# Properties of the solutions of rational matrix difference equations

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**Abstract**

We prove a comparison theorem for the solutions of a rational matrix difference equation, generalizing the Riccati difference equation, and existence and convergence results for the solutions of this equation. Moreover we present conditions ensuring that the corresponding algebraic matrix equation has a stabilizing or almost stabilizing solution.

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*Key words:* Rational matrix difference equations, generalized Riccati difference equations, comparison theorem, existence and convergence results.

## 1 Introduction

In this note we consider matrix difference equations of the form

$$X_t = A^* X_{t+1} A + Q + \Pi_1(X_{t+1}) - [S + A^* X_{t+1} B + \Pi_{12}(X_{t+1})] \\ \times [R + B^* X_{t+1} B + \Pi_2(X_{t+1})]^+ [S + A^* X_{t+1} B + \Pi_{12}(X_{t+1})]^* \tag{1.1}$$

and the corresponding algebraic equations

$$A^* X A - X + Q + \Pi_1(X) - [S + A^* X B + \Pi_{12}(X)] \\ \times [R + B^* X B + \Pi_2(X)]^+ [S + A^* X B + \Pi_{12}(X)]^* = 0, \tag{1.2}$$

where $Z^+$ is the Moore-Penrose inverse of a matrix $Z$ and $A$, $B$, $Q$, $R$ and $S$ are given matrices of sizes $n \times n$, $n \times m$, $n \times n$, $m \times m$ and $n \times m$, respectively, such that

$$T := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$$

is hermitian. We assume further that the operator $\Pi\colon \mathcal{H}^n \to \mathcal{H}^{n+m}$ with

$$\Pi(X) := \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{bmatrix}$$

is linear and positive, i.e. $X \geq 0$ implies $\Pi(X) \geq 0$. Here, $\mathcal{H}^n$ stands for the real vector space of hermitian matrices of size $n$, and by $X \geq 0$ (or $X > 0$) it is denoted that $X$ is positive semi-definite (or positive definite).

Notice that for $\Pi \equiv 0$ the equations (1.1) and (1.2) reduce to the Riccati difference equation and the discrete-time algebraic Riccati equation which both play an important role in control and filtering theory (see [1] for references).

Linearly perturbed Riccati equations appear in stochastic control theory – in particular in problems with stochastically jumping parameters – and have been studied among others in [2], [3], [4] and [5].

The continuous-time counterparts of (1.1) and (1.2) also arise in stochastic control problems and have been studied recently in [6], [7] and [8], where already existence and comparison theorems have been presented.

It is the main object of this paper to show that some of the nice properties of discrete-time Riccati equations remain valid for the more general rational matrix equations (1.1) and (1.2). In the Sections 2 – 4 we summarize all preliminary results and notations which are necessary to formulate and prove the main results of this paper. In Section 5 we prove a comparison theorem which shows that the solutions of (1.1) depend monotonically on $T$ and on a given initial or terminal value. As corollaries we derive two existence results for (1.1). The main contribution of Section 6 is Theorem 6.9 where we present sufficient conditions for the existence and the uniqueness of the stabilizing solution $X_+$ of (1.2). Furthermore we show in Section 7 that under adequate definiteness, stabilizability and detectability assumptions on the coefficients the solution of (1.1) converges for any positive semi-definite terminal value to the stabilizing solution of (1.2).

## 2 Positive operators in ordered Banach spaces

In this section we summarize for convenience of the reader some notations and preliminary results from the theory of positive operators, such as the Krein-Rutman theorem. Details on this topic can be found in [9]. For more information about properties of cones we refer the reader to [10].

**Definition 2.1** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{K}$ be a subset of $\mathcal{X}$. Then $\mathcal{K}$ is called an order cone if:*

*(i)* $\mathcal{K}$ *is closed, nonempty, and* $\mathcal{K} \neq \{0\}$*;*
*(ii) if* $x, y \in \mathcal{K}$*, then* $ax + by \in \mathcal{K}$ *for all* $a, b \geq 0$*;*
*(iii) if both* $x$ *and* $-x$ *are in* $\mathcal{K}$*, then* $x = 0$*.*

*The order cone* $\mathcal{K}$ *is called*

- *solid if* $\operatorname{int}\mathcal{K}$*, the interior of* $\mathcal{K}$*, is not empty,*
- *generating if* $\mathcal{X} = \mathcal{K} - \mathcal{K}$ *and*
- *total if* $\mathcal{X} = \overline{\mathcal{K} - \mathcal{K}}$*.*

*Note that* $\mathcal{K} - \mathcal{K} = \{\, x - y \mid x, y \in \mathcal{K} \,\}$

*Given* $x, y \in \mathcal{K}$ *we write* $x \leq y$ *if* $y - x \in \mathcal{K}$*. In particular, all the elements of* $\mathcal{K}$ *satisfy* $y \geq 0$*. By an ordered Banach space we mean a Banach space together with an order cone.*

Every solid order cone is generating (see [11]).

**Example 2.2** *Let* $\mathcal{X} = \mathcal{H}^n$ *denote the set of all hermitian matrices of size* $n$ *endowed with the inner product* $\langle A, B\rangle = \operatorname{tr} AB$ *and the Frobenius norm*

$$\|A\|_F = \langle A, A\rangle^{1/2} = \left[\sum_{i,k=1}^{n} |a_{ik}|^2\right]^{1/2}.$$

*Then*

$$\mathcal{H}^n_+ := \{\, A \in \mathcal{H}^n \mid x^*Ax \geq 0 \text{ for all } x \in \mathbb{C}^n \,\}$$

*is a solid order cone.*

**Definition 2.3** *Let* $\mathcal{X}'$ *be the dual of a real Banach space* $\mathcal{X}$*. If* $\mathcal{K}$ *is a total order cone in* $\mathcal{X}$*, then the set*

$$\mathcal{K}' = \{\, f \in \mathcal{X}' \mid f(x) \geq 0 \text{ for all } x \in \mathcal{K} \,\}$$

*is an order cone of* $\mathcal{X}'$ *which we call the dual order cone of* $\mathcal{K}$*.*

For example, if $\mathcal{K} = \mathcal{H}^n_+$, then $\mathcal{K}' = \mathcal{K}$ (see [12]).

**Definition 2.4** *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *be ordered Banach spaces. An operator* $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ *is called positive if* $x \geq 0$ *in* $\mathcal{X}$ *implies* $\mathcal{T}x \geq 0$ *in* $\mathcal{Y}$*. The operator* $\mathcal{T}$ *is called inverse-positive if the inverse* $\mathcal{T}^{-1}$ *exists and is positive.*

For any linear continuous operator $\mathcal{T}$ on $\mathcal{X}$ we denote by $\sigma(\mathcal{T})$ and $r(\mathcal{T})$ the spectrum and the spectral radius of $\mathcal{T}$, respectively.

**Theorem 2.5** *(Krein-Rutman, 1948) Let* $\mathcal{X}$ *be a real Banach space with the total order cone* $\mathcal{X}_+$*. Suppose that* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is linear, compact and positive,* *with $r(\mathcal{T}) > 0$. Then $r(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ and the dual operator $\mathcal{T}'$ with eigenvectors in $\mathcal{X}_+$ and $\mathcal{X}'_+$, respectively. If $\mathcal{X}$ is a Hilbert space then $r(T)$ is also an eigenvalue of the adjoint operator $\mathcal{T}^{adj}$.*

An important consequence of the previous theorem which was proved in [13] is

**Corollary 2.6** *(Schneider, 1965) Let $\mathcal{X}$ be a finite dimensional real Banach space with the solid order cone $\mathcal{X}_+$. Suppose that $\mathcal{S}, \mathcal{T} \colon \mathcal{X} \to \mathcal{X}$ are linear operators such that $\mathcal{S}$ is positive and either $\mathcal{T}$ is inverse-positive or $\mathcal{T}(\operatorname{int}\mathcal{X}_+) \cap \operatorname{int}\mathcal{X}_+ = \emptyset$. Then the following statements are equivalent:*

*(i)* *$\mathcal{T}$ is inverse-positive and $r(\mathcal{T}^{-1}\mathcal{S}) < 1$.*
*(ii)* *$\mathcal{T} - \mathcal{S}$ is inverse-positive.*
*(iii)* *$(\mathcal{T} - \mathcal{S})(\operatorname{int}\mathcal{X}_+) \cap \operatorname{int}\mathcal{X}_+ \neq \emptyset$.*

In particular, a positive linear operator $\mathcal{S} \colon \mathcal{X} \to \mathcal{X}$ is $d$-stable, i.e. $\sigma(\mathcal{S}) \subset \mathbb{D}$, if and only if $\mathcal{I} - \mathcal{S}$ is inverse-positive, and this holds if and only if

$$(\mathcal{I} - \mathcal{S})(\operatorname{int}\mathcal{X}_+) \cap \operatorname{int}\mathcal{X}_+ \neq \emptyset.$$

## 3 Stein equations and $d$-stability

It is well known that Stein equations play an important role in the analysis of discrete-time Riccati equations. In this section we consider the linearly perturbed algebraic Stein equation

$$X = A^*XA + \Pi_1(X) + Q, \tag{3.1}$$

where $A$ and $Q$ are given $n \times n$ matrices, $Q$ is hermitian and $\Pi_1 \colon \mathcal{H}^n \to \mathcal{H}^n$ is a positive linear operator.

Throughout this article we endow $\mathcal{H}^n$ with the scalar product $\langle A, B\rangle := \operatorname{tr} AB$ and the induced Frobenius norm $\|A\|_F := \langle A, A\rangle^{1/2}$. Notice that $\mathcal{H}^n$ is a Hilbert space with respect to $\langle\cdot,\cdot\rangle$; moreover $\mathcal{H}^n$ is ordered since the cone $\mathcal{H}^n_+$ of all positive semidefinite matrices defines an order relation on $\mathcal{H}^n$ by

$$A \geq B \iff A - B \in \mathcal{H}_n^+;$$

this order is used subsequently.

In the next lemma we recall some properties of the trace of a product of matrices.

**Lemma 3.1** *(i) For all matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, $\operatorname{tr} AB = \operatorname{tr} BA$.*

*(ii)* *Let* $A, B \in \mathcal{H}_+^n$ *with* $B > 0$. *Then,* $\operatorname{tr} AB \geq 0$, *with equality holding if and only if* $B = 0$.
*(iii)* *Let* $A, B \in \mathcal{H}_+^n$. *Then,* $\operatorname{tr} AB \geq 0$, *with equality holding if and only if* $AB = 0$.

We define the discrete-time Lyapunov operator $\mathcal{L}_A$ by

$$\mathcal{L}_A \colon \mathcal{H}^n \to \mathcal{H}^n, \;\; X \mapsto A^* X A.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ (counted with multiplicities), then the eigenvalues of $\mathcal{L}_A$, considered as an operator from $\mathbb{C}^{n \times n}$ to $\mathbb{C}^{n \times n}$, are the $n^2$ numbers $\lambda_j \overline{\lambda}_k$, $1 \leq j, k \leq n$. If all eigenvalues of $A$ lie in the open unit disc then $\mathcal{I} - \mathcal{L}_A$ is inverse-positive, and its inverse is given by

$$(\mathcal{I} - \mathcal{L}_A)^{-1}(X) = \sum_{j=0}^{\infty} (A^*)^j X A^j.$$

If $A$ has an eigenvalue $\lambda$ with $|\lambda| \geq 1$ then $(\mathcal{I} - \mathcal{L}_A)(\operatorname{int} \mathcal{H}_n^+) \cap \operatorname{int} \mathcal{H}_n^+ = \emptyset$ (see the remark to Corollary 2.6).

**Lemma 3.2** *Assume that* $\mathcal{I} - \mathcal{L}_A - \Pi_1$ *is inverse-positive. Then* (3.1) *has a unique solution* $X$ *and* $Q \geq 0$ *implies that* $X \geq 0$.

**PROOF.** If $\mathcal{I} - \mathcal{L}_A - \Pi_1$ is inverse-positive, then (3.1) has a unique solution

$$X = (\mathcal{I} - \mathcal{L}_A - \Pi_1)^{-1} Q$$

which is positive semi-definite if $Q \geq 0$.

The following theorem generalizes the discrete-time version of Lyapunovs stability theorem:

**Theorem 3.3** *The following statements are equivalent:*

*(i)* *All eigenvalues of* $A$ *lie in the open unit disk and*

$$r\left((\mathcal{I} - \mathcal{L}_A)^{-1} \Pi_1\right) < 1.$$

*(ii)* $\mathcal{I} - \mathcal{L}_A - \Pi_1$ *is inverse-positive.*
*(iii)* *There is some* $X > 0$ *such that* $(\mathcal{I} - \mathcal{L}_A - \Pi_1)(X) > 0$.
*(iv)* *If* $Q > 0$ *then* (3.1) *has a unique solution* $X > 0$.
*(v)* $\mathcal{L}_A + \Pi_1$ *is d-stable.*

*If any one of these conditions holds then* $A$ *is called* $d$-stable relative to $\Pi_1$.

**PROOF.** The equivalence of (i), (ii) and (iii) follows easily by an application of Corollary 2.6 with $\mathcal{X} := \mathcal{H}^n$, $\mathcal{K} := \mathcal{H}^n_+$, $\mathcal{S} := \Pi_1$ and $\mathcal{T} := \mathcal{I} - \mathcal{L}_A$. Since $\mathcal{L}_A + \Pi_1$ is a positive operator, the statements (ii) and (v) are also equivalent (see again the remark to Corollary 2.6). The fact, that (iv) implies (iii), is trivial. We show now that (ii) implies (iv). Therefore, let $Q > 0$ be arbitrarily. If $\mathcal{I} - \mathcal{L}_A - \Pi_1$ is inverse-positive, equation (3.1) has by Lemma 3.2 a unique solution $X \geq 0$. Hence $\tilde{Q} := Q + \Pi_1(X) > 0$. Since we have already proved that (i) and (ii) are equivalent, all eigenvalues of $A$ are contained in the open unit disk. From standard Lyapunov theory it follows now that $X$ – interpreted as a solution of $X = A^* X A + \tilde{Q}$ – is positive definite.

**Definition 3.4** *A pair $(A, B)$ of matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ is said to be $d$-*stabilizable relative to $\Pi$ *if there is a matrix $F$ such that $A + BF$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$.*

According to Theorem 3.3 $(A, B)$ is $d$-stabilizable relative to $\Pi$ if and only if the inequality

$$X - (A + BF)^* X (A + BF) - \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix} > 0$$

is fulfilled by a pair $(F, X)$ with $X > 0$.

**Definition 3.5** *A pair $(C, A)$ of matrices $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$ is said to be $d$-*detectable relative to $\Pi_1$ *if there is a matrix $L \in \mathbb{C}^{n\times m}$ such that $A + LC$ is $d$-stable relative to $\Pi_1$.*

**Lemma 3.6** *If there exist a positive semi-definite matrix $V \neq 0$ with $CV = 0$ and some $\lambda \geq 1$ such that*

$$(\mathcal{L}_A + \Pi_1)^{adj}(V) = \lambda V,$$

*then $(C, A)$ is not $d$-detectable relative to $\Pi_1$.*

**PROOF.** We assume that $(C, A)$ is $d$-detectable relative to $\Pi_1$. Then according to Theorem 3.3 there exist matrices $L \in \mathbb{C}^{n\times m}$ and $X > 0$ such that $(\mathcal{I} - \mathcal{L}_{A+LC} - \Pi_1)(X) > 0$. ¿From the hypotheses it follows now that

$$\begin{aligned} 0 \leq \langle V, (\mathcal{I} - \mathcal{L}_{A+LC} - \Pi_1)(X)\rangle &= \langle V, (\mathcal{I} - \mathcal{L}_A - \Pi_1)(X)\rangle \\ &= \langle (\mathcal{I} - \mathcal{L}_A - \Pi_1)^{adj}(V), X\rangle \\ &= (1 - \lambda)\langle V, X\rangle \leq 0. \end{aligned}$$

Hence $\langle V, X\rangle = 0$, and since $X$ is positive definite, it follows that $V = 0$, which contradicts the hypotheses.

**Lemma 3.7** *Suppose $Q \geq 0$ and (3.1) has a solution $X \geq 0$.*

*(i) If $Q > 0$ then $A$ is d-stable relative to $\Pi_1$ and we have $X > 0$.*
*(ii) If $(Q, A)$ is d-detectable relative to $\Pi_1$ then $A$ is d-stable relative to $\Pi_1$.*

**PROOF.** (i) Let us assume that $A$ is not $d$-stable relative to $\Pi_1$. Then from Theorem 3.3 it follows that $r(\mathcal{L}_A + \Pi_1) \geq 1$ and now Theorem 2.5 shows that there is some $\lambda \geq 1$ and a matrix $V \in \mathcal{H}_n^+ \setminus \{0\}$ such that $(\mathcal{L}_A + \Pi_1)^{adj}(V) = \lambda V$. So we have

$$0 \leq \langle V, Q \rangle = \langle V, (\mathcal{I} - \mathcal{L}_A - \Pi_1)(X) \rangle = (1 - \lambda) \langle V, X \rangle \leq 0.$$

Hence $\langle V, Q \rangle = 0$, and since $Q$ is positive definite, it follows that $V = 0$. Therefore, $A$ is $d$-stable relative to $\Pi_1$ and from Theorem 3.3 we obtain that the unique solution of (3.1) is positive definite.

(ii) If $A$ is not $d$-stable relative to $\Pi_1$, then it follows from the proof above that there is a matrix $V \geq 0$ such that $\langle V, Q \rangle = 0$. Since $V$ and $Q$ are both positive semi-definite we obtain $QV = 0$ which contradicts the $d$-detectability of $(Q, A)$ relative to $\Pi_1$.

## 4 The Schur complement

In this section we present some notations and preliminary results from matrix analysis.

**Definition 4.1** *The* Moore-Penrose inverse *of a $p \times q$ matrix $Z$ is the unique $q \times p$ matrix $Z^+$ satisfying the conditions*

*(i) $Z^+ Z Z^+ = Z^+$, $Z Z^+ Z = Z$,*
*(ii) $(Z^+ Z)^* = Z^+ Z$, $(Z Z^+)^* = Z Z^+$.*

If $Z$ is hermitian or positive semi-definite, then so is $Z^+$ (see [14], Proposition 12.8.3).

**Lemma 4.2 ([15], Theorem 9.17)** *Assume that $Z$ is a $m \times n$ matrix and $W$ is a $p \times n$ matrix. Then the following statements are equivalent:*

*(i)* $\operatorname{Ker} Z \subseteq \operatorname{Ker} W$.
*(ii) $W = W Z^+ Z$.*
*(iii) $W^+ = Z^+ Z W^+$.*

**Lemma 4.3 ([16], Theorem 1)** *Let $H$ be a hermitian matrix of size $n+m$ with*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix}$$

*where $L$ is $n \times n$ and $M$ is $m \times m$. Then $H$ is positive semi-definite if and only if*

$$M \geq 0, \quad L - NM^+N^* \geq 0 \quad \textit{and} \quad \operatorname{Ker} M \subseteq \operatorname{Ker} N.$$

*The matrix $H/M := L - NM^+N^*$ is called the* Schur complement *of $M$ in $H$.*

The following lemma generalizes Lemma 2.2 in [17] and provides the basis for the proof of a comparison theorem for rational matrix difference equations of the form (1.1); a proof can be found in [8].

**Lemma 4.4** *Let*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix} \quad \textit{and} \quad \tilde{H} = \begin{bmatrix} \tilde{L} & \tilde{N} \\ \tilde{N}^* & \tilde{M} \end{bmatrix}$$

*both be hermitian $(n+m) \times (n+m)$ matrices, where $L$ and $\tilde{L}$ are both $n \times n$. Define $H_d := H - \tilde{H}$, $M_d := M - \tilde{M}$ and $N_d := N - \tilde{N}$. If*

$$\operatorname{Ker} M \subseteq \operatorname{Ker} N, \quad \operatorname{Ker} \tilde{M} \subseteq \operatorname{Ker} \tilde{N} \quad \textit{and} \quad \operatorname{Ker} M_d \subseteq \operatorname{Ker} N_d,$$

*then*

$$H/M - \tilde{H}/\tilde{M} - H_d/M_d = (NM^+\tilde{M} - \tilde{N})(M_d^+ + \tilde{M}^+)(NM^+\tilde{M} - \tilde{N})^*.$$

**Corollary 4.5** *Given the hypotheses of Lemma 4.4 assume, in addition, that* $\operatorname{Ker} M \subseteq \operatorname{Ker} N$ *and* $\operatorname{Ker} \tilde{M} \subseteq \operatorname{Ker} \tilde{N}$. *If $H \geq \tilde{H}$ and $\tilde{M} \geq 0$, then the difference $H/M - \tilde{H}/\tilde{M}$ is positive semi-definite.*

## 5 Existence and comparison theorems

In this section we present a general comparison theorem which allows the comparison of solutions of two rational matrix difference equations. As corollaries we derive two existence results. To formulate the comparison theorem we define $D(\mathcal{R})$ as the set of all $X \in \mathcal{H}^n$ such that $R + B^*XB + \Pi_2(X) \geq 0$ and

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker}[S + A^*XB + \Pi_{12}(X)]$$

and the rational matrix operator $\mathcal{R}\colon D(\mathcal{R}) \to \mathcal{H}^n$ by

$$\mathcal{R}(X) = A^*XA + Q + \Pi_1(X) - [S + A^*XB + \Pi_{12}(X)] \\ \times [R + B^*XB + \Pi_2(X)]^+[S + A^*XB + \Pi_{12}(X)]^*. \qquad (5.1)$$

We have the following lemma:

**Lemma 5.1** *If* $\hat{X} \in D(\mathcal{R})$ *and* $\operatorname{Ker}\left[R + B^*\hat{X}B + \Pi_2(\hat{X})\right] \subseteq \operatorname{Ker} B$*, then*

$$X \in D(\mathcal{R}) \quad \textit{for all} \quad X \geq \hat{X}.$$

*In particular,* $\mathcal{H}_n^+$ *is contained in* $D(\mathcal{R})$ *if* $R \geq 0$ *and* $\operatorname{Ker} R \subseteq \operatorname{Ker} \left[\begin{smallmatrix} S \\ B \end{smallmatrix}\right]$.

**PROOF.** From $R + B^*\hat{X}B + \Pi_2(\hat{X}) \geq 0$ and $X \geq \hat{X}$ we infer that

$$R + B^*XB + \Pi_2(X) \geq R + B^*\hat{X}B + \Pi_2(\hat{X}) \geq 0,$$

and

$$R + B^*XB + \Pi_2(X) \geq B^*(X - \hat{X})B + \Pi_2(X - \hat{X}) \\ \geq \Pi_2(X - \hat{X}) \geq 0.$$

These inequalities imply that

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker}\left[R + B^*\hat{X}B + \Pi_2(\hat{X})\right] \qquad (5.2)$$

and

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker} \Pi_2(X - \hat{X}) \subseteq \operatorname{Ker} \Pi_{12}(X - \hat{X}),$$

where the last inclusion is obtained by applying Lemma 4.3 to the matrix $H := \Pi(X - \hat{X})$. Using (5.2), $\hat{X} \in D(\mathcal{R})$ and the assumptions fulfilled by $\operatorname{Ker}\left[R + B^*\hat{X}B + \Pi_2(\hat{X})\right]$ we get

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker}\left[S + A^*\hat{X}B + \Pi_{12}(\hat{X})\right]$$

and

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker} B \subseteq \operatorname{Ker}\left[A^*(X - \hat{X})B\right].$$

Combining the preceding relations we obtain finally

$$\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq [S + A^*XB + \Pi_{12}(X)],$$

and together with $R + B^*XB + \Pi_2(X) \geq 0$ it follows that $X \in D(\mathcal{R})$. If in particular $R \geq 0$ and $\operatorname{Ker} R \subseteq \operatorname{Ker} \left[\begin{smallmatrix} S \\ B \end{smallmatrix}\right]$, then $\hat{X} = 0$ fulfills the assumptions of the lemma. In this case $\mathcal{H}_n^+$ is contained in $D(\mathcal{R})$.

It is obvious that $\mathcal{R}(X) - X$ is the Schur complement of the so-called *dissipation matrix*

$$\Lambda(X) := \begin{bmatrix} A^*XA - X + Q + \Pi_1(X) & S + A^*XB + \Pi_{12}(X) \\ [S + A^*XB + \Pi_{12}(X)]^* & R + B^*XB + \Pi_2(X) \end{bmatrix}. \tag{5.3}$$

Consequently, by Lemma 4.3 the quadratic matrix inequality $\mathcal{R}(X) \geq X$ and the linear matrix inequality $\Lambda(X) \geq 0$ are equivalent on $D(\mathcal{R})$.

**Lemma 5.2** *If $X$ is a hermitian matrix such that*

$$\mathrm{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \mathrm{Ker}[S + A^*XB + \Pi_{12}(X)] \tag{5.4}$$

*then*

$$\mathcal{R}(X) = (A + BF)^*X(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* [T + \Pi(X)] \begin{bmatrix} I \\ F \end{bmatrix}, \tag{5.5}$$

*where*

$$F = F(X) := -[R + B^*XB + \Pi_2(X)]^+[S + A^*XB + \Pi_{12}(X)]^*. \tag{5.6}$$

**PROOF.** From Lemma 4.2 it follows that the condition (5.4) is equivalent to

$$-F^*[R + B^*XB + \Pi_2(X)] = S + A^*XB + \Pi_{12}(X).$$

So, if we rewrite $\mathcal{R}(X)$ as

$$\mathcal{R}(X) = A^*XA + Q + \Pi_1(X) - F^*[R + B^*XB + \Pi_2(X)]F$$

we obtain

$$\begin{aligned} &\mathcal{R}(X) - \begin{bmatrix} I \\ F \end{bmatrix}^* [T + \Pi(X)] \begin{bmatrix} I \\ F \end{bmatrix} \\ &\quad = A^*XA + F^*B^*XBF \\ &\qquad - \Big\{S + \Pi_{12}(X) + F^*[R + B^*XB + \Pi_2(X)]\Big\}F \\ &\qquad - F^*\Big\{[S + \Pi_{12}(X)]^* + [R + B^*XB + \Pi_2(X)]F\Big\} \\ &\quad = (A + BF)^*X(A + BF). \end{aligned}$$

**Lemma 5.3** *Let $X_1$ and $X_2$ be hermitian matrices such that*

$$\mathrm{Ker}[R + B^*X_iB + \Pi_2(X_i)] \subseteq \mathrm{Ker}[S + A^*X_iB + \Pi_{12}(X_i)], \quad i = 1, 2.$$

*For $i = 1, 2$, define*

$$F_i := F(X_i) = -[R + B^* X_i B + \Pi_2(X_i)]^+ [S + A^* X_i B + \Pi_{12}(X_i)]^*.$$

*Then the following identities hold:*

$$\begin{aligned}
\mathcal{R}(X_1) = (A + BF_2)^* X_1 (A + BF_2) & \\
- (F_2 - F_1)^* [R + B^* X_1 B + \Pi_2(X_1)](F_2 - F_1) & \\
+ \begin{bmatrix} I \\ F_2 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_2 \end{bmatrix} &
\end{aligned} \tag{5.7}$$

*and*

$$\begin{aligned}
\mathcal{R}(X_2) - \mathcal{R}(X_1) = (A + BF_2)^* (X_2 - X_1)(A + BF_2) & \\
+ (F_2 - F_1)^* [R + B^* X_1 B + \Pi_2(X_1)](F_2 - F_1) & \\
+ \begin{bmatrix} I \\ F_2 \end{bmatrix}^* \Pi(X_2 - X_1) \begin{bmatrix} I \\ F_2 \end{bmatrix}. &
\end{aligned} \tag{5.8}$$

**PROOF.** Using Lemma 5.2 we get

$$\begin{aligned}
\mathcal{R}(X_1) &= (A + BF_1)^* X_1 (A + BF_1) + \begin{bmatrix} I \\ F_1 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_1 \end{bmatrix} \\
&= (A + BF_2)^* X_1 (A + BF_2) \\
&\quad + Q + \Pi_1(X_1) + F_1^* [R + B^* X_1 B + \Pi_2(X_1)] F_1 \\
&\quad + [S + A^* X_1 B + \Pi_{12}(X_1)] F_1 - A^* X_1 B F_2 \\
&\quad + F_1^* [S + A^* X_1 B + \Pi_{12}(X_1)]^* - F_2^* B^* X_1 A \\
&\quad - F_2^* B^* X_1 B^* F_2 \\
&= (A + BF_2)^* X_1 (A + BF_2) \\
&\quad + Q + \Pi_1(X_1) - F_1^* [R + B^* X_1 B + \Pi_2(X_1)] F_1 \\
&\quad + \Big\{ S + \Pi_{12}(X_1) + F_1^* [R + B^* X_1 B + \Pi_2(X_1)] \Big\} F_2 \\
&\quad + F_2^* \Big\{ [S + \Pi_{12}(X_1)]^* + [R + B^* X_1 B + \Pi_2(X_1)] F_1 \Big\} \\
&\quad - F_2^* B^* X_1 B F_2 \\
&= (A + BF_2)^* X_1 (A + BF_2) \\
&\quad - (F_2 - F_1)^* [R + B^* X_1 B + \Pi_2(X_1)](F_2 - F_1) \\
&\quad + \begin{bmatrix} I \\ F_2 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_2 \end{bmatrix},
\end{aligned}$$

which proves (5.7). Subtracting this from (5.5) with $X := X_2$, we obtain (5.8).

To formulate the announced comparison theorem we introduce another rational matrix operator $\tilde{\mathcal{R}} \colon D(\tilde{\mathcal{R}}) \to \mathcal{H}^n$ with

$$\tilde{\mathcal{R}}(X) = A^*XA + \tilde{Q} + \Pi_1(X) - [\tilde{S} + A^*XB + \Pi_{12}(X)] \\ \times [\tilde{R} + B^*XB + \Pi_2(X)]^+ [\tilde{S} + A^*XB + \Pi_{12}(X)]^* \tag{5.9}$$

where we assume that $\tilde{Q}$ and $\tilde{R}$ are hermitian and where $D(\tilde{\mathcal{R}})$ denotes the set of all $X \in \mathcal{H}^n$ such that $\tilde{R} + B^*XB + \Pi_2(X) \geq 0$ and

$$\text{Ker}[\tilde{R} + B^*XB + \Pi_2(X)] \subseteq \text{Ker}[\tilde{S} + A^*XB + \Pi_{12}(X)].$$

For every $X \in D(\tilde{\mathcal{R}})$ we define the corresponding feedback matrix $\tilde{F}$ by

$$\tilde{F} = \tilde{F}(X) := -[\tilde{R} + B^*XB + \Pi_2(X)]^+ [\tilde{S} + A^*XB + \Pi_{12}(X)]^*.$$

With these notations we have

**Lemma 5.4** *Let $X \in D(\tilde{\mathcal{R}})$ be given. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix}, \tag{5.10}$$

*then*

$$X \in D(\mathcal{R}) \quad \textit{and} \quad \mathcal{R}(X) \geq \tilde{\mathcal{R}}(X).$$

**PROOF.** Inequality (5.10) implies, in particular, that $R \geq \tilde{R}$, and consequently

$$R + B^*XB + \Pi_2(X) \geq \tilde{R} + B^*XB + \Pi_2(X) \geq 0. \tag{5.11}$$

Furthermore, we have

$$R + B^*XB + \Pi_2(X) \geq R - \tilde{R} \geq 0.$$

¿From these two inequalities it follows that

$$\begin{aligned} \text{Ker}[R + B^*XB + \Pi_2(X)] &\subseteq \text{Ker}\left[\tilde{R} + B^*XB + \Pi_2(X)\right] \\ &\subseteq \text{Ker}\left[\tilde{S} + A^*XB + \Pi_{12}(X)\right] \end{aligned}$$

and

$$\text{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \text{Ker}\left[R - \tilde{R}\right] \subseteq \text{Ker}\left[S - \tilde{S}\right],$$

where the last inclusion follows from the fact that (5.10) also implies that $\text{Ker}\left[R - \tilde{R}\right] \subseteq \text{Ker}\left[S - \tilde{S}\right]$ (see Lemma 4.3). Combining the two relations above we get

$$\text{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \text{Ker}\left[S + A^*XB + \Pi_{12}(X)\right],$$

and together with (5.11), we obtain $X \in D(\mathcal{R})$.

If we associate the matrix

$$\tilde{\Lambda}(X) := \begin{bmatrix} A^*XA - X + \tilde{Q} + \Pi_1(X) & \tilde{S} + A^*XB + \Pi_{12}(X) \\ [\tilde{S} + A^*XB + \Pi_{12}(X)]^* & \tilde{R} + B^*XB + \Pi_2(X) \end{bmatrix}$$

with (5.9), just as $\Lambda(X)$ is associated with (5.1), it follows from (5.10) that $\Lambda(X) \geq \tilde{\Lambda}(X)$, and now an application of Corallary 4.5 yields the statement of the lemma.

Recall that a *discrete interval* is a (not necessarily finite) set of successive integers. A sequence $\{X_t\}_{t\in\mathcal{I}}$ defined on a discrete interval $\mathcal{I} \subseteq \mathbb{Z}$ is said to be a solution of $X_{t+1} = \mathcal{R}(X_t)$ if it satisfies the difference equation (1.1) and the additional condition $X_t \in D(\mathcal{R})$ for $t \in \mathcal{I}$.

**Theorem 5.5** *(Comparison theorem) Let $\mathcal{I} \subseteq \mathbb{Z}$ be some discrete interval and $t_f \in \mathcal{I}$. Assume that $\{X_t^2\}$ and $\{X_t^1\}$ are on $\mathcal{I}$ solutions of $X_t^2 = \mathcal{R}(X_{t+1}^2)$ and $X_t^1 = \tilde{\mathcal{R}}(X_{t+1}^1)$, respectively. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix},$$

*then $X_{t_f}^2 \geq X_{t_f}^1$ implies that $X_t^2 \geq X_t^1$ for all $t \in \mathcal{I}$ with $t \leq t_f$.*

**PROOF.** Define $X_t := X_t^2 - X_t^1$ and $F_t^i := F(X_t^i)$, $i = 1, 2$. According to Lemma 5.4 we have $X_t^1 \in D(\mathcal{R})$ for $t \in \mathcal{I}$, and using (5.8), we infer that $\{X_t\}_{t\in\mathcal{I}}$ is a solution of the difference equation

$$\begin{aligned} X_t &= \mathcal{R}(X_{t+1}^2) - \tilde{\mathcal{R}}(X_{t+1}^1) \\ &= \mathcal{R}(X_{t+1}^2) - \mathcal{R}(X_{t+1}^1) + \mathcal{R}(X_{t+1}^1) - \tilde{\mathcal{R}}(X_{t+1}^1) \\ &= \hat{A}_{t+1}^* X_{t+1} \hat{A}_{t+1} + \hat{Q}_{t+1} + \hat{\Pi}_{t+1}(X_{t+1}), \end{aligned}$$

where

$$\hat{A}_t = A + BF_t^2, \quad \hat{\Pi}_t(X) = \begin{bmatrix} I \\ F_t^2 \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F_t^2 \end{bmatrix}$$

and

$$\begin{aligned} \hat{Q}_t := &\mathcal{R}(X_t^1) - \tilde{\mathcal{R}}(X_t^1) \\ &+ (F_t^2 - F_t^1)^*[R + B^* X_t^1 B + \Pi_2(X_t^1)](F_t^2 - F_t^1). \end{aligned}$$

Now Lemma 5.4 implies that $\hat{Q}_t \geq 0$ for all $t \in \mathcal{I}$, therefore by induction it follows that $X_t \geq 0$ for all $t \in \mathcal{I}$ with $t \leq t_f$ which proves the theorem.

Theorem 5.5 shows that the solutions of (1.1) depend monotonically on $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$ and on the terminal value $X_f$; this generalizes the corresponding result for discrete-time Riccati equations (see [18], [19]). We mention that it is also possible to use a Fréchet derivative based approach for proving Theorem 5.5 – in the special case $\Pi \equiv 0$ this approach has been used in [20].

Subsequently we present two corollaries showing how the comparison theorem can be used to derive existence results and upper and lower bounds for the solutions of (1.1).

**Corollary 5.6** *Let $\mathcal{I} \subseteq \mathbb{Z}$ be some discrete interval and $t_f \in \mathcal{I}$. Assume that $\{X_t^\ell\}$, $\{X_t^u\}$ are on $\mathcal{I}$ solutions of the difference inequalities $X_t^\ell \leq \mathcal{R}^d(X_{t+1}^\ell)$ and $X_t^u \geq \mathcal{R}^d(X_{t+1}^u)$, respectively, with*

$$\operatorname{Ker}\left[R + B^* X_t^\ell B + \Pi_2\left(X_t^\ell\right)\right] \subseteq \operatorname{Ker} B \quad \textit{for all} \quad t \in \mathcal{I} \cap (-\infty, t_f]. \tag{5.12}$$

*Then $X_{t_f}^\ell \leq X_f \leq X_{t_f}^u$ implies that the solution $\{X_t\}$ of*

$$X_t = \mathcal{R}^d(X_{t+1}), \quad X_{t_f} = X_f, \tag{5.13}$$

*exists for all $t \in \mathcal{I}$ with $t \leq t_f$ and fulfills there the inequality*

$$X_t^\ell \leq X_t \leq X_t^u. \tag{5.14}$$

**PROOF.** The solution $\{X_t\}$ exists a priori only on a certain discrete interval $\{t^-, \ldots, t_f\}$ with some unknown $t^-$. By the hypotheses, there exists a sequence $\{Q_t^\ell\}$ of positive semi-definite matrices such that

$$X_t^\ell = \mathcal{R}^d(X_{t+1}^\ell) - Q_{t+1}^\ell.$$

We define $\tilde{Q}_t := Q - Q_t^\ell$, $\tilde{R} := R$ and $\tilde{S} := S$. Since $X_f \geq X_{t_f}^\ell$ we obtain from Theorem 5.5 (which holds also in the time-varying case – see Remark 5.8) that $X_t^\ell \leq X_t$ for $t = t^-, \ldots, t_f$. Substituting $\{Q_t^\ell\}$ by an adequate sequence $\{Q_t^u\}$ of negative semi-definite matrices the right inequality from (5.14) follows analogously. Hence, the sequence $\{X_t\}$ is bounded from below and above and therefore it follows that $\{t^-, \ldots, t_f\} \supseteq \mathcal{I} \cap (-\infty, t_f]$.

It remains to show that $X_t \in D(\mathcal{R})$ for $t \in \mathcal{I}$ with $t \leq t_f$. Using Lemma 5.1 this results immediately from (5.12) and the fact that $X_t \geq X_t^\ell$ for all $t \in \mathcal{I} \cap (-\infty, t_f]$.

**Corollary 5.7** *Assume that* $\operatorname{Ker} R \subseteq \operatorname{Ker} B$ *and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{5.15}$$

*If* $X_f \geq 0$ *then the solution* $\{X_t\}$ *of* (5.13) *exists for all* $t \in \mathbb{Z}$ *with* $t < t_f$ *and fulfills there the inequality*

$$0 \leq X_t \leq X_t^u$$

*where* $\{X_t^u\}$ *is the solution of*

$$X_t^u = A^* X_{t+1}^u A + Q + \Pi_1(X_{t+1}^u), \quad X_{t_f}^u = X_f. \tag{5.16}$$

**PROOF.** We compare the solution of (5.13) with the solutions of the difference equations (5.16) and

$$X_t^\ell = \mathcal{R}^d(X_{t+1}^\ell) + SR^+S^* - Q, \quad X_{t_f}^\ell = 0. \tag{5.17}$$

Since (5.16) is a linear difference equation the solution $\{X_t^u\}$ exists for all $t \in \mathbb{Z}$ with $t \leq t_f$. Since $X_f \geq 0$ it follows by induction that $X_t^u \geq 0$ for all $t \in \mathbb{Z}$ with $t \leq t_f$. This yields in particular that $X_t^u \geq \mathcal{R}^d(X_{t+1}^u)$ for all $t \in \mathbb{Z}$ with $t \leq t_f$.

The solution of (5.17) is the trivial solution, and from (5.15) it follows with Lemma 4.3 that it satisfies the difference inequality $X_t^\ell \leq \mathcal{R}^d(X_{t+1}^\ell)$. Since we assume that $\operatorname{Ker} R \subseteq \operatorname{Ker} B$ the assertion of the corollary results from Corollary 5.6.

**Remark 5.8** *We mention that all the results obtained in Section 5 remain valid if the coefficients of* (1.1) *depend on* $t$ *and the assumptions used are valid for all* $t$.

## 6 Stabilizing and almost stabilizing solutions

In this section we present results concerning stabilizing and almost stabilizing solutions of the algebraic matrix equation (1.2). Using an iterative procedure which can be viewed as a slight modification of the Newton-Kantorovich method applied to the equation $\mathcal{R}(X) = X$ we prove an existence theorem for an almost stabilizing solution of (1.2). For a detailed representation in the continuous-time case the reader is referred to [8].

**Definition 6.1** *Let* $\hat{X} \in D(\mathcal{R})$ *be a solution of* $\mathcal{R}(X) = X$. *If* $F = F(\hat{X})$ *denotes the corresponding feedback matrix and* $\hat{\Pi}$ *the positive linear operator* *defined by* $\hat{\Pi}(X) := \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]^* \Pi(X) \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$ *for* $X \in \mathcal{H}^n$ *then* $\hat{X}$ *is called stabilizing (resp. almost stabilizing) if* $\sigma(\mathcal{L}_{A+BF} + \hat{\Pi})$ *is contained in the open (resp. closed) left half-plane.*

The following theorem generalizes a well-known result for discrete-time algebraic Riccati equations (see [1], Theorem 13.1.1). A continuous-time version of this result was already derived in [8] (see also [6] and [21]).

**Theorem 6.2** *Assume that* $(A, B)$ *is d-stabilizable relative to* $\Pi$ *and that there exists a solution* $\hat{X}$ *of* $\mathcal{R}(X) \geq X$ *for which*

$$\operatorname{Ker}[R + B^* \hat{X} B + \Pi_2(\hat{X})] \subseteq \operatorname{Ker} B.$$

*Then there exists an almost stabilizing solution* $X_+$ *of* $\mathcal{R}(X) = X$*, and we have* $X_+ \geq X$ *for all solutions of* $\mathcal{R}(X) \geq X$ *with* $\operatorname{Ker}[R+B^*XB+\Pi_2(X)] \subseteq \operatorname{Ker} B$.

**PROOF.** By the hypotheses, there exists a hermitian matrix $\hat{X} \in D(\mathcal{R})$ with

$$\mathcal{R}(\hat{X}) = \hat{X} + Q - \hat{Q} \tag{6.1}$$

where $\hat{Q}$ is a hermitian matrix such that $\hat{Q} \leq Q$.

Since $(A, B)$ is $d$-stabilizable relative to $\Pi$, there is an $F_0$ such that $A_0 := A + BF_0$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$. Let $X_1$ be the unique solution of the linearly perturbed Stein equation

$$A_0^* X_1 A_0 - X_1 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_0 \end{bmatrix} + I = 0$$

If we define $\hat{F} := F(\hat{X})$ then as in the proof of (5.7) we get

$$\begin{aligned} \mathcal{R}(\hat{X}) = {} & A_0^* \hat{X} A_0 - (F_0 - \hat{F})^* [R + B^* \hat{X} B + \Pi_2(\hat{X})](F_0 - \hat{F}) \\ & + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(\hat{X})] \begin{bmatrix} I \\ F_0 \end{bmatrix}. \end{aligned}$$

This yields that

$$\begin{aligned} X_1 - \hat{X} &= A_0^* X_1 A_0 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_0 \end{bmatrix} + I - \mathcal{R}(\hat{X}) + Q - \hat{Q} \\ &= A_0^* (X_1 - \hat{X}) A_0 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* \Pi(X_1 - \hat{X}) \begin{bmatrix} I \\ F_0 \end{bmatrix} + Q - \hat{Q} \\ &\quad + (F_0 - \hat{F})^* [R + B^* \hat{X} B + \Pi_2(\hat{X})](F_0 - \hat{F}) + I. \end{aligned} \tag{6.2}$$

Now $R + B^*\hat{X}B + \Pi_2(\hat{X}) \geq 0$ by hypothesis and $Q \geq \hat{Q}$; hence

$$A_0^*(X_1 - \hat{X})A_0 - (X_1 - \hat{X}) + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* \Pi(X_1 - \hat{X}) \begin{bmatrix} I \\ F_0 \end{bmatrix} < 0.$$

Since $A_0$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$, part (iv) of Theorem 3.3 shows that $X_1 > \hat{X}$. Consequently, according to Lemma 5.1, we have $X_1 \in D(\mathcal{R})$.

Starting with $A_0$, $F_0$, $X_1$, induction is used to construct three sequences of matrices $\{A_i\}_{i=0}^{\infty}$, $\{F_i\}_{i=0}^{\infty}$, $\{X_i\}_{i=1}^{\infty}$, with certain properties (given below). Thus, assume that for some $m \geq 1$ we have already determined matrices $\{A_i\}_{i=0}^{m-1}$, $\{F_i\}_{i=0}^{m-1}$, $\{X_i\}_{i=1}^{m}$ with $X_i = X_i^*$,

$$X_1 > X_2 > \ldots > X_m > \hat{X},$$

$$A_i = A + BF_i, \quad i = 0, 1, \ldots, m-1,$$

where

$$F_i = -[R + B^*X_iB + \Pi_2(X_i)]^+[S + A^*X_iB + \Pi_{12}(X_i)]^*,$$

$$A_i^*X_{i+1}A_i - X_{i+1} + \begin{bmatrix} I \\ F_i \end{bmatrix}^* [T + \Pi(X_{i+1})] \begin{bmatrix} I \\ F_i \end{bmatrix} + \frac{1}{i+1} I = 0 \qquad (6.3)$$

and the matrices $A_i$ are $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]$, $i = 0, 1, \ldots, m-1$. Now define

$$F_m := -[R + B^*X_mB + \Pi_2(X_m)]^+[S + A^*X_mB + \Pi_{12}(X_m)]^*,$$

$$A_m := A + BF_m.$$

It has to be shown that $A_m$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$. Letting $X_1 := X_m$, $X_2 := X_{m-1}$ in (5.7) and applying (6.3), we get

$$\mathcal{R}(X_m) = X_m - (F_m - F_{m-1})^*[R + B^*X_mB + \Pi_2(X_m)] \times (F_m - F_{m-1}) - \frac{1}{m} I.$$

Together with (5.5) it follows that

$$A_m^*X_mA_m - X_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(X_m)] \begin{bmatrix} I \\ F_m \end{bmatrix} + \frac{1}{m} I + (F_m - F_{m-1})^*[R + B^*X_mB + \Pi_2(X_m)](F_m - F_{m-1}) = 0. \qquad (6.4)$$

Next, use (5.7) again with $X_1 := \hat{X}$, $X_2 := X_m$ and apply (6.1) to get

$$A_m^*\hat{X}A_m - \hat{X} + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(\hat{X})] \begin{bmatrix} I \\ F_m \end{bmatrix} - (Q - \hat{Q}) - (F_m - \hat{F})^*[R + B^*\hat{X}B + \Pi_2(\hat{X})](F_m - \hat{F}) = 0.$$

Subtracting this from (6.4), we obtain

$$
\begin{aligned}
&A_m^*(X_m - \hat{X})A_m - (X_m - \hat{X}) + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_m - \hat{X}) \begin{bmatrix} I \\ F_m \end{bmatrix} \\
&\quad = -(Q - \hat{Q}) - (F_m - F_{m-1})^*[R + B^* X_m B + \Pi_2(X_m)](F_m - F_{m-1}) \\
&\qquad - (F_m - \hat{F})^*[R + B^* \hat{X} B + \Pi_2(\hat{X})](F_m - \hat{F}) - \frac{1}{m} I < 0.
\end{aligned}
$$

Since $X_m > \hat{X}$ it follows from Theorem 3.3 that $A_m$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$.

We now define $X_{m+1}$ as the unique solution (necessarily hermitian) of the linearly perturbed Stein equation

$$
A_m^* X_{m+1} A_m - X_{m+1} + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(X_{m+1})] \begin{bmatrix} I \\ F_m \end{bmatrix} + \frac{1}{m+1} I = 0. \quad (6.5)
$$

As in (6.2) it is found that

$$
\begin{aligned}
&A_m^*(X_{m+1} - \hat{X})A_m - (X_{m+1} - \hat{X}) + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_{m+1} - \hat{X}) \begin{bmatrix} I \\ F_m \end{bmatrix} \\
&\quad = \hat{X} - \mathcal{R}(\hat{X}) - (F_m - \hat{F})^*[R + B^* \hat{X} B + \Pi_2(\hat{X})](F_m - \hat{F}) \\
&\qquad - \frac{1}{m+1} I < 0. \qquad (6.6)
\end{aligned}
$$

Next it will be shown that $X_m > X_{m+1}$. Subtracting (6.5) from (6.4) we get

$$
\begin{aligned}
&A_m^*(X_m - X_{m+1})A_m - (X_m - X_{m+1}) + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_m - X_{m+1}) \begin{bmatrix} I \\ F_m \end{bmatrix} \\
&\quad = -(F_m - F_{m-1})^*[R + B^* X_m B + \Pi_2(X_m)](F_m - F_{m-1}) \\
&\qquad - \frac{1}{m(m+1)} I < 0.
\end{aligned}
$$

The last two equations, together with the fact, that $A_m$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$ imply that $X_m > X_{m+1} > \hat{X}$.

We have obtained a nonincreasing sequence $\{X_i\}_{i=0}^{\infty}$ of hermitian matrices bounded below by $\hat{X}$. Hence

$$
X_+ := \lim_{i \to \infty} X_i
$$

exists and is a hermitian matrix with $X_+ \geq \hat{X}$, and $X_+ \in D(\mathcal{R})$. Passing to the limit in (6.5) when $m \to \infty$, and writing $F_+ := F(X_+)$, it is found that

$$
A_+^* X_+ A_+ - X_+ + \begin{bmatrix} I \\ F_+ \end{bmatrix}^* [T + \Pi(X_+)] \begin{bmatrix} I \\ F_+ \end{bmatrix} = 0,
$$

which, in view of Lemma 5.2, can be rewritten as $X_+ = \mathcal{R}(X_+)$.

Since $X_+$ is independent of $\hat{X}$, we have $X_+ \geq X$ for every solution $X$ of $\mathcal{R}(X) \geq X$ for which $\operatorname{Ker}[R + B^*XB + \Pi_2(X)] \subseteq \operatorname{Ker} B$.

Finally, since $A_m$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$ for all $m \geq 0$, the solution $X_+$ is almost stabilizing.

**Corollary 6.3** *Assume that* $\operatorname{Ker} R \subseteq \operatorname{Ker} B$, $(A, B)$ *is d-stabilizable relative to* $\Pi$ *and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{6.7}$$

*Then* $\mathcal{R}(X) = X$ *has an almost stabilizing solution* $X_+$*, and we have* $X_+ \geq 0$.

**PROOF.** According to Lemma 4.3 condition (6.7) implies that $\hat{X} = 0$ is a solution of $\mathcal{R}(X) \geq X$. Therefore an application of Theorem 6.2 yields the statement of the corollary.

**Corollary 6.4** *Assume that* $(A, B)$ *is d-stabilizable relative to* $\Pi$ *and that there exists a solution* $\hat{X}$ *of* $\mathcal{R}(X) > X$ *for which*

$$\operatorname{Ker}[R + B^*\hat{X}B + \Pi_2(\hat{X})] \subseteq \operatorname{Ker} B.$$

*Then there exists a stabilizing solution* $X_+$ *of* $\mathcal{R}(X) = X$*, and* $X_+ > \hat{X}$.

**PROOF.** Passing to the limit in (6.6) when $m \to \infty$ we obtain

$$\begin{aligned} A_+^*(X_+ - \hat{X})A_+ - (X_+ - \hat{X}) + \begin{bmatrix} I \\ F_+ \end{bmatrix}^* \Pi(X_+ - \hat{X}) \begin{bmatrix} I \\ F_+ \end{bmatrix} \\ = \hat{X} - \mathcal{R}(\hat{X}) - (F_+ - \hat{F})^*[R + B^*\hat{X}B + \Pi_2(\hat{X})](F_+ - \hat{F}) < 0. \end{aligned}$$

Since $X_+ \geq \hat{X}$ it follows now from Lemma 3.7, (i), that $A_+$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_+ \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_+ \end{smallmatrix}\right]$ and that $X_+ > \hat{X}$ holds.

**Lemma 6.5** *If* $\mathcal{R}(X) = X$ *has a stabilizing solution* $X_s$*, then* $X_s \geq X$ *for every solution* $X$ *of* $\mathcal{R}(X) \geq X$*. In particular,* $X_s$ *is the maximal solution of* $\mathcal{R}(X) = X$.

**PROOF.** Let $X_s$ be a stabilizing solution of $\mathcal{R}(X) = X$ and denote the corresponding feedback matrix by $F_s = F(X_s)$. For every $\hat{X}$ with $\mathcal{R}(\hat{X}) \geq \hat{X}$

there is a matrix $\hat{Q} \le Q$ such that $\mathcal{R}(\hat{X}) = \hat{X} + Q - \hat{Q}$. If $\hat{F} = F(\hat{X})$, then an application of (5.8) with $X_1 := \hat{X}$ and $X_2 := X_s$ yields the equation

$$\begin{aligned}
&(A + BF_s)^*(X_s - \hat{X})(A + BF_s) - (X_s - \hat{X}) + Q - \hat{Q} \\
&\quad + (F_s - \hat{F})^*[R + B^*\hat{X}B + \Pi_2(\hat{X})](F_s - \hat{F}) \\
&\quad + \begin{bmatrix} I \\ F_s \end{bmatrix}^* \Pi(X_s - \hat{X}) \begin{bmatrix} I \\ F_s \end{bmatrix} = 0.
\end{aligned}$$

Hence, the difference $X_s - \hat{X}$ fulfills a linearly perturbed Stein equation where $Q \ge \hat{Q}$, $R + B^*\hat{X}B + \Pi_2(\hat{X}) \ge 0$ and $A + BF_s$ is $d$-stable relative to $\left[\begin{smallmatrix} I \\ F_s \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_s \end{smallmatrix}\right]$. Applying Theorem 3.3 and Lemma 3.2 we obtain $X_s \ge \hat{X}$.

**Lemma 6.6** *Assume that the following conditions hold:*

*(i)* $R > 0$, $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \ge 0$,

*(ii)* $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ *is d-detectable relative to* $\left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]$.

*Then every positive semi-definite solution of* $\mathcal{R}(X) = X$ *is stabilizing.*

**PROOF.** Let $X \ge 0$ be a solution of $\mathcal{R}(X) = X$ and denote by $F = F(X)$ the corresponding feedback matrix. From Lemma 5.2 we know that $X$ is also a solution of the linearly perturbed Stein equation

$$(A + BF)^*X(A + BF) - X + \hat{Q} + \hat{\Pi}(X) = 0 \tag{6.8}$$

with

$$\begin{aligned}
\hat{Q} &= \begin{bmatrix} I \\ F \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\
&= \begin{bmatrix} I \\ F \end{bmatrix}^* \begin{bmatrix} I & SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^* & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^* & I \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\
&= Q - SR^{-1}S^* + \left(F + R^{-1}S^*\right)^* R \left(F + R^{-1}S^*\right)
\end{aligned} \tag{6.9}$$

and

$$\hat{\Pi}(X) = \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix}.$$

We assume now that $A + BF$ is not $d$-stable relative to $\hat{\Pi}$. Then it follows from Theorem 3.3, (v), and Theorem 2.5 that there is a matrix $V \in \mathcal{H}_+^n \setminus \{0\}$ and some $\lambda \ge 1$ such that

$$\left(\mathcal{L}_{A+BF} + \hat{\Pi}\right)^{adj}(V) = \lambda V. \tag{6.10}$$

But then

$$0 \leq \langle V, \hat{Q} \rangle = -\langle V, (\mathcal{I} - \mathcal{L}_{A+BF} - \hat{\Pi})(X) \rangle = (1-\lambda)\langle V, X \rangle \leq 0$$

and we conclude that $\langle V, \hat{Q} \rangle = 0$. From $Q \geq SR^{-1}S^*$ and $R > 0$ it follows that

$$\left\langle V, Q - SR^{-1}S^* \right\rangle = \left\langle V, \left(F + R^{-1}S^*\right)^* R \left(F + R^{-1}S^*\right) \right\rangle = 0, \tag{6.11}$$

and this equations implies

$$\left(Q - SR^{-1}S^*\right) V = 0 \tag{6.12}$$

and

$$\left(F + R^{-1}S^*\right) V \left(F + R^{-1}S^*\right)^* = 0.$$

Since $V$ has a positive semi-definite square root, we obtain $FV = -R^{-1}S^*V$ and an easy calculation now yields

$$\left\langle V, \mathcal{L}_{A+BF}(X) + \hat{\Pi}(X) \right\rangle = \left\langle V, \mathcal{L}_{A-BR^{-1}S^*}(X) + \check{\Pi}(X) \right\rangle$$

with

$$\check{\Pi}(X) = \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}.$$

¿From (6.10) it follows finally that

$$\left(\mathcal{L}_{A-BR^{-1}S^*} + \check{\Pi}\right)^{adj}(V) = \lambda V,$$

and together with (6.12) this contradicts the presupposed detectability. Hence $A + BF$ is $d$-stable relative to $\hat{\Pi}$.

In the special case where $\Pi_{12} \equiv 0$ and $\Pi_2 \equiv 0$, the assumption $R > 0$ in Lemma 6.6 can be weakened to $\operatorname{Ker} R \subseteq \operatorname{Ker} B$; we obtain in this case:

**Corollary 6.7** *Assume that* $\operatorname{Ker} R \subseteq \operatorname{Ker} B$*,*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0,$$

*and that* $(Q - SR^+S^*, A - BR^+S^*)$ *is d-detectable relative to* $\Pi_1$*. Then every positive semi-definite solution of*

$$\begin{aligned} &A^*XA - X + Q + \Pi_1(X) \\ &\quad - (S + A^*XB)(R + B^*XB)^+(S + A^*XB)^* = 0 \end{aligned} \tag{6.13}$$

*is stabilizing.*

**PROOF.** Let $X \geq 0$ be a solution of (6.13) and

$$F := -(R + B^*XB)^+(S + A^*XB)^*$$

the corresponding feedback matrix. Then $\operatorname{Ker} R \subseteq \operatorname{Ker} B$ implies that

$$\operatorname{Ker} R^+ = \operatorname{Ker} R \subseteq \operatorname{Ker}(R + B^*XB) = \operatorname{Ker}(R + B^*XB)^+ \subseteq \operatorname{Ker} F^*,$$

since $\operatorname{Ker} Z^+ = \operatorname{Ker} Z^*$ for every $Z \in \mathbb{C}^{m \times n}$. According to Lemma 4.2 this is equivalent to $R^+RF = F$.

We precede as in the proof of Lemma 6.6. Since $R \geq 0$ we obtain from (6.11) and Lemma 3.1, (iii), that

$$V(F + R^+L^*)^*R(F + R^+L^*)V = 0.$$

Using $R \geq 0$ we get

$$R(F + R^+L^*)V = 0,$$

and multiplication from the left with $R^+$ yields now that $FV = -R^+S^*V$. Continuing as in the proof of Lemma 6.6 the statement of the corollary follows.

**Lemma 6.8** *Assume that*

$$R \geq 0, \quad \operatorname{Ker} R \subseteq \operatorname{Ker} S \quad \textit{and} \quad Q > SR^+S^*. \tag{6.14}$$

*If $X \geq 0$ is a solution of $\mathcal{R}(X) = X$, then $X$ is stabilizing and positive definite.*

**PROOF.** Using the notation from the proof of Lemma 6.6 we obtain $\hat{Q} > 0$ (this can be easily derived from formula (6.9) where $R^{-1}$ has to be replaced by $R^+$). Applying Lemma 3.7, (i), to the equation (6.8) we now obtain the statement of the lemma.

¿From Corollary 6.3, Lemma 6.5 and Lemma 6.6 we infer:

**Theorem 6.9** *Assume that $(A, B)$ is d-stabilizable relative to $\Pi$, $R > 0$ and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0.$$

*If $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ is d-detectable relative to $\begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}^* \Pi \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}$ then $\mathcal{R}(X) = X$ has a unique positive semi-definite solution $X_+$. Moreover, $X_+$ is stabilizing and maximal among all solutions of $\mathcal{R}(X) = X$.*

## 7 Convergence theorems

As an application of Theorem 5.5 we obtain the following monotonicity property which generalizes the corresponding results for continuous- and discrete-time Riccati equations (see [22]).

**Lemma 7.1** *Let $\mathcal{I} \subseteq \mathbb{Z}$ be some discrete interval and $t_f \in \mathcal{I}$. Assume that $\{X_t\}$ is on $\mathcal{I}$ a solution of $X_t = \mathcal{R}(X_{t+1})$. Then $X_{t_f} \geq X_{t_f-1}$ ($X_{t_f} \leq X_{t_f-1}$) implies $X_t \geq X_{t-1}$ ($X_t \leq X_{t-1}$) for all $t \in \mathcal{I}$ with $t \leq t_f$.*

**PROOF.** Define $X_t^2 := X_t$, $X_t^1 := X_{t-1}$ and apply Theorem 5.5.

**Lemma 7.2** *Assume that*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$$

*and that $\{X_t^\ell\}$ is a solution of the difference equation $X_t = \mathcal{R}(X_{t+1})$ with $X_0^\ell = 0$. If $\mathcal{R}(X) = X$ has no solution in the cone of positive semi-definite matrices then $\lim_{t\to-\infty} \|X_t^\ell\| = \infty$. Otherwise $X_{\min} = \lim_{t\to-\infty} X_t^\ell$ exists and is the minimal positive semi-definite solution of $\mathcal{R}(X) = X$.*

**PROOF.** From Corollary 5.6 we know that $X_t^\ell \geq 0$ exists for all $t \leq 0$. Since $X_{-1}^\ell = \mathcal{R}(0) = Q - SR^{-1}S^* \geq 0$, it follows from Lemma 7.1 that $X_{t-1}^\ell \geq X_t^\ell$ for all $t \leq 0$. So if $\lim_{t\to-\infty} \|X_t^\ell\| < \infty$ then the monotonicity of $\{X_t^\ell\}$ implies that $X_{\min} = \lim_{t\to-\infty} X_t^\ell$ exists. Obviously $X_{\min}$ is positive semi-definite and a solution of $\mathcal{R}(X) = X$. If $X$ is another positive semi-definite solution of $\mathcal{R}(X) = X$, it follows from Theorem 5.5 that $X_t^\ell \leq X$ for all $t \leq 0$. Passing to $t \to -\infty$ we obtain $X_{\min} \leq X$.

**Theorem 7.3** *Assume that the following hypotheses hold:*

*(i)* $R > 0$, $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \geq 0$,
*(ii)* $(A, B)$ *ist d-stabilizable relative to* $\Pi$,
*(iii)* $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ *ist d-detectable relative to*

$$\left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right].$$

*Then $\lim_{t\to-\infty} X_t = X_+$ for any solution $\{X_t\}$ of $X_t = \mathcal{R}(X_{t+1})$ with $X_0 \geq 0$.*

**PROOF.** We choose a matrix $\tilde{Q} \in \mathcal{H}^n$ such that

$$\tilde{Q} \geq Q \quad \text{and} \quad \tilde{Q} > SR^{-1}S^*.$$

Then it follows from Corollary 6.3 that the algebraic equation

$$\begin{aligned} X = A^*XA + \tilde{Q} + \Pi_1(X) - [S + A^*XB + \Pi_{12}(X)] \\ \times [R + B^*XB + \Pi_2(X)]^+[S + A^*XB + \Pi_{12}(X)]^* \end{aligned} \tag{7.1}$$

has a solution $\tilde{X}_+ \geq 0$ which in view of Lemma 6.8 is stabilizing and positive definite. Now there exists some $\lambda > 1$ such that $X_0 \leq \lambda \tilde{X}_+$. We consider the solution $\{X_t^u\}$ of $X_t = \mathcal{R}(X_{t+1})$ with $X_0^u = \lambda \tilde{X}_+$. If $\{X_t^\ell\}$ denotes the solution of the same difference equation with $X_0^\ell = 0$, then from Theorem 5.5 we obtain the inequality

$$X_t^\ell \leq X_t \leq X_t^u \quad \text{for all} \quad t \leq 0. \tag{7.2}$$

Under the given hypotheses the equation $\mathcal{R}(X) = X$ has a unique positive semi-definite solution $X_+$. Because of Lemma 7.2 it follows that $X_t^\ell \to X_+$ (monotonically) for $t \to -\infty$. We show now that $\{X_t^u\}$ is monotonically decreasing as $t$ is decreasing. If we multiply (7.1) with $\lambda$ and substitute $X$ by $\tilde{X}_+$, then (having in mind that $\tilde{Q} \geq Q$) we obtain an inequality of the form $\tilde{\mathcal{R}}(\lambda \tilde{X}_+) \leq \lambda \tilde{X}_+$, where $\tilde{\mathcal{R}}$ is a rational matrix operator with $\tilde{T} = \lambda T$. On the other hand we have $X_{-1}^u = \mathcal{R}(X_0^u) = \mathcal{R}(\lambda \tilde{X}_+)$. Since $\tilde{T} > T$ it follows with Lemma 5.4 that

$$X_0^u - X_{-1}^u = \lambda \tilde{X}_+ - \mathcal{R}(\lambda \tilde{X}_+) \geq \tilde{\mathcal{R}}(\lambda \tilde{X}_+) - \mathcal{R}(\lambda \tilde{X}_+) \geq 0.$$

Using again Lemma 7.1 this proves that $\{X_t^u\}$ is a monotonically decreasing sequence with $X_t^u \geq X_t^\ell \geq 0$ for all $t \leq 0$. Hence $\lim_{t \to -\infty} X_t^u = X_+$ since $X_+$ is the unique positive semi-definite solution. Together with (7.2) this proves the assertion of the theorem.

**Remark 7.4** *(i) The proof of Theorem 7.3 shows how the comparison theorem 5.5 in combination with the monotonicity theorem 7.1 can be used to derive existence, convergence and monotonicity results for the solutions of* (1.1) *and for the solutions of the corresponding algebraic equation* (1.2)*. For example it it also possible to derive results on the monotonicity of the maximal or the minimal positive semi-definite solution of* (1.2) *(if they exist) on the matrix* $T$*.*

*(ii) Usually it is not possible to determine the solutions of* (1.2) *explicitly. On the other hand it follows from Theorem 7.3 and the proof of Theorem 6.2, respectively, how the solution* $X_{\min}$ *of* (1.2) *and maximal solution* $X_+$ *can be determined numerically (if they exist). Alternatively one can solve* (1.2)

*numerically by a slight modification of the algorithms presented in [3], [23] and [24], we omit details.*

*(iii) The control-theoretical background for equations of the class studied in this note can be found in [25].*

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