynamic programming and differential games. Most of these topics been discussed in the following textbooks, where the reader can find the derivation of various types of Riccati equations and a lot of further references:
Reid [Reid72] presents the theory of symmetric and nonsymmetric matrix Riccati equations from a general mathematical point of view, Ando [Ando88] is mainly focused on Hermitian algebraic Riccati equations and their applications in control theory, the books by Lasiecka and Triggiani [LaTr00] and Bensoussan, Da Prato, Delfour and Mitter [BPDM92] are devoted to distributed parameter systems and contain various results on operator Riccati equations, Basar and Olsder [BaOl95] contains contributions on the application of coupled and uncoupled matrix Riccati equations in the theory of differential games, in the books by Basar and Bernhard [BaBe91] and Knobloch, Isidori and Flockerzi [KIF93] one can find results on $H^\infty-$type symmetric algebraic Riccati equations and their application in H^∞-control theory and Zelikin [Zeli98] studies complex matrix Riccati differential equations and their geometric properties in connection with problems from calculus of variations.

Nonsymmetric operator Riccati equations with unbounded coefficients have been studied in [Kuip94], [Juan92] and [Juan95]; see [BPDM92] and [LaTr00] and the references cited therein for the theory of symmetric operator Riccati equations and their applications in the control theory of distributed parameter systems.
It is also worth while to mention the paper [Schu68] of Schumitzky, who showed (without recourse to any physical model) that the solution to every matrix Riccati equation can be generated by the resolvent of a certain Fredholm integral operator and, in turn this resolvent can be determined from the solution of the corresponding Riccati equation.

We conclude this section by an example from the theory of noncooperative differential games which also leads to nonsymmetric Riccati differential equations.

Consider a two-player linear quadratic differential game

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2,\ x(0) = x_0,$$
$$x \in \mathbb{R}^n,\ u_1 \in \mathbb{R}^{r_1},\ u_2 \in \mathbb{R}^{r_2},$$

with cost functionals

$$J_1 = \frac{1}{2} x^T(t_f) K_{1f} x(t_f) + \frac{1}{2}\int_0^{t_f} [x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2],$$
$$J_2 = \frac{1}{2} x^T(t_f) K_{2f} x(t_f) + \frac{1}{2}\int_0^{t_f} [x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2] dt,$$

where all weighting matrices are assumed to be symmetric and R_{11}, R_{22} are positive semidefinite.
It is known (see [BaOl95], Theorem 6.12), that the game has a so-called *open loop Nash solution* (i.e. the control u depends explicitly only on t) which is given by

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) \Phi(t,0), x_0,$$
$$u_2^*(t) = -T_{22}^{-1} B_2^T K_2(t) \Phi(t,0) x_0,$$

provided the unique solutions $K_1(t), K_2(t)$ of the coupled nonsymmetric Riccati equations

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2,\ K_1(t_f) = K_{1f},$$
$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_{22} K_2 + K_2 S_{11} K_1,\ K_2(t_f) = K_{2f}, \tag{2.9}$$

exist for $t \in [0, t_f]$. Here $S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$ and $\Phi(t,0)$ is the transition matrix of the corresponding closed loop system, i.e.

$$\dot{\Phi}(t,0) = (A - S_{11} S_{22})\Phi(t,0),\ \Phi(t,t) = I.$$

Notice that the rectangular matrix $K = \binom{K_1}{K_2}$ is the solution of the following terminal value problem for a nonsymmetric Riccati equation:

$$\dot{K} = -\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} K - KA - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + K(S_{11} S_{22})K,\ K(t_f) = \begin{pmatrix} K_{1f} \\ K_{2f} \end{pmatrix}. \tag{2.10}$$

Usually the solution K of (2.10) may have finite escape time; in Subsection 3.5 we give sufficient conditions for the existence of $K(t)$, for all $t \le t_f$.
The open loop Nash and the open loop Stackelberg strategies for $N(\ge 2)$ player differential games are often also determined by the solutions of terminal or boundary value problems for coupled Riccati equations which can be transformed to nonsymmetric Riccati equations (see [BaOl95], [AFJ93a], [AFJ93b]); an interesting sample of such results can be found in [FJL01], where new sufficient conditions for an open loop Stackelberg equilibrium in terms of simultaneous solvability of two symmetric matrix Riccati differential equations and one nonsymmetric matrix Riccati differential equation have been derived.

On the other hand the corresponding *closed loop Nash strategies* (i.e. the control u depends explicitly on t and $x(t)$) are defined by solutions of the coupled Riccati equations

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_{11} K_1 + K_1 S_{22} K_2 + K_2 S_{22} K_1,\ K_1(t_f) = K_{1f},$$
$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_{22} K_2 + K_2 S_{11} K_1 + K_1 S_{11} K_2,\ K_2(t_f) = K_{2f}. \tag{2.11}$$

The solutions of the later system are symmetric (for symmetric K_{if}) but this system cannot be reduced (by Radon's Lemma) to an equivalent linear system. Therefore these coupled systems are (in our notation) not Riccati equations and ask for a separate special treatment (see [FJA96b] and - for a similar type of coupled equations - [FJL99]). Notice that by some authors *any quadratic matrix differential equation* is called matrix Riccati equation - we do not adopt this notation. We propose to denote only those quadratic differential equations which can be written in the form (RDE) as Riccati equations, since these can be transformed by Radons Lemma (see Theorem 3.1) to a locally equivalent linear system of differential equations.
Further types of quadratic matrix differential equations resulting from coupled or nested systems of Riccati equations have been investigated in connection with isospectral flows in [dMSh88], [Broc94] and [Helm91]; such quadratic differential equations appear also in the theory of inverse spectral problems (see [Jurk91]).

3 Results on Nonsymmetric Riccati Equations

3.1 The corresponding linear system

Let $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ and $Y = \binom{Q}{P}$ with real or complex $n \times n$ and $m \times n$ matrices Q and P, respectively.

It is known at least since the work of Radon [Rado27], [Rado28] and it can be verified by an elementary calculation that each Riccati differential equation

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W \tag{RDE}$$

is equivalent to the corresponding linear system

$$\dot{Y} = M(t)Y \tag{L}$$

in the following sense :

3.1 Theorem (Radons Lemma - version 1)

(i) Let I be the $n \times n$ identity matrix and let W be on some interval $J \subset \mathbb{R}$ a solution of (RDE) with $W(t_0) = W_0$. If Q is for $t \in J$ the unique solution of the initial value problem

$$\dot{Q} = (M_{11}(t) + M_{12}(t)W(t))Q,\ Q(t_0) = I,$$

and $P(t) := W(t)\, Q(t)$, then $Y(t) = \binom{Q(t)}{P(t)}$ defines for $t \in J$ the solution of (L) with $Y(t_0) = \binom{I}{W_0}$.

(ii) If $Y = \binom{Q}{P}$ is on some interval $J \subset \mathbb{R}$ a solution of (L) such that $\det Q(t) \neq 0$ for $t \in J$ then

$$W : J \to \mathbb{C}^{m\times n},\ t \mapsto P((t)Q^{-1}(t) =: W(t) \tag{3.1}$$

is a solution of (RDE); in particular $W(t_0) = P(t_0)Q(t_0)^{-1}$.

Version 1 of Radons Lemma shows that any *initial value problem* for a matrix Riccati equation is locally equivalent to an *initial value problem* for the linear system (L). Theorem 3.1 also yields that the the Riccati flow has very nice properties and that matrix Riccati equations play an exceptional role among nonlinear differential equations (see [Shay85], [FrJa95]), moreover, this result can be used to represent the solutions of (RDE) explicitely (see Theorem 3.8).

On the other hand it is clear from (3.1) that the solutions of (RDE) may show the finite escape time phenomenon; therefore one needs sufficient conditions ensuring that the solutions of (RDE) do not blow up on a given interval (see Section 3.5).

In variational calculus, in control theory and game theory, e.g. in Stackelberg differential games (see [BaOl95], Chapter 7), necessary conditions for an equilibrium are determined by the solvability of a linear boundary value problem in the time interval $[t_0, t_f]$

$$\frac{d}{dt}\begin{pmatrix}\psi \\ x\end{pmatrix} = M(t)\begin{pmatrix}\psi \\ x\end{pmatrix} = \begin{pmatrix}M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t)\end{pmatrix}\begin{pmatrix}\psi \\ x\end{pmatrix},\quad x(t_0) = x_0,\quad \psi(t_f) \;=\; K_f x(t_f), \tag{3.2}$$

$x(t) \in \mathbb{R}^m,\ \ \psi(t) \in \mathbb{R}^n,\ \ K_f \in \mathbb{R}^{n\times m}$.

We mention that in control theory the matrix $M(t)$ turns out to be Hamiltonian which is indeed a rather specific situation.

The following variant of Radon's Lemma shows how the solution of the *boundary value problem* (3.3) can be derived from the existence of a solution of an *initial value problem* for (RDE); usually this initial value problem is more adequate for applying numerical solution algorithms.

We give below a proof of this theorem since it describes one of the most important applications of matrix Riccati differential equations.

3.2 Theorem (Radons Lemma - version 2) *If the following initial value problem for the Riccati differential equation*

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W,\quad W(t_0) = 0,\ \ W(t) \in \mathbb{R}^{m\times n} \tag{3.3}$$

admits a solution W in $[t_0, t_f]$ and if

$$\det(K_f W(t_f) - I_n) \neq 0$$

then the boundary value problem (3.2) has a unique solution.

Proof. If the initial value problem (3.3) has a solution W in $[t_0, t_f]$ then we let $Q(t) \in \mathbb{R}^{n\times n}$ denote the solution of

$$\dot{Q} = (M_{11} + M_{12}W(t))Q, \quad Q(t_0) = I_n \tag{3.4}$$

in $[t_0, t_f]$ and define there

$$P(t) := W(t)Q(t) \in \mathbb{R}^{m\times n}. \tag{3.5}$$

Notice that (3.4) implies $\det Q(t) \neq 0$ for all $t \in [t_0, t_f]$. Then it follows that $\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} \in \mathbb{R}^{(m+n)\times n}$ solves the linear differential equation in (3.2) and consists of n linearly independent solutions of that system with

$$\begin{pmatrix} Q \\ P \end{pmatrix}(t_0) = \begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

Let

$$\phi(t) := \begin{pmatrix} \phi_{11} & Q \\ \phi_{21} & P \end{pmatrix}, \quad \phi(t_0) = \begin{pmatrix} * & I_n \\ I_n & 0 \end{pmatrix}$$

denote a fundamental matrix of (3.2) then all solutions of the boundary value problem (3.2) are determined by

$$\begin{pmatrix} \psi \\ x \end{pmatrix}(t) = \phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

for some constant vectors $c_1 \in \mathbb{R}^n,\ c_2 \in \mathbb{R}^m$.
We shall now prove that c_1, c_2 are uniquely determined. At t_0 we obtain

$$\begin{pmatrix} \psi \\ x \end{pmatrix}(t_0) = \begin{pmatrix} \psi(t_0) \\ x_0 \end{pmatrix} = \begin{pmatrix} * & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

hence $c_1 = x_0$.
At t_f, from the equality

$$K_f x(t_f) = K_f(\phi_{21}(t_f)x_0 + P(t_f)c_2) = \psi(t_f) = \phi_{11}(t_f)x_0 + Q(t_f)c_2$$

we obtain the equation

$$(I_n - K_f W(t_f))Q(t_f)c_2 = (K_f\phi_{21}(t_f) - \phi_{11}(t_f))x_0.$$

Since $W(t) = P(t)Q^{-1}(t)$ exists in $[t_0, t_f]$ we infer together with (3.4) that $Q(t_f)$ is regular and since $(I_n - K_f W(t_f))$ is assumed to be regular too, c_2 is also uniquely determined. Hence the boundary value problem has a unique solution. □

The impact of Radons Lemma on the so-called Riccati flow is described in more detail in Section 3.4.

3.2 Algebraic Riccati Equations

In the case of constant coefficients M_{ij} one is in particular interested in the constant solutions of (RDE), i.e. in the solutions of the corresponding algebraic Riccati equation

$$0 = M_{21} + M_{22}W - WM_{11} - WM_{12}W. \qquad (ARE)$$

In principle the solutions of (ARE) could be determined directly by applying Theorem 3.1 (see the comments following Theorem 3.8). If one is only interested in the solutions of (ARE) it is easier to use the following elementary but basic theorem.

3.3 Theorem *There is a one-to-one correspondence between the set of all (real or complex) solutions W of (ARE) and the set of n-dimensional M-invariant subspaces which are in $\mathbb{C}^{n\times m}$ complementary to the m-dimensional subspace $Im\binom{0}{I_m}$; this correspondence is defined by*

$$W \leftrightarrow Im \begin{pmatrix} I_n \\ W \end{pmatrix} =: S(W);$$

Here I_n denotes the n-dimensional unit matrix - in the sequel we suppress n. W is called the solution of (ARE) corresponding to the subspace $S(W)$.
Moreover $M_{11} + M_{12}W$ is the matrix of the restriction of the linear map, defined by M to $S(W)$ with respect to the basis defined by the columns of $\binom{I}{W}$.

Proof. (i) Let S be an n-dimensional M-invariant subspace which is complimentary to Im $\binom{0}{I_m}$. Then there exists a $m \times n$ matrix W such that $S = \text{Im } \binom{I}{W}$ and a $n \times n$ matrix R verifying

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I \\ W \end{pmatrix} = \begin{pmatrix} I \\ W \end{pmatrix} R. \qquad (3.6)$$

The first equation of (3.6) yields $R = M_{11} + M_{12}W$. Using this expression of R in the second equation gives

$$M_{21} + M_{22}W = WM_{11} + WM_{12}W,$$

which shows that W is a solution of (ARE).
(ii) If W is a solution of (ARE) then, similarly,

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I \\ W \end{pmatrix} = \begin{pmatrix} I \\ W \end{pmatrix} (M_{11} + M_{12}W),$$

which shows that $S(W)$ is M invariant and that $M_{11} + M_{12}W$ is the matrix of the restriction of M to $S(W)$. □

Let $V = (v_1, \ldots, v_{n+m}) = \begin{pmatrix} x_1 \ldots x_{n+m} \\ y_1 \ldots y_{n+m} \end{pmatrix}$ be a Jordan basis (matrix) of M, i.e.

$$V^{-1}MV = J_M \begin{pmatrix} T_1 & * & 0 \\ & & * \\ 0 & & T_{n+m} \end{pmatrix} (\text{ with } x \in \{0,1\}) \qquad (3.7)$$

for some Jordan canonical form J_M of M.
If Im $\begin{pmatrix} x_{i_1} & \ldots & x_{i_n} \\ y_{i_1} & \ldots & y_{i_n} \end{pmatrix}$ is M-invariant and if $(x_{i_1}, \ldots, x_{i_n}) \in \mathbb{C}^{n\times n}$ ist nonsingular, then

$$W_0 := (y_{i_1}, \ldots, y_{i_n})(x_{i_1}, \ldots, x_{i_1})^{-1} \qquad (3.8)$$

is a solution of (ARE). Moreover any solution of (ARE) can be represented in this way for some adequate subsystem of an adequate Jordan basis of M.

Remark a) *If M has eigenvalues of geometric multiplicity greater than one* then there exists always an uncountable number of n-dimensional M-invariant subspaces - in this case (ARE) may also have an uncountable set of solutions. Moreover in this case it is in general not possible to represent each solution of (ARE) in the form (3.8) by choosing $\begin{pmatrix} x_{i_1} \\ y_{i_1} \end{pmatrix}, \ldots, \begin{pmatrix} x_{i_n} \\ y_{i_n} \end{pmatrix}$ from a *fixed* Jordan basis of M since in this case M has n-dimensional subspaces which cannot be represented as $\mathrm{Im} \begin{pmatrix} x_{i_1} & \ldots & x_{i_n} \\ y_{i_1} & \ldots & y_{i_n} \end{pmatrix}$ using only generalized eigenvectors from a fixed, apriori selected Jordan basis.

b) *If M is semisimple*, i.e. if the geometric multiplicity of each eigenvalue of M is one, then the number N_M of all n-dimensional M-invariant subspaces, which is an upper bound for the number of all solutions of (ARE) is $\leq \binom{n+m}{n}$. Moreover $N_M = \binom{n+m}{n}$ if and only if all eigenvalues of M are simple.

c) If the solution W_0 of (ARE) is defined by (3.8) then the restriction of the map, defined by M on $\mathbb{C}^{n+m}$, to the subspace $S(W_0)$ has (counting multiplicity) the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_n}$. Consequently, according to Theorem 3.1, the matrix $M_{11} + M_{12}W_0 \in \mathbb{C}^{n \times n}$ has also the same eigenvalues. This elucidates the influence of the choice of W_0 on the spectral factorization defined by the corresponding Riccati transformation with the transformation matrix $T_0 = \begin{pmatrix} I & 0 \\ W_0 & I \end{pmatrix}$ (see Subsection 2.2); in particular one of the corresponding spectral factor has in this case the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_n}$ determined by W_0.

From the preceding results is follows that the solutions of (ARE) can be determined in two steps:

Step 1: Determine (using some numerically stable method) a Jordan basis (matrix)

$$V = (v_1, \ldots, v_{n+n}) = \begin{pmatrix} x_1, & \ldots, & x_{n+m} \\ y_1, & \ldots, & y_{n+m} \end{pmatrix} \text{ of } M.$$

Step 2: If Im $(v_{i_1}, \ldots, v_{i_n})$ is M invariant and if the *complimentary condition* $\det(x_{i_1}, \ldots, x_{i_n}) \neq 0$ is fulfilled, then W_0, defined by (3.8), is a solution of (ARE).

Many different *numerical methods for the solution of* (ARE) have been proposed in the literature; an interesting survey of the numerical methods for hermitian (or real symmetric) algebraic Riccati equations has been given in the recent book of Sima [Sima96].
Numerical methods for the solution of nonsymmetric algebraic Riccati equations have been proposed among others by Beavers and Denman [BeDe74] (matrix-sign-function), by Adomian, Pandolfi and Rach [APR88] (decomposition method) and by Ghavimi, Kenney and Laub [GKL92], [GhLa95] (conjugate gradient method, sensitivity analysis).

The set of all hermitian (symmetric) solutions of the hermitian (symmetric) algebraic Riccati equation

$$0 = -A^*P - PA - Q + PSP, \qquad (HARE)$$

with $Q = Q^*$ and $S = S^*$ has been studied extensively in the literature. Therefore we resign to address to this topic in detail and refer the interested reader to the survey articles in [BLW91] and to the excellent textbook of Lancaster and Rodman [LaRo95].

3.3 Hermitian Riccati differential equations

In this subsection we recall for convenience of the reader some important results concerning hermitian Riccati differential equations, since these results are of great importance for applications and from methodical point of view.
In the theory of linear-quadratic optimal control and filtering and also in H^∞-control one is interested in the solutions of hermitian (or symmetric) Riccati differential equations

$$\dot{P} = -A^*(t)P - PA(t) - Q(t) + PS(t)P,\ P(t_f) = P_f, \qquad (HRDE)$$

with hermitian (or real symmetric) $n \times n$ matrices Q, S and P_f; in many applications the coefficients A, Q and S are constant, then one is mainly interested in the existence of a (unique) stabilizing solution of the corresponding algebraic Riccati equation ($HARE$).
The global existence of the (hermitian) solution $P(t)$ of ($HRDE$) for $t \in (-\infty, t_f)$ or $t \in [t_0, t_f]$ for some $[t_0, t_f]$ can be guaranteed under rather weak assumptions (compare the existence results given in Section 3.3 with those stated in Section 3.5 for (RDE)).
The main reason for this fact is the following comparison theorem which shows that the solutions of ($HRDE$) depend monotonically on the initial value and on the coefficients of the differential equation.

3.4 Comparison Theorem. *Let $D \subset \mathbb{R}$ be some interval, $t_0 \in D$. If $P_i, 1 \leq i \leq 2$, are on D solutions of*

$$\dot{P}_i = -A^*(t)P_i - P_iA(t) - Q_i(t) + P_iS(t)P_i$$

with $P_1(t_0) \leq P_2(t_0)$ then

$$\begin{pmatrix} Q_1 & A_1^* \\ A_1 & -S_1 \end{pmatrix}(t) \leq \begin{pmatrix} Q_2 & A_2^* \\ A_2 & -S_2 \end{pmatrix}(t) \text{ for } t \in D$$

implies that $P_1(t) \leq P_2(t)$ for $t \in D_1(-\infty, t_0]$.

This theorem (see [FJA96a], Theorem 2.1) is a slight generalization of a theorem stated by Coppel (see [Copp65], pp. 51/52) where $S_i(t) \geq 0$ was assumed. Several variants of this theorem and of its analog version for hermitian algebraic Riccati equations and also of its infinite dimensional version (see [ErMy82], [BPDM92], Proposition 2.2, [vEHe90] and [CuRo90]) can be found in the literature.

It has been proved by Stokes [Stok74] that the hermitian Riccati differential equation is for $n > 1$ the only matrix differential equation of the form $W' = F(t, W)$ having the *order preserving property.*

As an immediate consequence of Theorem 3.4 we get the following existence theorem

3.5 Theorem *If $S(t)$, $Q(t) \geq 0$ for $t \leq t_0$ then the (unique) solution P of the Riccati differential equation*

$$\dot{P} = -A^*(t)P - PA(t) - Q(t) + PS(t)P, \quad P(t_0) = P_0 \geq 0 \qquad (3.9)$$

with piecewise continuous and locally bounded coefficients exists for $t \leq t_0$ with

$$0 \leq P(t) \leq \widetilde{P}(t) \quad \text{for} \quad t \leq t_0; \qquad (3.10)$$

here $\widetilde{P}$ is the solution of the linear equation

$$\dot{\widetilde{P}} = -A^*(t)\widetilde{P} - \widetilde{P}A(t) - Q(t), \quad \widetilde{P}(t_0) = P_0.$$

Proof. Since P_0, $S(t)$, $Q(t) \geq 0$ the Comparison Theorem 3.4 implies that P is for $t \leq t_0$ bounded from above by $\widetilde{P}$ and from below by the trivial solution 0 of the initial value problem

$$\dot{P}_0 = -A^*(t)P_0 - P_0A(t) + P_0S(t)P_0, \quad P_0(t_0) = 0.$$

Consequently $P(t)$ exists for all $t \leq t_0$ and satisfies (3.10). □

Theorem 3.5 shows in particular why the solutions of the standard symmetric Riccati equations (with $Q(t), S(t), P_f \geq 0$) do not blow up for $t \leq t_f$. If $S(t)$ or/and $Q(t)$ or/and P_f are no longer positive semidefinite it is usually much more complicated to ensure that $P(t)$ exists for all $t \leq t_f$ (see [FJLA96] for an elementary study of the phase portrait of $(HRDE)$).
The following existence result is also a consequence of the Comparison Theorem 3.4, combined with the monotonicity preserving property of hermitian Riccati equations, see [FrJa96].

3.6 Theorem *Let $A_i, Q_i = Q_i^*,\ S_i = S_i^* \in \mathbb{C}^{n\times n},\ 1 \leq i \leq 2$, be such that*

$$\begin{pmatrix} Q_1 & A_1 \\ A_1^* & -S_1 \end{pmatrix} \leq \begin{pmatrix} Q(t) & A(t) \\ A^*(t) & -S(t) \end{pmatrix} \leq \begin{pmatrix} Q_2 & A_2 \\ -A_2^* & -S_2^* \end{pmatrix} \text{ for } t \leq t_f.$$

If there exist hermitian matrices P_1, P_2 with

$$P_1 \leq P_f \leq P_2$$

and

$$-A^*P_i - P_iA - Q_i + P_iS_iP_i \begin{cases} \leq 0 & \text{for } i = 1, \\ \geq 0 & \text{for } i = 2, \end{cases} \tag{3.11}$$

then the solution $P(t)$ of $(HRDE)$ exists for $t \leq t_f$, moreover

$$P_1 \leq P(t) \leq P_2 \ \text{ for } t \leq t_f.$$

Notice that the algebraic Riccati inequalities (3.11) are solvable if the corresponding hermitian algebraic Riccati equations have a hermitian solution. The solvability of (3.11) could also be considered as a feasibility problem for a linear matrix inequality (LMI); for applications of LMI's in systems and control see [BEFB94].

In the oscillation theory for second order linear differential equations

$$Y'' + Q(t)Y = 0 \tag{3.12}$$

with a real continuous matrix coefficient Q one is interested in comparison theorems for special matrix Riccati equations of the form

$$Z' + Z^2 + Q(t) = 0. \tag{3.13}$$

Here $Z(t) = Y'Y^{-1}$ satisfies (3.13) if and only if Y is an invertible solution of (3.12) and one has to compare the solutions of differential equations of the form (3.12) with different coefficients Q. If Q is symmetric then one can use the Comparison Theorem 3.4 - in the nonsymmetric case the situation is more involved and one can use an *elementwise* comparison theorem (see for example [BJW85]) for the proof of nonoscillation theorems.

A different approach for the proof of existence results for hermitian or nonhermitian solutions of hermitian Riccati differential equations with *constant coefficients* A, Q, S has been presented in [FrJa96], Section 5;

here we reproduce a sample of these results.

Assumption A: Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}^{n\times n}$ be a symplectic Jordan basis matrix of $H = \begin{pmatrix} A & -S \\ -Q & -A^* \end{pmatrix}$, where H has no purely imaginary eigenvalues and where Im $\begin{pmatrix} a \\ c \end{pmatrix}$ is the stable subspace of H; moreover let $P_0 \in \mathbb{C}^{n\times n}$ such that $\det Z_1 \neq 0$. Here

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := V^{-1} \begin{pmatrix} I \\ P_0 \end{pmatrix}.$$

3.7 Theorem *If under* Assumption A *there exists some $\rho \leq 0$ with $S + \rho bb^* > 0$ and*

$$P_0(S + \rho bb^*)P_0^* - P_0(A + \rho bd^*) - (A^* + \rho db^*)P_0^* - Q + \rho dd^* \leq 0,$$

then solution of

$$\dot{P} = -A^*P - PA - Q + PSP, \; P(0) = P_0$$

does not blow up on the interval $(-\infty, 0]$.

Notice that the terminal value P_0 in Theorem 3.7 is not necessarily hermitian, therefore Theorem 3.7 gives also a non-blow-up condition for nonhermitian solutions of hermitian Riccati differential equations. Several variants of Theorem 3.7 are given in [FrJa96], Corollary 5.5.

We mention that a part of the results stated in this subsection for hermitian Riccati equations (in particular a comparison theorem) remains valid for the following class of nonlinear differential equations:

$$-\dot{X} = XA + A^*X + \prod\nolimits_{11}(x) + Q$$

$$-[S + XB + \prod\nolimits_{12}(X)][R + \prod\nolimits_{22}(X)]^{-1}[S + XB + \prod\nolimits_{12}(X)]^*,$$

where $\prod(X) = \begin{pmatrix} \prod_{11} X & \prod_{12}(X) \\ \prod_{12}(X)^* & \prod_{22}(X) \end{pmatrix}$ is a positive linear operator in X.
Equations of this type and the corresponding algebraic equations appear in stochastic control problems. Their solutions show a part of the nice properties of the solutions of hermitian matrix Riccati differential equations and have been studied in among others in [Wonh68], [DrMo97], [FCdS98], [DaHi99], [DrMo01], [FrHo01].

3.4 Flows on Graßmann manifolds and a representation formula for the solutions of (RDE)

a) The Riccati flow on G(n,n+m) Let V_0 be a complex vector space of dimension α. The Graßmann manifold $G(k, V_0)$ is defined to be the set of all k-dimensional linear subspaces of V_0; we write $G(n, n+m)$ for $G(n, \mathbb{C}^{n+m})$. $G(n, n+m)$ is a complex manifold and is equivalent to the set of all $(n-1)$-planes in the projective space $\mathbb{P}^{n+m-1}$.
Let $\psi : \mathbb{C}^{m\times n} \to G(n, n+m)$ be defined by $\psi(K) =$ Im $\begin{pmatrix} I_n \\ K \end{pmatrix}$ and let $G_0(n, n+m)$ consist of those subspaces in $G(n, n+m)$ which are complementary to the m-dimensional subspace Im $\begin{pmatrix} 0 \\ I_m \end{pmatrix}$, then ψ imbeds $\mathbb{C}^{m\times n}$ in $G(n, n+m)$ as the open and dense subset $G_0(n, n+m)$. In fact $(G_0(n, n+m), \psi^{-1})$ is one

of the standard charts for the manifold $G(n, n+m)$. Thus $G(n, n+m)$ can be viewed as a compactification of $\mathbb{C}^{m\times n}$.
Let $W(\cdot; W_0)$ denote the solution of (RDE) with $W(0; W_0) = W_0$, and define a flow (the so-called Riccati flow) on $G(n, n+m)$ by $S(t; S_0) = e^{Mt}(S_0) = \textit{Image of the subspace } S_0 \textit{ under } e^{Mt} \in Gl\ (n+m, \mathbb{C})$. Then we have

$$\psi(W(t; W_0)) = S(t; \psi(W_0)),$$

whenever $W(t; W_0)$ exists. This means that (RDE) is the local expression with respect to the chart $(G_0(n, n+m), \psi^{-1})$ for the differential equation on $G(n, n+m)$ which corresponds to the flow S. If we use the embedding ψ to identify $\mathbb{C}^{m\times n}$ with $G_0(n, n+m)$, then the restriction of S to $G_0(n, n+m)$ is identified with (RDE); $W(t; W_0)$ ceases to exist (in $\mathbb{C}^{m\times n}$) precisely when $S(t; \psi(W_0))$ leaves the subset $G_0(n, n+m)$.
By the extended Riccati differential equation $(ERDE)$, we mean the differential equation on $G(n, n+m)$ whose flow is given by S, thus the flow is given by the action of a one-parameter subgroup of $Gl\ (n+m, \mathbb{C})$ on $G(n, n+m)$.
For hermitian (symmetric) matrix Riccati differential equations [Shay85] contains a detailed analysis of the phase portrait of RDE) and $(ERDE)$. For general nonsymmetric (RDE) the phase portrait has been analyzed by Medanic [Meda82] and, in more detail, in [FrJa91], [FrJa95]; the two latter papers are based on the following representation formula.

b) The fundamental representation formula

Subsequently we present - for the time-invariant case - a detailed representation formula for the general solution of (RDE).
Let $V = (v_1, \ldots, v_{n+m}) \in \mathbb{C}^{(n+m)\times(n+m)}$ be the matrix defined by a Jordan basis of generalized eigenvectors of M such that

$$V^{-1}M\ V = J = \operatorname{diag}(J_1, \ldots, J_p) = \begin{pmatrix} \lambda_1 & * & 0 \\ \ddots & \ddots & * \\ 0 & \ddots & \lambda_{n+m} \end{pmatrix}$$

with $* \in \{0, 1\}$ and (without loss of generality)

$$\operatorname{Re}(\lambda_1) \le \operatorname{Re}(\lambda_2) \le \ldots \le \operatorname{Re}(\lambda_{n+m}),$$

and where J is a Jordan canonical form of M with Jordan-blocks

$$J_\nu = \begin{pmatrix} \mu_\nu & 1 & \ldots & 0 \\ 0 & \mu_\nu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & \mu_\nu \end{pmatrix} \in \mathbb{C}^{j_\nu \times j_\nu},\ \mu_\nu \in \{\lambda_1, \ldots, \lambda_{n+m}\}, 1 \le \nu \le p.$$

Let an initial value $W_0 \in \mathbb{C}^{m\times n}$ be given and let

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+m} \end{pmatrix} = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{n+m,1} & \ldots & c_{n+m,n} \end{pmatrix} := V^{-1}\begin{pmatrix} I_n \\ W_0 \end{pmatrix}$$

and

$$(x_1(t), \ldots, x_{n+m}(t))\operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_{n+m} t}) := V\ e^{Jt},$$

then the unique matrix-solution of (L), satisfying $Y(0)=\binom{I_n}{W_0}$ has the form

$$Y(t)=V\,e^{Jt}C=\begin{pmatrix}Q(t)\\P(t)\end{pmatrix}$$

with $Q(t)\in\mathbb{C}^{n\times n}$ since $W(t)\in\mathbb{C}^{m\times n}$ and $Y(0)=VC$.
Further, to every Jordan-block $J_\nu, 1\le\nu\le p$, there correspond j_ν solution-vectors of the form

$$\begin{aligned}&e^{\mu_\nu t}v_{j_1+\ldots+j_{\nu-1}+1}=:e^{\mu_\nu t}x_{j_1+\ldots+j_{\nu-1}+1}(t),\\&\qquad\vdots\\&e^{\mu_\nu t}\sum_{i=1}^{j_\nu}\frac{t^{j_\nu-i}}{(j_\nu-i)!}v_{j_1+\ldots+j_{\nu-1}+i}=:e^{\mu_\nu t}x_{j_1+\ldots+j_\nu}(t),\end{aligned}\tag{3.14}$$

$(1\le\nu\le p$ and $j_0:=0)$; notice that $x_j(t\equiv v_j$ for $1\le j\le n+m$ if M is semisimple.
Let $x_\nu(t)=\begin{pmatrix}x_{\nu 1}(t)\\\vdots\\x_{\nu,n+m}(t)\end{pmatrix}, 1\le\nu\le n+m$, be the polynomials defined by (3.14), $1\le j\le n$ and $1\le\ell\le m$, then we set

$$\tilde{x}_\nu:=\begin{pmatrix}x_{\nu 1}\\\vdots\\x_{\nu n}\end{pmatrix},\ x_\nu(\ell,j):=(x_{\nu 1},\ldots,x_{\nu,j-1},x_{\nu,n+\ell},x_{\nu,j+1},\ldots,x_{\nu n})^T;$$

similarly we define $\tilde{v}_\nu$ and $v_\nu(\ell,j)$. Using these notations we obtain explicit formulas for $Q(t)$, $P(t)$ and $P(t)Q(t)^{-1}$ (see [FrJa95] for details), hence version 1 of Radon's Lemma yields:

3.8 Theorem (Fundamental representation formula for the solutions of (RDE))
The solution $W(\cdot,W_0)=(w_{\ell,\alpha})$ of (RDE) satisfying $W(0)=W_0$ has the explicit representation

$$w_{\ell\alpha}(t)=\frac{\displaystyle\sum_{1\le k_1<\ldots<k_n\le n+m}e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t}|x_{k_1}(\ell,\alpha),\ldots,x_{k_n}(\ell,\alpha)|(t)\begin{vmatrix}c_{k_1}\\\vdots\\c_{k_n}\end{vmatrix}}{\displaystyle\sum_{1\le k_1<\ldots<k_n\le n+m}e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t}|\tilde{x}_{k_1},\ldots,\tilde{x}_{k_n}|(t)\begin{vmatrix}c_{k_1}\\\vdots\\c_{k_n}\end{vmatrix}},\tag{3.15}$$

$1\le\ell\le m, 1\le\alpha\le n$; *here t may be complex.*

Notice that W is here (and also in the case of arbitrary entire coefficients) a meromorphic function; in particular its only possible singularities are its poles which are situated in the zeros of det $Q(t)$, which is the denominator of (3.15).
In [FrJa91] an analogous (asymptotic) formula has been derived for Riccati equations with polynomial coefficients and in [FrHo99] the representation formula and the Comparison Theorem 3.4 have been used to derive in an elegant way convergence results for hermitian Riccati differential equations.
We mention that formula (3.15) nicely displays the influence of the initial value W_0 and of the Jordan structure of M on the corresponding Riccati flow; moreover (3.15) contains the whole information on the phase-portrait of (RDE); we give below only some short comments, for further details see [FrJa95].

The determinants $\begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}$ can be considered as the homogeneous coordinates of the initial value W_0 and also of the subspace $\mathrm{Im} \begin{pmatrix} I_n \\ W_0 \end{pmatrix}$ with respect to the basis(-matrix) V. The determinants

$$|x_{k_1}(\ell,\alpha),\ldots,x_{k_n}(\ell,\alpha)|(t) \text{ and } |\tilde{x}_{k_1},\ldots,\tilde{x}_{k_n}|(t)$$

are polynomials with coefficients depending only on the eigenvectors v_k of M and on the partial multiplicities of the eigenvalues of M.
If M is semisimple then these polynomials are constant and (3.15) has the simple form

$$w_{\ell\alpha}(t) = \frac{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t}|v_{k_1}(\ell,\alpha),\ldots,v_{k_n}(\ell,\alpha)| \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}{\sum\limits_{1\le k_1<\ldots<k_n\le n+m} e^{(\lambda_{k_1}+\ldots+\lambda_{k_n})t}|\tilde{v}_{k_1},\ldots,\tilde{v}_{k_n}| \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}.$$

Consequently in this case each matrix $W = (w_{\ell\alpha})_{1\le\ell\le m, 1\le\alpha\le n}$, with

$$w_{\ell\alpha}(t) = \frac{|v_{k_1}(\ell,\alpha),\ldots,v_{k_n}(\ell,\alpha)|}{|\tilde{v}_{k_1},\ldots,\tilde{v}_{k_n}|}$$

defines (for $|\tilde{v}_{k_1},\ldots,\tilde{v}_{k_n}| \neq 0$) a solution of (ARE) which is generated by the M-invariant subspace $\mathrm{Im}\ (v_{k_1},\ldots,v_{k_n}) =: \mathrm{Im} \begin{pmatrix} Q \\ P \end{pmatrix}$; moreover $W = PQ^{-1}$ and $\mathrm{Im}\ e^{Mt} \begin{pmatrix} Q \\ P \end{pmatrix} = \mathrm{Im} \begin{pmatrix} Q \\ P \end{pmatrix}$ $\forall t$.
From Theorem 3.8 it follows that any solution of (RDE) is for $t \to +\infty$ (with respect to the chordal metric) asymptotically either almost periodic or periodic or constant.
Furthermore (3.15) can be used to prove (see [FrJa91] and [FrJa95]) that $\lim_{r\to\infty} W(re^{i\phi}) =: W_\phi$ exists for almost all $\phi \in [0, 2\pi)$ and that each of these limits W_ϕ is a solution of (ARE) - since any solution of (ARE) can be obtained in this way, (3.15) yields a parametrization of the set of all solutions of (ARE); another parametrization was derived recently by Ferrante, Pavon and Pinzoni [FPP01].
Using Theorem 3.8 it can also be shown that there exists at most one anti-stable (and also at most one stable) equilibrium of (RDE) which is called dichotomic (anti-dichotomic) solution of (RDE). It is known from [Meda82], [FrJa95] that the dichotomic (anti-dichotomic) solution of (RDE) - if it exists - is the only equilibrium solution of time-invariant (RDE) having a neighborhood being negative (positive) invariant under (RDE).
For example the (exponentially) anti-dichotomic solution is defined by $W_{ad} = PQ^{-1}$, where $\begin{pmatrix} Q \\ P \end{pmatrix} := (v_{m+1},\ldots,v_{n+m})$ provided $\mathrm{Re}(\lambda_{m+1}) < \mathrm{Re}(\lambda_{n+m})$ and $|\tilde{v}_{m+1},\ldots,\tilde{v}_{n+m}| \neq 0$.
If $\lambda_m = \alpha + i\ \beta_m,\quad \lambda_{m+1} = \alpha + i\ (\beta_m + \beta)$ and $\mathrm{Re}(\lambda_{m-1}) < \mathrm{Re}(\lambda_m)(= \mathrm{Re}(\lambda_{m+1})) < \mathrm{Re}(\lambda_{m+2})$ then (RDE) has for $\beta \neq 0$ no anti-dichotomic solution; if in addition

$$|\tilde{v}_{m-1},\tilde{v}_{m+2},\ldots,\tilde{v}_{m+n}| \neq 0 \neq |\tilde{v}_{m+1},\ldots,\tilde{v}_{m+n}|,$$

then(3.15) yields that

$$w_{\ell\alpha}(t) = \frac{c_1|v_m(\ell,\alpha), v_{m+2}(\ell,\alpha),\cdots v_{m+n}(\ell,\alpha)| + c_2\ e^{i\beta t}|v_{m+1}(\ell,\alpha),\cdots v_{m+n}(\ell,\alpha)|}{c_1|\tilde{v}_m,\tilde{v}_{m+2},\cdots\tilde{v}_{m+n}| + c_2\ e^{i\beta t}|\tilde{v}^T_{m+1},\cdots\tilde{v}_{m+n}|}$$

defines a one-parameter family of periodic (meromorphic) solutions of (RDE). In general it follows from (3.15) that for the flow, defined in the autonomous case by (RDE), there are various invariant sets containing constant and/or periodic and/or almost periodic solutions.

3.5 Existence results for nonsymmetric Riccati equations

Since in most applications one is mainly interested in solutions W of (RDE) which are bounded on some given interval J, the main open problem in the theory of nonsymmetric matrix Riccati differential equations is to find sufficient conditions on the initial value $W(t_0)$ ensuring that $W(t)$ exists for all $t \in J$. Unfortunately there exists no general, unified existence theory for nonsymmetric matrix Riccati differential equations.
Subsequently we summarize the most important existence results for (RDE) obtained up to now; we mention that each of the criteria given below can only be applied in certain special cases.

First results on the global existence for the solutions of nonsymmetric (RDE) for $t \geq t_0$ (or $t \leq t_0$) were obtained by Redheffer [Redh59], [Redh60] and Reid [Reid60]. Redheffer and Volkmann [Redh75], [ReVo79] obtained - as far as we know for the first time - conditions for the existence of an invariant ball for operator differential equations, which include (RDE) as a special case. Kuiper [Kuip94] gave another proof for the existence of an invariant ball for (RDE), his result has been slightly modified in [FJS00], where the following version of the invariance result was presented:

3.9 Theorem. *If for some constants a, $\gamma > 0$, a positive definite matrix $C \in \mathbb{C}^{m\times m}$ and every $t \leq t_0$ there holds*

$$\eta^* \begin{pmatrix} -a(M_{11}(t) + M_{11}^*(t)) & M_{21}^*(t)C - aM_{12}(t) \\ CM_{21}(t) - aM_{12}^*(t) & CM_{22}(t) + M_{22}^*(t)C \end{pmatrix} \eta \geq \gamma \|\eta\|^2$$

for all $\eta \in \mathbb{C}^{n+m}$, then the set $\{W \in \mathbb{C}^{m\times n}|$ spectral radius of $W^*CW < \sqrt{a}\}$ is negative invariant under (RDE).

In [FJS00] it has been discussed how this theorem should be applied for obtaining existence results for (RDE) - in particular it has been proposed to apply Theorem 3.9 in the autonomous case only after a transformation $W \to W - W_d$ if the dichotomic solution W_d of (RDE) exists. Notice that a, γ, and C can be used here as parameters in order to achieve a maximal negative invariant ellipsoid; moreover, in the same way one can check the existence of a positive invariant ellipsoid.

In the special case of square matrices W (i.e. $n = m$) Knobloch and Pohl [KnPo97] have derived the following existence result which is based on a special maximum principle for second order linear differential equations.

3.10 Theorem. *Given the Riccati differential equation*

$$\dot{X} + XR(t)X = -A^T(t)X - XA(t) - Q(t), \tag{3.16}$$

where all (matrix-) coefficients A, R, Q are real continuous and locally integrable functions of t and $R(t) = R^T(t)$ is positive definite. Let $A_1 = A_1^T = \sqrt{R}$,

$$B = \dot{A}_1(A^{-1})^T - A_1 A^T A_1^{-1}$$

and assume that

$$[-\sqrt{R}\, Q\sqrt{R} - \frac{1}{2}(B + B^T) + \frac{1}{4}(B - B^T)^2 + BB^T](t) \geq 0 \textit{ for } t \geq 0$$

and that there exists some initial value $X = X(0)$ such that the inequality

$$[-\frac{1}{2}(B + B^T) + \sqrt{R}\, X\sqrt{R}](t) \geq 0 \tag{3.17}$$

holds for $t = 0$.
Then the solution $X(t)$ of (3.16) with this initial value exists and is bounded for $t \geq 0$ and satisfies (3.17) for $t \geq 0$.

Notice that the coefficient Q in (3.16) may be nonsymmetric. The inequality (3.17) provides a *lower bound* for $X(t)$ and the term $XR(t)X$ is essential for obtaining an *upper bound* for $t \geq 0$.

It is worthwhile to mention that already Reid (see §9 of [Reid72]) used an *elementwise* comparison method in order to prove an existence theorem for nonsymmetric Riccati differential equations; this method has been applied, modified and extended by Gewert [BJW85], [Gewe94] and also by Juang and Lee [JuLe94], [Juan92], [Juan98]; an algorithm for the iterative solution of a special class of nonsymmetric Riccati equations has been proposed in [GuLa00].
Moreover, quantitative bounds for the interval of existence of the solution of (RDE) have been derived in [JoNa93] (see also [JoPo95] and the work of Jodar et al. cited therein).

In [FJS00] the autors use another approach - a Lyapunov-type function, which is applied to (L) - in order to prove global existence for the solution of (RDE) for a big class of initial values; see [FJS00] for a proof of the following theorem and for comments on its application.

3.11 Theorem. *Let $M_{11}, M_{12}, M_{21}, M_{22}$ be piecewise continuous and locally integrable on $(-\infty, T]$. If for some matrices $C \in \mathbb{C}^{n\times n}$, with $C^* = C$, $D \in \mathbb{C}^{n\times m}$, with*

$$L = \begin{pmatrix} CM_{11} + DM_{21} & CM_{12} + M_{11}^* D + DM_{22} \\ 0 & M_{12}^* D \end{pmatrix}$$

the condition

$$L(t) + L^*(t) \leq 0,$$

holds for all $t \leq t_0 (\leq T)$ and if

$$C + DW_0 + W_0^* D^* > 0,$$

for some $W_0 \in \mathbb{C}^{m\times n}$, then the solution $W(\cdot, W_0)$ of (RDE) with $W(t_0, W_0) = W_0$ exists for all $t \leq t_0$.

3.6 Further topics

i) Nonlinear superposition formulas.
In the study of first order systems of ordinary differential equations

$$\dot{y} = f(t, y),\ y(t) \in \mathbb{R}^n, \tag{3.18}$$

one has tried to generalize the concept of linear superposition.
If the general solution of (3.18) can be written as a function

$$y(t) = F(c_1, \ldots, c_n, y^1(t), \ldots, y^p(t)) \tag{3.19}$$

of p particular solutions and n significant constants, we shall say that (3.18) allows a (nonlinear) superposition formula.
In particular it has been shown (see e.g. Winternitz et al. [AHW82], [HWA83], [SoWi85], [dORW87] and also [Egor93], [BuWi93], [PadA97] and the literature cited therein) that matrix Riccati equations do

have a nonlinear superposition formula - the number p of particular solutions needed in the superposition formula depends on the structure of the Riccati equation under consideration. These superposition formulas generalize the well known fact that if three particular solutions y_1, y_2, y_3 of the scalar Riccati differential equation

$$\dot{y} = a(t)y^2 + b(t)y + c(t) \tag{3.20}$$

are known then one can write the formula for any solution y of (3.20) by equating the cross-ratio of these four solutions to a constant c:

$$\frac{y - y_1}{y - y_2} : \frac{y_3 - y_1}{y_3 - y_2} = c.$$

For real symmetric matrix Riccati equations it is known [Levi59] that the matrix anharmonic ratio

$$R = (W_2 - W_3)^{-1}(W_3 - W_1)(W_1 - W)-1(W - W_2)$$

of four solutions $W_1,\ W_2,\ W_3,\ W$ is conjugate to a constant matrix, here $R = QUQ^{-1}$ for some known $Q\ \in$SL$(n, \mathbb{R})$. Therefore the general solution W is determined by three independent solutions and a matrix U which is determined by the initial condition satisfied by W.
As a sample of more general results we state the following theorem which was proved in [HWA83]:

3.12 Theorem *Let $m \geq n \geq 2$ and let $W_1,\ W_2,\ W_3,\ W_4,\ W_5$ be solutions of (RDE) satisfying for some initial time $t_0 \in \mathbb{R}$*

1. *det $(W_k - W_1) \neq 0,\ 2 \leq k \leq 5$ and det $(W_2 - W_3) \neq 0$.*
2. *All eigenvalues of $R_4 := (W_2 - W_3)^{-1}(W_3 - W_1)(W_1 - W_4)^{-1}(W_4 - W_2)$ are distinct.*
3. *$R_5 := (W_2 - W_3)^{-1}(W_3 - W_1)(W_1 - W_5)^{-1}(W_5 - W_2)$ (regarded as a linear map) leaves none of the irreducible invariant subspaces of R_4 invariant.*

Then there exists a matrix Q which is unique (up to an immaterial constant factor) such that the general solution W of (RDE) is represented in one of the two forms

$$W = [W_1(W_3 - W_1)^{-1}(W_2 - W_3)QU + W_2Q][(W_3 - W_1)^{-1}(W_2 - W_3)QU + Q]^{-1},$$

or

$$W = [W_2QV + W_1(W_3 - W_1)^{-1}(W_2 - W_3)Q][QV + (W_3 - W_1)^{-1}(W_2 - W_3)Q]^{-1}.$$

Here U and V determine the initial value $W(t_0)$; moreover both formulas are equivalent if det $U \neq 0 \neq$ det V, then $U = V^{-1}$.

The proof of nonlinear superposition formulas is based on the application of basic properties of Lie groups. The connection between Lie theory and matrix Riccati equations can be established by first introducing a nonlinear realization of a matrix group and then showing that systems of linear differential equations become Riccati equations in the nonlinear realization.

ii) Complex Riccati equations

Readers who are interested in complex matrix Riccati equations (with complex coefficients and complex t) are referred to the recent book of Zelikin [Zeli98] who nicely describes complex Riccati equations as flows on Cartan-Siegel homogeneity domains - moreover [Zeli98] contains several chapters on the geometric theory of matrix Riccati equations in connection with the classical calculus of variations. This geometrical aspect of the theory of matrix Riccati equations has its origins in the papers of Siegel [Sieg43], [Sieg50] and Hua Luo-Keng [HuaL59] on analytic functions of several complex variables.

iii) Further applications of Riccati equation

For convenience of the reader we give here some further references on possible applications of matrix Riccati equations.

A discussion of Riccati equations whose solutions can be treated as probabilities was began by Ambartsumyan [Amba43], for further details on this topic we refer the reader to the survey paper by Zakhar-Itkin [Zakh73], where also the importance of Riccati equations in connection with Gelfands screw method has been described.

A short survey of applications of Riccati equations in invariant embedding and in scattering theory is given in [Denm86], see also [Vasu73], [Wang67] and [Reid72]. The book of Reid [Reid72] contains also a summary of the work of Redheffer and of Reid on this topic and on the Mycielski-Pazkowski problem; in particular it is explained that the semi-group property of the solutions of matrix Riccati equations can be expressed in terms of the Redheffer $*$-product.

Thompson and Volz [ThVo75] study nonsymmetric Riccati differential equations in connection with the solution of linear quadratic control problems with constraints.

De Moor and David [dMDa92] show that the solutions of certain least squares problems satisfy a nonsymmetric algebraic Riccati equation; such algebraic equations appear also in the study of the solution of suboptimal control problems [Teod86].

We mention that general partial Riccati-type differential equations play also a central role in the study of Bäcklund transformations, see [Goll89] for further details.

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