Applicable Analysis, Vol. 81, pp. 1283–1305

Spectral Analysis for an Indefinite Singular Sturm–Liouville Problem

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Communicated by P.L. Butzer

(Received 15 February 2002; In final form 17 April 2002)

Direct and inverse problems of spectral analysis are studied for an indefinite singular boundary value problem coming from astrophysics. We establish properties of the spectrum, prove completeness and expansion theorems and investigate the inverse problem of recovering the differential equation from the given spectral characteristics.

Keywords: Boundary value problems; Singularities and turning points; Spectrum; Completeness and expansion theorems; Inverse spectral problem

AMS Subject Classifications: 34B24; 34L10; 34A55; 47E05

1. INTRODUCTION

Consider the boundary value problem $\mathcal{L}$ of the form

$$-(p(x)Y'(x))' + q(x)Y(x) = \lambda s(x)Y(x), \quad -1 < x < 1, \tag{1}$$

$$Y(x) = O(1), \quad x \to \pm 1, \tag{2}$$

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ISSN 0003-6811 print; ISSN 1563-504X online
DOI:10.1080/0003681021000035560

where $p(x) = (1 - x^2)p_0(x)$, $s(x) = (x - x_0)s_0(x)$, $x_0 \in (-1, 1)$, $p_0(x), s_0(x) \in C^2[-1, 1]$, $q(x) \in C[-1, 1]$, $p_0(x)s_0(x) \neq 0$ for all $x \in [-1, 1]$, and λ is the spectral parameter. In this article we study direct and inverse problems of spectral analysis for the boundary value problem $\mathcal{L}$. We establish properties of the spectrum, prove completeness and expansion theorems and specify spectral characteristics which uniquely determine the differential equation.

This research was stimulated by the problem of the acceleration of charged particles at relativistic shock fronts [1–3]. Highly relativistic particles occur in various sites in astrophysics, e.g. cosmic rays, solar flares and nonthermal radio sources. In the mathematical model, the central role is played by a boundary value problem for the partial differential equation

$$s_0(x)(x - x_0)\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}\left(p_0(x)(1 - x^2)\frac{\partial f}{\partial x}\right) + q(x)f, \quad -1 < x < 1, \quad -\infty < z < \infty$$

with the boundary conditions $f(x, z) = O(1)$ as $x \to \pm 1$, and additional conditions with respect to z. After separation of variables we come to the boundary value problem $\mathcal{L}$ of the form (1)–(2). Problems similar to $\mathcal{L}$ also appears in other problems of natural sciences. For example, in the particular case $p_0(x) = s_0(x) \equiv 1$, $q(x) \equiv 0$, $x_0 = 0$, the problem $\mathcal{L}$ describes electron scattering in an one-dimensional slab configuration [4,5].

In order to study the problem $\mathcal{L}$, we use here the contour integral method of integrating the resolvent along expanding contours in the complex plane of the spectral parameter. In this method an important role is played by special fundamental systems of solutions of the differential equation having desirable analytic, asymptotic and structural properties. For constructing such fundamental systems of solutions we use ideas from [6,7]. Using these fundamental systems of solutions and properties of the corresponding Stokes multipliers, we carry out an investigation of the behavior of the Green's function which allow us to study direct and inverse problems for the boundary value problem $\mathcal{L}$ even for the nonselfadjoint case. Another method was used in [8,9] where some aspects of direct problems for a particular case of $\mathcal{L}$ have been considered. We note that direct and inverse problems of spectral analysis for various classes of differential equations with singularities and turning points were studied in many works (see [10–16] and the references therein).

Some words about the structure of the article. In Section 2 we formulate the main results of the article. Fundamental systems of solutions and properties of the characteristic function and the eigenvalues are studied in Section 3. A theorem on the completeness of the system of eigen- and associated functions (e.a.f.) in adequate Banach spaces is proved in Section 4. In Section 5 an expansion theorem is obtained. Section 6 is devoted to the inverse problem of spectral analysis for $\mathcal{L}$.

2. MAIN RESULTS

Consider the function

$$r(x) := \frac{s(x)}{p(x)} = \frac{(x - x_0)s_0(x)}{(1 - x^2)p_0(x)}, \quad -1 < x < 1. \tag{3}$$

Let for definiteness $s_0(x) > 0$, $p_0(x) > 0$, and let $r(x) = R^2(x)$, where $R(x) > 0$ for $x > x_0$, and $-iR(x) > 0$ for $x < x_0$. Denote

$$R_- = \int_{-1}^{x_0} |R(s)|\, ds, \qquad R_+ = \int_{x_0}^{1} |R(s)|\, ds.$$

The following theorem, which will be proved in Section 3, concerns the existence and the asymptotic behavior of the eigenvalues of the boundary value problem $\mathcal{L}$.

THEOREM 1 *The boundary value problems* (1) *and* (2) *have a countable set of eigenvalues* $\{\lambda_n\}_{n=-\infty}^{\infty}$ (*counting with multiplicities*), *and*

$$\lambda_n = \pm\left(\left(n + \frac{1}{2}\right)\frac{\pi}{R_\pm}\right)^2 + O(1) \quad as\ n \to \pm\infty. \tag{4}$$

Example Let $p(x) = 1 - x^2$, $s(x) = x$, $q(x) = 0$, $x_0 = 0$. Then $\lambda_{-n} = -\lambda_n$, $n > 0$, and the eigenvalue $\lambda = 0$ has multiplicity 2 with the eigenfunction $Y_0(x)=1$ and with the associated function $Y_0^0(x)=-x/2$. This special case was studied in [5].

Let α be a real number and let $1 \leq p < \infty$. We consider the Banach spaces $B_{\alpha,p} := \{f(x): f(x)(1-x^2)^{-\alpha} \in L_p(-1,1)\}$ with the norm $\|f\|_{\alpha,p} = \|f(x)(1-x^2)^{-\alpha}\|_p$, where $\|.\|_p$ is the norm in the space $L_p(-1,1)$. We denote by $B^*_{\alpha,p}$ the dual space of $B_{\alpha,p}$. Clearly $B^*_{\alpha,p} = B_{-\alpha,q}$ $(p^{-1}+q^{-1}=1,\ p>1)$. Let us show that

$$B_{\alpha,p} \subseteq B_{\beta,s}, \quad 1 \leq s \leq p < \infty, \ \ \beta-\alpha < s^{-1}-p^{-1}, \tag{5}$$

(here the symbol $\subseteq$ denotes dense embedding [17, p.9]).

Indeed, for $\alpha \geq \beta$, $s \leq p$, we have $B_{\alpha,p} \subseteq B_{\beta,p}$, $B_{\beta,p} \subseteq B_{\beta,s}$ and, consequently, (5) is obvious. Assume now that $\alpha < \beta$, $s < p$. We consider the function $f(x) \in B_{\alpha,p}$. Let $r = p/s$, $r' = p/(p-s)$; then $r^{-1}+(r')^{-1}=1$. Since $\beta-\alpha<s^{-1}-p^{-1}$, we have $(\alpha-\beta)sr'>-1$. Applying Hölder's inequality, we obtain $\|f(x)(1-x^2)^{-\beta}\|_s \leq \|f(x)(1-x^2)^{-\alpha}\|_{sr}\times \|(1-x^2)^{\alpha-\beta}\|_{sr'}$, and consequently $\|f\|_{\beta,s} \leq C\|f\|_{\alpha,p}$. Since $B_{\alpha,p}$ is dense in $B_{\beta,s}$, it follows that (5) holds. In particular, it follows from (5) that $B_{\alpha,p} \subseteq L_s(-1,1)$ for $1 \leq s \leq p < \infty$, $\alpha > p^{-1}-s^{-1}$; moreover $Y_n(x) \in B_{\alpha,p}$ for $p \geq 1$, $\alpha < 1/p$.

Let E be a separable Banach space. Then the system $\{e_n: n \in \mathbf{Z}\} \subset E$ is called complete in E if there does not exist any nontrivial linear functional $F \in E^*$ such that $F(e_n)=0$ for all $n \in \mathbf{Z}$ (here E^* is the dual space for E).

THEOREM 2 *The system of e.a.f.* $\{Y_n(x)\}_{n=-\infty}^{\infty}$ *of the boundary-value problem* $\mathcal{L}$ *is complete in the space* $B_{\alpha,p}$ for $p \geq 1$, $\alpha < 1/p$. *Moreover, for* $n \neq k$,

$$\int_{-1}^{1} s(x)\,Y_n(x)\,Y_k(x)\,dx = 0. \tag{6}$$

In particular, it follows from Theorem 2 that the system of e.a.f. $\{Y_n(x)\}_{n=-\infty}^{\infty}$ of the boundary-value problem $\mathcal{L}$ is complete in $L_p(-1,1)$ for $p \geq 1$. Theorem 2 will be proved in Section 4.

Denote

$$\alpha_n := \int_{-1}^{1} s(x)\,Y_n^2(x)\,dx. \tag{7}$$

It follows from Theorem 2 that $\alpha_n \neq 0$. Let us formulate an expansion theorem, which will be proved in Section 5.

THEOREM 3 *Let* $g(x),\ x \in [-1,1]$, *be an absolutely continuous function having an absolutely continuous derivative. Then*

$$g(x) = \sum_{n=-\infty}^{\infty} a_n Y_n(x), \qquad a_n := \frac{1}{\alpha_n}\int_{-1}^{1} s(x)g(x)Y_n(x)\,dx, \tag{8}$$

and the series converges uniformly on $[-1,1]$.

In Section 5 we also provide a description of the class of functions g, for which the spectral expansion on the whole interval $[-1,1]$ contains only the e.a.f.'s related to the eigenvalues with negative (nonnegative) real parts. We note that the problem of removing a half of the spectral modes from the spectral expansion is important in the above-mentioned astrophysical problems (see [1–3,18]).

We will call $\{\lambda_n, \alpha_n\}_{n\in\mathbf{Z}}$ the spectral data of $\mathcal{L}$. In Section 6 the inverse spectral problem is considered and the following global uniqueness theorem is proved.

THEOREM 4 *The specification of the spectral data* $\{\lambda_n, \alpha_n\}_{n\in\mathbf{Z}}$ *uniquely determines* $q(x),\ -1 < x < 1$, *provided that* $s(x)$ *and* $p(x)$ *are known a priori.*

Consider the particular case when $x_0 = 0, q(-x) = q(x), p(-x) = p(x)$, $s(-x) = -s(x)$. For this symmetrical case the following global uniqueness theorem is valid.

THEOREM 5 *If* $x_0 = 0$, $q(-x) = q(x)$, $p(-x) = p(x)$, $s(-x) = -s(x)$, *then the specification of the spectrum* $\{\lambda_n\}_{n\in\mathbf{Z}}$ *uniquely determines* $q(x)$, $-1 < x < 1$, *provided that* $s(x)$ *and* $p(x)$ *are known a priori.*

We note that for the case $p(x) = 1 - x^2$, $s(x) = x$, a local version of Theorem 5 in a neighborhood of the zero potential was proved in [19].

3. PROPERTIES OF THE SPECTRUM

We transform (1)–(2) by means of the replacement

$$z(x) = \sqrt{p(x)}\,Y(x) \tag{9}$$

to the boundary value problem $\mathcal{L}_1$ of the form

$$-z''(x) + \chi(x)z(x) = \lambda r(x)z(x), \quad -1 < x < 1, \tag{10}$$

$$z(x) = O(\sqrt{1 \pm x}) \quad \text{as } x \to \mp 1, \tag{11}$$

where $r(x)$ is defined by (3), and

$$\chi(x) = h'(x) + h^2(x) + h_1(x) = -\frac{1}{(1-x^2)^2} - \frac{xh_0(x)}{1-x^2} + \frac{q(x)}{(1-x^2)p_0(x)} + h_0'(x) + h_0^2(x),$$

$$h(x) := \frac{p'(x)}{2p(x)}, \quad h_0(x) := \frac{p_0'(x)}{2p_0(x)}, \quad h_1(x) := \frac{q(x)}{p(x)}.$$

Clearly, the spectrum of $\mathcal{L}$ coincides with the spectrum of $\mathcal{L}_1$. Indeed, if λ^* is an eigenvalue of $\mathcal{L}$ and $Y^*(x)$ is a nontrivial solution of (1)–(2) with $\lambda = \lambda^*$, then $z^*(x) := \sqrt{p(x)}Y^*(x)$ is a nontrivial solution of (10)–(11) with the same $\lambda = \lambda^*$, i.e. λ^* is an eigenvalue of $\mathcal{L}_1$. And vice versa, if λ^* is an eigenvalue of $\mathcal{L}_1$, then λ^* is an eigenvalue $\mathcal{L}$. It is more convenient for us to deal with the problem $\mathcal{L}_1$ instead of $\mathcal{L}$. Theorems 1–5 for the problem $\mathcal{L}$ will follow from the corresponding facts for the problem $\mathcal{L}_1$.

Let $\lambda = \rho^2$, and let for definiteness $\Re\rho \geq 0$, i.e. $\rho \in \overline{S_0 \cup S_{-1}}$, where

$$S_j = \left\{\rho : \ \arg(\rho) \in \left(\frac{\pi j}{2}, \frac{\pi(j+1)}{2}\right)\right\}, \quad j = -1, 0.$$

Denote

$$\xi_{-1} = \int_{-1}^{x} |R(s)|\,ds, \quad \xi_1 = \int_{x}^{1} |R(s)|\,ds, \quad \xi_0 = \int_{x_0}^{x} |R(s)|\,ds,$$

$$J_j = \{x : \ |\rho\xi_j| \leq 1\}, \quad J = J_{-1} \cup J_0 \cup J_1, \quad I_{-1} = (-1, x_0) \setminus J,$$
$$I_1 = (x_0, 1) \setminus J, \quad I = I_{-1} \cup I_1,$$

here we suppress the dependence of ξ_j and J_j, I_j, J, I on x and ρ, respectively. Fix $\varepsilon > 0$ and consider the intervals $\theta_{-1,\varepsilon} := (-1, x_0 - \varepsilon]$,

$\theta_{0,\varepsilon}:=[-1+\varepsilon,1-\varepsilon]$, $\theta_{1,\varepsilon}:=[x_0+\varepsilon,1)$, $\theta_\varepsilon=\theta_{0,\varepsilon}\setminus[x_0-\varepsilon,x_0+\varepsilon]$. Let $[1]=1+O(\rho^{-1})$, $[1]_j=1+O((\rho\xi_j)^{-1})$ for $|\rho\xi_j|\geq 1, x\in\theta_{j,\varepsilon}$ (i.e. $f(x,\rho)=[1]_j$ means that $|f(x,\rho)-1|\leq C_\varepsilon|\rho\xi_j|^{-1}$ for $|\rho\xi_j|\geq 1, x\in\theta_{j,\varepsilon}$. Denote $[\tilde{1}]=[1]_j$ for $|\rho\xi_j|\geq 1, x\in\theta_{j,\varepsilon}$.

Lemma 1 *There exists a unique solution* $u(x,\lambda)$, $x\in(-1,1)$ *of Eq.* (10) *such that* (i) *for each fixed* λ, *the functions* $u^{(m)}(x,\lambda)$, $m=0,1$ *are absolutely continuous with respect to* x; (ii) *for each fixed* $x\in(-1,1)$ *the functions* $u^{(m)}(x,\lambda)$, $m=0,1$ *is entire in* λ; (iii) for $\rho\in\overline{S_0\cup S_{-1}}$, $m=0,1$, *the following estimates hold*

$$|u^{(m)}(x,\lambda)|\leq C(1+x)^{1/2-m},\quad x\in J_{-1}, \tag{12}$$

$$u^{(m)}(x,\lambda)=(\rho|R(x)|)^{m-1/2}\Big(\exp(\rho\xi_{-1})[\tilde{1}]+i(-1)^{j+m}\exp(-\rho\xi_{-1})[\tilde{1}]\Big),\\ \rho\in\overline{S_j},\quad x\in I_{-1}, \tag{13}$$

$$|u^{(m)}(x,\lambda)|\leq C|\rho|^{-1/3}\exp(|\Re\rho|R_-),\quad x\in J_0, \tag{14}$$

$$u^{(m)}(x,\lambda)=\big(\rho|R(x)|\big)^{m-1/2}i^m\exp(i\pi/4)(\exp(\rho R_-)[1]+\exp(-\rho R_-)[1]\big)\\ \times\big((-1)^m\exp(-i\rho\xi_0)[\tilde{1}]-i\exp(i\rho\xi_0)[\tilde{1}]),\quad x\in I_{+1}, \tag{15}$$

$$|u^{(m)}(x,\lambda)|\leq C(1-x)^{1/2-m}|\ln|\rho\xi_1/2||\exp(|\Im\rho|R_+)\exp(|\Re\rho|R_-),\\ x\in J_1. \tag{16}$$

Lemma 2 *There exists a unique solution* $v(x,\lambda)$, $x\in(-1,1)$ *of Eq.* (10) *such that* (i) *for each fixed* λ, *the functions* $v^{(m)}(x,\lambda)$, $m=0,1$ *are absolutely continuous with respect to* x; (ii) *for each fixed* $x\in(-1,1)$ *the functions* $v^{(m)}(x,\lambda)$, $m=0,1$ *is entire in* λ; (iii) *for* $\rho\in\overline{S_0\cup S_{-1}}$, $m=0,1$, *the following estimates hold*

$$|v^{(m)}(x,\lambda)|\leq C(1+x)^{1/2-m}|\ln|\rho\xi_{-1}/2||\exp(|\Im\rho|R_+)\exp(|\Re\rho|R_-),\\ x\in J_{-1}, \tag{17}$$

$$v^{(m)}(x,\lambda) = (\rho|R(x)|)^{m-1/2}\exp(i\pi/4)(\exp(-iR_+)[1] + \exp(i\rho R_+)[1]) \\ \times \Big((-1)^m \exp(-\rho\xi_0)[\tilde{1}] + i(-1)^j \exp(\rho\xi_0)[\tilde{1}]\Big), \\ \rho \in \overline{S_j}, \quad x \in I_{-1}, \tag{18}$$

$$|v^{(m)}(x,\lambda)| \le C|\rho|^{-1/3}\exp(|\Im\rho|R_+), \quad x \in J_0, \tag{19}$$

$$v^{(m)}(x,\lambda) = (\rho|R(x)|)^{m-1/2} i^m \Big((-1)^m \exp(i\rho\xi_1)[\tilde{1}] + i\exp(-i\rho\xi_1)[\tilde{1}]\Big), \\ x \in I_{+1}, \tag{20}$$

$$|v^{(m)}(x,\lambda)| \le C(1-x)^{1/2-m}, \quad x \in J_1. \tag{21}$$

Proof of Lemmas 1 and 2 Let for definiteness $\rho \in \overline{S_0}$ (the arguments are similar for $\rho \in \overline{S_{-1}}$). According to [7] (see also [6]), for $x \in \theta_{0,\varepsilon}$ there exist solutions $w_j(x,\rho)$, $j = 1,2$, of Eq. (10) such that $w_j^{(m)}(x,\rho)$, $m = 0,1$, are absolutely continuous on $\theta_{0,\varepsilon}$, and furthermore, for $x \in \theta_{0,\varepsilon}\backslash J_0$, $m = 0,1$,

$$\left.\begin{aligned} w_1^{(m)}(x,\rho) &= \rho^m |R(x)|^{m-1/2}\exp(\rho\xi_0)[1]_0, \quad x < x_0, \\ w_2^{(m)}(x,\rho) &= \rho^m |R(x)|^{m-1/2}\big((-1)^m \exp(-\rho\xi_0)[1]_0 + i\exp(\rho\xi_0)[1]_0\big), \quad x < x_0, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} w_1^{(m)}(x,\rho) &= (i\rho)^m |R(x)|^{m-1/2}\exp(i\pi/4)\big((-1)^m \exp(-i\rho\xi_0)[1]_0 \\ &\quad - i\exp(i\rho\xi_0)[1]_0\big), \quad x > x_0, \\ w_2^{(m)}(x,\rho) &= (i\rho)^m |R(x)|^{m-1/2}\exp(i\pi/4)\exp(i\rho\xi_0)[1]_0, \quad x > x_0. \end{aligned}\right\} \tag{23}$$

Moreover,

$$|w_j^{(m)}(x,\rho)| \le C|\rho|^{1/6} \quad \text{for} \ \ x \in \theta_{0,\varepsilon} \cap J_0, \ \ j = 1,2, \ \ m = 0,1, \tag{24}$$

$$\det\Big[w_j^{(m)}(x,\rho)\Big]_{j=1,2}^{m=0,1} \equiv -2\rho[1]. \tag{25}$$

Analogously one can construct fundamental systems of solutions in $\theta_{-1,\varepsilon}$ and $\theta_{1,\varepsilon}$. For $x \in \theta_{-1,\varepsilon}$, there exist solutions $u_j(x,\lambda)$, $j = 1,2,$ of Eq. (10) such that the function $u_1^{(m)}(x,\lambda)$, $m = 0,1,$ is entire in λ, and

$$u_1^{(m)}(x,\lambda) = (\rho|R(x)|)^{m-1/2}\big(\exp(\rho\xi_{-1})[1]_{-1} + i(-1)^m \exp(-\rho\xi_{-1})[1]_{-1}\big),\\ x \in \theta_{-1,\varepsilon}\backslash J_{-1}, \tag{26}$$

$$u_2^{(m)}(x,\lambda) = (-\rho)^m |R(x)|^{m-1/2} \exp(-\rho\xi_{-1})[1]_{-1}, \quad x \in \theta_{-1,\varepsilon}\backslash J_{-1}, \tag{27}$$

$$|u_1^{(m)}(x,\lambda)| \le C(1+x)^{1/2-m}, \quad x \in \theta_{-1,\varepsilon} \cap J_{-1}, \tag{28}$$

$$|u_2^{(m)}(x,\lambda)| \le C(1+x)^{1/2-m}|\rho|^{1/2}|\ln(\rho\xi_{-1}/2)|, \quad x \in \theta_{-1,\varepsilon} \cap J_{-1}, \tag{29}$$

$$\det\Big[u_j^{(m)}(x,\lambda)\Big]_{j=1,2}^{m=0,1} \equiv -2\rho^{1/2}[1]. \tag{30}$$

For $x \in \theta_{1,\varepsilon}$, there exist solutions $v_j(x,\lambda)$, $j = 1,2,$ of Eq. (10) such that the function $v_1^{(m)}(x,\lambda)$, $m = 0,1,$ is entire in λ, and

$$v_1^{(m)}(x,\lambda) = (\rho|R(x)|)^{m-1/2} i^m \big((-1)^m \exp(i\rho\xi_1)[1]_1 + i\exp(-i\rho\xi_1)[1]_1\big),\\ x \in \theta_{1,\varepsilon}\backslash J_1, \tag{31}$$

$$v_2^{(m)}(x,\lambda) = (-i\rho)^m |R(x)|^{m-1/2} \exp(i\rho\xi_1)[1]_1, \quad x \in \theta_{1,\varepsilon}\backslash J_1, \tag{32}$$

$$|v_1^{(m)}(x,\lambda)| \le C(1-x)^{1/2-m}, \quad x \in \theta_{1,\varepsilon} \cap J_1, \tag{33}$$

$$|v_2^{(m)}(x,\lambda)| \le C(1-x)^{1/2-m}|\rho|^{1/2}|\ln(\rho\xi_1/2)|, \quad x \in \theta_{1,\varepsilon} \cap J_1, \tag{34}$$

$$\det\Big[v_j^{(m)}(x,\lambda)\Big]_{j=1,2}^{m=0,1} \equiv 2\rho^{1/2}[1]. \tag{35}$$

We extend $u_j(x,\lambda)$, $v_j(x,\lambda)$ to the whole interval $(-1,1)$ as smooth solutions of (10) and put $u(x,\lambda) := u_1(x,\lambda)$, $v(x,\lambda) := v_1(x,\lambda)$. Using the fundamental system of solutions $\{w_j(x,\rho)\}$ one can write

$$u(x,\lambda) = a_1(\rho)w_1(x,\rho) + a_2(\rho)w_2(x,\rho). \tag{36}$$

In view of (25), according to Cramer's rule one gets

$$a_j(\rho) = (-1)^j(2\rho)^{-1}[1]\det\Big[u^{(m)}(x,\lambda), w_{3-j}^{(m)}(x,\rho)\Big]_{m=0,1}, \quad j=1,2, \tag{37}$$

and the determinant in (37) does not depend on x. Fix $x_* \in (-1, x_0)$. Substituting (22) and (26) into (37) we calculate

$$a_1(\rho) = \rho^{-1/2}(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]), \tag{38}$$

$$a_2(\rho)=\rho^{-1/2}\left(i\exp(-\rho R_-)[1]+O(\rho^{-1})\exp(\rho R_-)\exp\left(-2\rho\int_{x_*}^{x_0}|R(s)|ds\right)\right). \tag{39}$$

Let $x \in \theta_{0,\varepsilon}\backslash J_0$. Substituting (22), (23), (38), and (39), into (36) we obtain (13) and (15) for $x \in \theta_{0,\varepsilon}\backslash J_0$. Together with (26) this yields (13) for all $x \in I_{-1}$. In order to extend (15) to the whole interval I_1, we can use the fundamental system of solutions $\{v_j(x,\lambda)\}$:

$$u(x,\lambda) = b_1(\lambda)v_1(x,\lambda) + b_2(\lambda)v_2(x,\lambda). \tag{40}$$

According to Cramer's rule and (35),

$$b_j(\lambda) = (-1)^j(2\rho^{1/2})^{-1}[1]\det\Big[v_{3-j}^{(m)}(x,\lambda), u^{(m)}(x,\lambda)\Big]_{m=0,1}, \quad j=1,2, \tag{41}$$

and the determinant in (41) does not depend on x. Fix $x_* \in (x_0, 1)$. It follows from (15), (31), (32) and (41) that

$$b_1(\lambda) = -\exp(i\pi/4)\exp(i\rho R_+)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]), \tag{42}$$

$$\begin{aligned} b_2(\lambda) = \rho^{-1/2}&\exp(i\pi/4)(\exp(i\rho R_+)[1] + \exp(-i\rho R_+)[1]) \\ &\times(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]). \end{aligned} \tag{43}$$

Substituting (31), (32), (42), and (43), into (40) we arrive at (15) for $x \in \theta_{1,\varepsilon} \backslash J_1$. Since (15) is also valid for $x \in \theta_{0,\varepsilon} \backslash J_0$, it follows that (15) holds for the whole interval $x \in I_1$. Furthermore, the relations (24), (36), (38), and (39) imply (14). Estimate (16) follows from (33), (34), (40), (42), and (43), and estimate (12) follows from (28). Similarly, using (22)–(35), one can get the relations (17)–(21). Hence Lemmas 1 and 2 are proved. ■

Denote

$$\Delta(\lambda) = \det[u^{(m)}(x, \lambda), v^{(m)}(x, \lambda)]_{m=0,1}. \tag{44}$$

By virtue of Liouville's theorem, the Wronskian determinant in (44) does not depend on x. The function $\Delta(\lambda)$ is called the characteristic function of the boundary value problem $\mathcal{L}_1$. It follows from (44) and Lemmas 1 and 2 that the function $\Delta(\lambda)$ is entire in λ, and the zeros of $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem $\mathcal{L}_1$, and consequently, coincide with the eigenvalues of the boundary value problem $\mathcal{L}$.

Indeed, if λ^* is a zero of $\Delta(\lambda)$, then $v(x, \lambda^*) = \beta^* u(x, \lambda^*)$. Hence, by virtue of (12) and (21), the functions $u(x, \lambda^*)$ and $v(x, \lambda^*)$ are eigenfunctions, and λ^* is an eigenvalue of $\mathcal{L}_1$. Conversly, if λ^* is an eigenvalue with an eigenfunction $u^*(x)$, then $u^*(x) = \beta_1^* u(x, \lambda^*)$ and $u^*(x) = \beta_2^* v(x, \lambda^*)$, and consequently, $\Delta(\lambda^*) = 0$.

Fix $x \in (x_0, 1)$. Then $x \in I_1$ for sufficiently large $|\rho|$. Substituting (15) and (20) into (44) we calculate

$$\begin{aligned}\Delta(\lambda) = &-2\exp(i\pi/4)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]) \\ &\times (\exp(-i\rho R_+)[1] + \exp(i\rho R_+)[1]), \\ &\rho \in \overline{S_0 \cup S_{-1}}, \quad |\rho| \to \infty.\end{aligned} \tag{45}$$

Using Rouché's theorem [20, p. 125] we get that the function $\Delta(\lambda)$ has a countable set of zeros $\lambda_n = \rho_n^2$, $n \in \mathbf{Z}$ such that

$$\rho_n = \left(n + \frac{1}{2}\right)\frac{\pi}{R_+} + O\left(\frac{1}{n}\right), \quad n \geq 0, \ \ n \to +\infty,$$
$$\rho_n = \left(n + \frac{1}{2}\right)\frac{\pi i}{R_-} + O\left(\frac{1}{n}\right), \quad n \leq -1, \ \ n \to -\infty,$$

and consequently, Theorem 1 is proved. We note that for sufficiently large $|n|$, all zeros of $\Delta(\lambda)$ are simple.

Fix $\delta > 0$, and denote

$$G_\delta := \left\{\rho \in \overline{S_0 \cup S_{-1}} : \ |\rho - \rho_n| \geq \delta, \ n \in \mathbf{Z}\right\}.$$

It follows from (45) that

$$|\Delta(\lambda)| \geq C \exp(|\Im\rho| R_+) \exp(|\Re\rho| R_-), \quad \rho \in G_\delta, \tag{46}$$

where C depends on δ. We note that the functions $u(x, \lambda_n)$ and $v(x, \lambda_n)$ are eigenfunctions of $\mathcal{L}_1$, and

$$v(x, \lambda_n) = \beta_{n0} u(x, \lambda_n), \quad \beta_{n0} \neq 0.$$

Moreover, if κ_n is the multiplicity of the zero λ_n of $\Delta(\lambda)$ ($\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+\kappa_n - 1}$), then we infer from the general theory of associated functions for multiple eigenvalues that the e.a.f. of $\mathcal{L}_1$ have the form (see [23])

$$u_{n+j}(x) = \frac{\partial^j}{\partial\lambda^j} u(x, \lambda)_{|\lambda=\lambda_j}, \quad v_{n+j}(x) = \frac{\partial^j}{\partial\lambda^j} v(x, \lambda)_{|\lambda=\lambda_j}, \ \ j = \overline{0, \kappa_n - 1},$$

and

$$v_{n+j}(x) = \sum_{k=0}^{j} \binom{j}{k} \beta_{n,j-k} u_{n+k}(x).$$

4. COMPLETENESS THEOREM

In this section we prove the completeness of the system of e.a.f. of the boundary-value problem $\mathcal{L}_1$ in the Banach spaces $B_{\alpha,p}$. As a consequence this will give us the proof of Theorem 2 for the boundary value problem $\mathcal{L}$.

THEOREM 6 *The system of e.a.f.* $\{u_n(x)\}_{n=-\infty}^{\infty}$ *of the boundary-value problem* $\mathcal{L}_1$ *is complete in the space* $B_{\alpha,p}$ *for* $p \geq 1, \ \alpha < 1/2 + 1/p$.

Proof Let a function $f(t),\ t \in (-1, 1)$ be such that

$$f(t)\sqrt{1-t^2} \in L(-1,1), \qquad \int_{-1}^{1} f(t)u_n(t)\,dt = 0, \quad n \in \mathbf{Z}. \tag{47}$$

Then we consider the function

$$z(x,\lambda) = \frac{1}{\Delta(\lambda)}\left(v(x,\lambda)\int_{-1}^{x} f(t)u(t,\lambda)\,dt + u(x,\lambda)\int_{x}^{1} f(t)v(t,\lambda)\,dt\right). \tag{48}$$

It is easy to verify by differention that $z(x, \lambda)$ satisfies the differential equation

$$-z'' + \chi(x)z - \lambda r(x)z + f(x) = 0. \tag{49}$$

Fix $\delta > 0$ and $\varepsilon > 0$. Let us show that for $\rho \in G_\delta,\ x \in \theta_\varepsilon$,

$$|z(x,\lambda)| \le \frac{C}{|\rho|^{1/2}}\int_{-1}^{1} |f(t)|\sqrt{1-t^2}\,dt. \tag{50}$$

For definiteness let $x < x_0$ (for $x > x_0$ the arguments are similar). Then $x \in I_{-1}$ for sufficiently large $|\rho|$. It follows from (13), (18), and (46) that for $x \in I_{-1},\ \rho \in G_\delta$,

$$\left.\begin{aligned} |(\Delta(\lambda))^{-1}u(x,\lambda)| &\le C(\rho|R(x)|)^{-1/2}\exp(-|\Im\rho|R_+)\exp(|\Re\rho|\xi_0),\\ |(\Delta(\lambda))^{-1}v(x,\lambda)| &\le C(\rho|R(x)|)^{-1/2}\exp(-|\Re\rho|\xi_-). \end{aligned}\right\} \tag{51}$$

Furthermore, according to (3),

$$|R(t)|^{-1} \le C\left|\frac{1-t^2}{t-x_0}\right|^{1/2}. \tag{52}$$

We divide the integrals in (48) into parts corresponding to the intervals J_{-1}, I_{-1}, J_0, I_1, and J_1. Using (51), (52) and Lemmas 1 and 2, we calculate

$$|z(x,\lambda)| \le \frac{C}{|\rho|^{1/2}}\Bigg(\int_{J_{-1}} |f(t)|\sqrt{1+t}\,dt + \frac{1}{|\rho|^{1/2}}\int_{I_{-1}} |f(t)|\cdot\left|\frac{1+t}{t-x_0}\right|^{1/4} dt + \frac{1}{|\rho|^{1/3}}\int_{J_0} |f(t)|\,dt + \frac{1}{|\rho|^{1/2}}\int_{I_1} |f(t)|\cdot\left|\frac{1-t}{t-x_0}\right|^{1/4} dt + \int_{J_1} |f(t)|\sqrt{1-t}\,dt\Bigg), \quad \rho\in G_\delta. \tag{53}$$

Consider the integral over I_{-1}. For $t\in I_{-1}$ we have

$$|t-x_0|^{3/2}|\rho| \ge C, \qquad |1+t|^{1/2}|\rho| \ge C.$$

Take $\gamma = (-1+x_0)/2$. Then

$$|t-x_0|^{-1/4} \le C|\rho|^{1/6}, \qquad |1+t|^{-1/4} \le C \quad \text{for} \quad t\in I_{-1},\ \ t\ge\gamma,$$
$$|t-x_0|^{-1/4} \le C, \qquad |1+t|^{-1/4} \le C|\rho|^{1/2} \quad \text{for} \quad t\in I_{-1},\ \ t\le\gamma,$$

and consequently,

$$\frac{1}{|\rho|^{1/2}}\int_{I_{-1}} |f(t)|\cdot\left|\frac{1+t}{t-x_0}\right|^{1/4} dt \le C\int_{I_{-1}} |f(t)|\sqrt{1+t}\,dt. \tag{54}$$

Similarly,

$$\frac{1}{|\rho|^{1/2}}\int_{I_1} |f(t)|\cdot\left|\frac{1-t}{t-x_0}\right|^{1/4} dt \le C\int_{I_1} |f(t)|\sqrt{1-t}\,dt. \tag{55}$$

Substituting (54) and (55) into (53), we arrive at (50).

On the other hand, it follows from (47) and (48) that for each fixed $x\in(-1,1)$, the function $z(x,\lambda)$ is entire in λ. Indeed, since the functions $\Delta(\lambda)$, $u(x,\lambda)$, and $v(x,\lambda)$ are entire in λ, it follows from (48) that for each fixed $x\in(-1,1)$, the function $z(x,\lambda)$ is meromorphic in λ with poles at the points $\lambda = \lambda_n$. If for a certain n, λ_n is a simple zero of $\Delta(\lambda)$, it follows from (48) that

$$\operatorname*{Res}_{\lambda=\lambda_n} z(x,\lambda) = \frac{1}{\dot{\Delta}(\lambda_n)}\left(v(x,\lambda_n)\int_{-1}^{x} f(t)u(t,\lambda_n)\,dt + u(x,\lambda_n)\int_{x}^{1} f(t)v(t,\lambda_n)\,dt\right),$$

where $\dot{\Delta}(\lambda) := (d/d\lambda)\Delta(\lambda)$. Since $v(x,\lambda_n) = \beta_{n0}u(x,\lambda_n)$, we obtain

$$\operatorname*{Res}_{\lambda=\lambda_n} z(x,\lambda) = \frac{\beta_{n0}}{\dot{\Delta}(\lambda_n)}u(x,\lambda_n)\int_{-1}^{1} f(t)u(t,\lambda_n)\,dt.$$

By virtue of (47), this yields $\operatorname{Res}_{\lambda=\lambda_n} z(x,\lambda) = 0$, hence the function $z(x,\lambda)$ has a removable singularity at the point $\lambda = \lambda_n$. The general case, when λ_n is a zero of $\Delta(\lambda)$ of an arbitrary multiplicity, can be treated by the same way with the help of formulas for e.a.f.'s at the end of Section 3. Thus, for each fixed $x \in (-1,1)$, the function $z(x,\lambda)$ is entire in λ. Together with (50) and Liouville's theorem [20, p.77] this yields $z(x,\lambda) \equiv 0$ for $x \in \theta_\varepsilon$. Since ε is arbitrary, we get $z(x,\lambda) \equiv 0$ for $x \in (-1,1)$. Then, by virtue of (49), $f(x) = 0$ a.e. on $(-1,1)$.

Thus, we have proved that for each $p \geq 1$, the system of e.a.f. $\{u_n(x)\}_{n=-\infty}^{\infty}$ of the boundary-value problem $\mathcal{L}_1$ is complete in the space $B_{1/2,p}$. Fix $p \geq 1$ and take $\alpha < 1/2 + 1/p$. Then $\alpha - 1/2 < 1/p - 1/s$ for sufficiently large s. Then, according to (5), $B_{1/2,s} \subseteq B_{\alpha,p}$. Therefore, the system of e.a.f. $\{u_n(x)\}_{n=-\infty}^{\infty}$ of the boundary-value problem $\mathcal{L}_1$ is complete in the space $B_{\alpha,p}$ for $p \geq 1, \alpha < 1/2 + 1/p$. Theorem 6 is proved. ■

By the well-known method (see, for example, [21, Section 3]) one can obtain the relation

$$\int_{-1}^{1} r(x)u_n(x)u_k(x)\,dx = 0 \quad \text{for } n \neq k. \tag{56}$$

In view of (9), the system $\{Y_n(x)\}_{n=-\infty}^{\infty}$ of e.a.f. of the boundary value problem $\mathcal{L}$ has the form

$$Y_n(x) = \frac{1}{\sqrt{p(x)}}u_n(x). \tag{57}$$

According to (3) and (57), relation (56) is equivalent to (6). Thus, Theorem 2 follows from Theorem 6 and (56).

5. EXPANSION THEOREM

In this section we prove a theorem on the expansion in a uniform convergent series with respect to the e.a.f. of the boundary value problem $\mathcal{L}_1$. As a consequence this will give us the proof of Theorem 3 for the boundary value problem $\mathcal{L}$.

Let the numbers α_n be defined by (7). Then, by virtue of (3) and (57),

$$\alpha_n = \int_{-1}^{1} r(x)u_n^2(x)\,dx. \tag{58}$$

It follows from (56) and Theorem 6 that $\alpha_n \neq 0$ for all n.

By virtue of (4), in the λ-plane there exist closed contours $\Gamma_N,\ N \geq 1$ such that

(1) for sufficiently small $\delta > 0,\ \Gamma_N \subset G_\delta$ for all N;
(2) $d_N := \operatorname{dist}(\Gamma_N, 0) \geq C_0 N^2,\ C_0 > 0$;
(3) $\ell_N \leq CN^2$, where ℓ_N is the length of Γ_N;
(4) for sufficiently large N, the set $D_{N+1} \backslash D_N$ (where $D_N := \operatorname{int}\Gamma_N$) contains exactly one zero of $\Delta(\lambda)$.

Theorem 7 *Let a function $f(x)$ be such that the function $g(x) := (p(x))^{-1/2} f(x)$ is absolutely continuous on $[-1, 1]$ and has an absolutely continuous derivative on $[-1, 1]$. Then*

$$\lim_{N\to\infty} \sup_{-1<x<1} \frac{1}{\sqrt{p(x)}} \left| f(x) - \sum_{\lambda_n \in D_N} a_n u_n(x) \right| = 0, \tag{59}$$

where

$$a_n := \frac{1}{\alpha_n} \int_{-1}^{1} r(x) f(x) u_n(x)\,dx, \tag{60}$$

and α_n is defined by (58).

Proof We consider the function

$$Z(x,\lambda)=\frac{1}{\Delta(\lambda)}\left(v(x,\lambda)\int_{-1}^{x}f(t)r(t)u(t,\lambda)\,dt+u(x,\lambda)\int_{x}^{1}f(t)r(t)v(t,\lambda)\,dt\right). \tag{61}$$

Since the functions $u(x,\lambda)$ and $v(x,\lambda)$ satisfy Eq. (10), we rewrite (61) as follows

$$Z(x,\lambda)=\frac{1}{\lambda\Delta(\lambda)}\left(v(x,\lambda)\int_{-1}^{x}f(t)(-u''(t,\lambda)+\chi(t)u(t,\lambda))\,dt\right.$$
$$\left.+u(x,\lambda)\int_{x}^{1}f(t)\big(-v''(t,\lambda)+\chi(t)v(t,\lambda)\big)\,dt\right).$$

In the terms containing second derivatives we perform two integration by parts. In view of (44), we obtain

$$Z(x,\lambda)=\frac{f(x)}{\lambda}+\frac{1}{\lambda}(Q_0(x,\lambda)+Q_1(x,\lambda)), \tag{62}$$

where

$$Q_0(x,\lambda)=\frac{v(x,\lambda)}{\Delta(\lambda)}(f(t)u'(t,\lambda)-f'(t)u(t,\lambda))_{|t=-1}$$
$$-\frac{u(x,\lambda)}{\Delta(\lambda)}(f(t)v'(t,\lambda)-f'(t)v(t,\lambda))_{|t=1}, \tag{63}$$

$$Q_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\left(v(x,\lambda)\int_{-1}^{x}f_1(t)u(t,\lambda)\,dt+u(x,\lambda)\int_{x}^{1}f_1(t)v(t,\lambda)\,dt\right), \tag{64}$$

$$f_1(t):=-f''(t)+\chi(t)f(t).$$

Let us show that $f_1(t)\in L(-1,1)$. Indeed, the function $g_1(t):=(1+t)^{-1/2}f(t)$ is absolutely continuous for $x\in[-1,0]$ and has an absolutely continuous derivative. Then we calculate

$$\left.\begin{aligned}f(t)=(1+t)^{1/2}g_1(t),\quad f'(t)=(1+t)^{1/2}g_1'(t)+\tfrac{1}{2}(1+t)^{-1/2}g_1(t),\\ f''(t)=(1+t)^{1/2}g_1''(t)+(1+t)^{-1/2}g_1'(t)-\tfrac{1}{4}(1+t)^{-3/2}g_1(t).\end{aligned}\right\} \tag{65}$$

Since $\chi_1(t) := (1+t)\chi(t) + 1/4(1+t)^{-1} \in C[-1,0]$, it follows from (65) that

$$f_1(t) = -(1+t)^{1/2} g_1''(t) - (1+t)^{-1/2} g_1'(t) + (1+t)^{-1/2} g_1(t)\chi_1(t) \in L(-1,0).$$

Similarly, one gets $f_1(t) \in L(0,1)$, and consequently, $f_1(t) \in L(-1,1)$.
Furthermore, it follows from (65) that

$$\begin{aligned} f(t)u'(t,\lambda) - f'(t)u(t,\lambda) = \big((1+t)^{1/2} u'(t,\lambda) - \tfrac{1}{2}(1+t)^{-1/2} u(t,\lambda)\big) g_1(t) \\ - (1+t)^{1/2} u(t,\lambda) g_1'(t) \to 0 \quad \text{as } t \to 1. \end{aligned}$$

Analogously we obtain

$$f(t)v'(t,\lambda) - f'(t)v(t,\lambda) \to 0 \quad \text{as } t \to 1.$$

Hence, according to (63),

$$Q_0(x,\lambda) = 0. \tag{66}$$

Let us show that

$$\sup_{-1<x<1} \frac{1}{\sqrt{p(x)}} |Q_1(x,\lambda)| \le C|\rho|^{-1/3}, \quad \rho \in G_\delta. \tag{67}$$

For definiteness, let $x \in I_{-1}$ (for the other cases the arguments are similar). We divide the integrals in (64) into parts corresponding to the intervals J_{-1}, I_{-1}, J_0, I_1, and J_1. Using (51), (52) and Lemmas 1 and 2, we calculate

$$\begin{aligned} |Q_1(x,\lambda)| \le \frac{C}{|\rho|^{1/2}} \left|\frac{1+x}{x-x_0}\right|^{1/4} \\ \times \Bigg(\int_{J_{-1}} |f_1(t)|\sqrt{1+t}\,dt + \frac{1}{|\rho|^{1/2}} \int_{I_{-1}} |f_1(t)| \cdot \left|\frac{1+t}{t-x_0}\right|^{1/4} dt \\ + \frac{1}{|\rho|^{1/3}} \int_{J_0} |f_1(t)|\,dt + \frac{1}{|\rho|^{1/2}} \int_{I_1} |f_1(t)| \cdot \left|\frac{1-t}{t-x_0}\right|^{1/4} dt \\ + \int_{J_1} |f_1(t)|\sqrt{1-t}\,dt \Bigg), \quad \rho \in G_\delta,\ x \in I_{-1}. \end{aligned}$$

Since $\sqrt{1+t} \le C|\rho|^{-1}$ for $t \in J_{-1}$, $\sqrt{1-t} \le C|\rho|^{-1}$ for $t \in J_1$ and $|t - x_0|^{-1/4} \le C|\rho|^{1/6}$ for $t \in I$, we get

$$\frac{1}{\sqrt{p(x)}} |Q_1(x,\lambda)| \le C|\rho|^{-5/6}|x - x_0|^{-1/4}|1 + x|^{-1/4} \int_{-1}^{1} |f_1(t)|\, dt \\ \le C|\rho|^{-1/3}, \quad \rho \in G_\delta, \ x \in I_{-1},$$

and we arrive at (67).

It follows from (62), (66), and (67) that

$$\lim_{N\to\infty} \sup_{-1<x<1} \frac{1}{\sqrt{p(x)}} \left| f(x) - \frac{1}{2\pi i} \int_{\Gamma_N} Z(x,\lambda)\, d\lambda \right| = 0. \tag{68}$$

Using (68) and calculating the integral in (61) by the residue theorem we arrive at (59). Theorem 7 is proved. ■

By virtue of (3) and (57),

$$\int_{-1}^{1} r(x) f(x) u_n(x)\, dx = \int_{-1}^{1} s(x) g(x) Y_n(x)\, dx,$$

and consequently, (59) and (60) yield (8). Thus, Theorem 3 follows from Theorem 7.

Since

$$G_1(x,t,\lambda) = \begin{cases} v(x,\lambda)u(t,\lambda)/\Delta(\lambda), & -1 < t < x, \\ u(x,\lambda)v(t,\lambda)/\Delta(\lambda), & x < t < 1, \end{cases}$$

is the Green's function of the boundary value problem $\mathcal{L}_1$ it follows that

$$G(x,t,\lambda) = \frac{1}{\sqrt{p(x)p(t)}} G_1(x,t,\lambda)$$

is the Green's function of the boundary value problem $\mathcal{L}$.

Let us introduce the projection operators $P_\pm : W_1^2[-1,1] \to W_1^2[-1,1]$ via

$$(P_+ g)(x) := \sum_{\Re\lambda_n \ge 0} a_n Y_n(x), \qquad (P_- g)(x) := \sum_{\Re\lambda_n < 0} a_n Y_n(x),$$

where the coefficients a_n are defined in (8). Denote

$$\mathcal{P}_\pm := \{g \in W_1^2[-1,1]\colon P_\pm g = g\}.$$

The following theorem gives the description of the sets $\mathcal{P}_\pm$.

THEOREM 8 *$g \in \mathcal{P}_+$ ($g \in \mathcal{P}_-$) if and only if for each fixed $x \in [-1,1]$ the function*

$$Y(x,\lambda) := \int_{-1}^{1} G(x,t,\lambda)s(t)g(t)\,dt$$

is an analytic function in the half-plane $\Re\lambda < 0$ ($\Re\lambda \geq 0$).

Proof Clearly

$$Y(x,\lambda) = \frac{1}{\sqrt{p(x)}} Z(x,\lambda), \tag{69}$$

where $Z(x,\lambda) = \int_{-1}^{1} G_1(x,t,\lambda)r(t)f(t)\,dt$, and $f(x) = (p(x))^{1/2} g(x)$. Denote $\mathcal{Q}_\pm = \{f(x)\colon f(x)(p(x))^{1/2} \in \mathcal{P}_\pm\}$. Using the representation of the main part of Laurent's series of the Green's function $G_1(x,t,\lambda)$ in neighborhoods of the eigenvalues λ_n (see [22,23]) one can obtain that $f \in \mathcal{Q}_+$ ($f \in \mathcal{Q}_-$) if and only if for each fixed $x \in [-1,1]$, the function $Z(x,\lambda)/\sqrt{p(x)}$ is an analytic function in the half-plane $\Re\lambda < 0$ ($\Re\lambda \geq 0$ respectively). In view of (69) this proves the theorem. ∎

COROLLARY 1 *$g \in \mathcal{P}_\pm$ if and only if for any $R > 0$ and sufficiently small $\beta > 0$*

$$\int_{\Gamma^{\mp}_{R,\beta}} Y(x,\lambda)\,d\lambda \equiv 0,$$

where $\Gamma^{\mp}_{R,\beta} = \partial Q^{\mp}_{R,\beta}$ *and* $Q^{\mp}_{R,\beta} := K_R \backslash \Pi^{\pm}_{\beta}$, $K_R := \{\lambda\colon |\lambda| \leq R\}$, $\Pi^{+}_{\beta} := \{\lambda\colon \Re\lambda > -\beta\}$, $\Pi^{-}_{\beta} := \{\lambda\colon \Re\lambda < -\beta\}$.

6. INVERSE PROBLEM

In this section we consider the inverse problem of recovering coefficients of differential equations from the given spectral data

$\{\lambda_n, \alpha_n\}_{n\in\mathbf{Z}}$ and prove the corresponding uniqueness theorem. For this purpose we agree that together with $\mathcal{L}_1 = \mathcal{L}_1(\chi, r)$ we consider a boundary value problem $\tilde{\mathcal{L}}_1 = \mathcal{L}_1(\tilde{\chi}, r)$ of the same form but with a different coefficient $\tilde{\chi}$. If a symbol α denotes an object related to $\mathcal{L}_1$, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{\mathcal{L}}_1$.

THEOREM 9 *If* $\tilde{\lambda}_n = \lambda_n$, $\tilde{\alpha}_n = \alpha_n$ *for* $n \in \mathbf{Z}$, *then* $\tilde{\chi}(x) = \chi(x)$ *for* $x \in (-1, 1)$.

Proof Consider the functions

$$P_m(x,\lambda) = \frac{1}{\tilde{\Delta}(\lambda)}\big(u(x,\lambda)\tilde{v}^{(m)}(x,\lambda) - v(x,\lambda)\tilde{u}^{(m)}(x,\lambda)\big), \quad m = 0, 1. \quad (70)$$

Fix $\varepsilon > 0$ and $\delta > 0$. It follows from (44), (70) and Lemmas 1 and 2 that for $x \in \theta_\varepsilon$, $\rho \in G_\delta$,

$$|P_0(x,\lambda)| \le C|\rho|^{-1}, \qquad |P_1(x,\lambda)| \le C.$$

On the other hand, it is easy to verify (see, [24, Chapter 1]) that under the assumptions of Theorem 9, the functions $P_m(x,\lambda)$ are entire in λ for each fixed x. Hence, by Liouville's theorem,

$$P_0(x,\lambda) \equiv 0, \qquad P_1(x,\lambda) \equiv P_1(x).$$

In particular this yields

$$u(x,\lambda)\tilde{v}(x,\lambda) \equiv v(x,\lambda)\tilde{u}(x,\lambda),$$

and consequently,

$$P_1(x,\lambda)\tilde{u}(x,\lambda) \equiv u(x,\lambda).$$

By virtue of Lemma 1, the functions $u(x,\lambda)$ and $\tilde{u}(x,\lambda)$ have the same asymptotic behavior as $|\rho| \to \infty$. Hence, $P_1(x,\lambda) \equiv 1$ and $\tilde{u}(x,\lambda) \equiv u(x,\lambda)$. This yields $\tilde{\chi}(x) = \chi(x)$ for $x \in (-1,1)$, and Theorem 9 is proved. ■

We note that Theorem 4 obviously follows from Theorem 9. Theorem 5 is proved similarly to Theorem 9 (see also [24 Section 1.4]); so we omit the proof.

Remark Using the method of spectral mappings [24,25] one can also obtain a constructive procedure for the global solution of the inverse problem considered along with necessary and sufficient conditions for its solvability.

Acknowledgment

This research was supported in part by DAAD and by the Russian Foundation for Basic Research (Grant No. 00-01-00741).

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