

# FULL- AND HALF-RANGE EXPANSIONS FOR AN INDEFINITE SINGULAR STURM-LIOUVILLE PROBLEM CONNECTED WITH AN ASTROPHYSICAL PROBLEM OF PARTICLE ACCELERATION AROUND SHOCKS

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**Abstract.** Problems of spectral analysis are studied for an indefinite singular boundary value problem coming from astrophysical theory of particle acceleration around shocks. This leads to a non-classical initial-boundary value problem for a partial differential equation that can be reduced by separation of variables to an indefinite Sturm-Liouville problem for which we establish Riesz basis properties of the eigen- and associated functions and formulate completeness and expansion theorems.

## 1. Introduction

According to modern astrophysical theory, very high energy particles are accelerated around astrophysical shocks, with energies occasionally exceeding  $10^{20}$  eV . From there, they diffuse through the interstellar medium to reach observing stations on Earth. This diffusion process has some unusual, and interesting, mathematical properties.

The diffusion process is described as follows [12] (see also [2], [8], [9] for further details on this topic) . The process is idealized as occurring around a plane-parallel, steady-state shock located at  $z = 0$  ; matter enters the shock from the left ( $z < 0$ ) half-space, and exits toward the right half-space ( $z > 0$ ) . The fluid speed with respect to the shock in units of  $c$  is  $u$  , and its associated Lorentz factor  $\gamma = \sqrt{1 - u^2}$  . The particle distribution function  $f$  depends on the distance from the shock  $z$  , and  $\mu$  , the cosine of the angle between its direction of motion and the shock normal in the **fluid** frame, not the shock's: this is done to allow immediate comparison of  $f$  with observations. The distribution function  $f$  is determined by the equation

$$\gamma(u + \mu) \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left( D(\mu)(1 - \mu^2) \frac{\partial f}{\partial \mu} \right) ,$$

where the scattering coefficient  $D(\mu)$  is supposed known and arbitrarily smooth, though astrophysical theory has not progressed enough to determine it, yet.

The above equation applies for  $z \geq 0$  . It can be seen as a transport equation for  $f$  . In order to impose suitable boundary conditions, we remark that the particle speed with respect to the shock is given by

$$v_{sh} = \frac{u + \mu}{1 + u\mu}$$

by the special relativistic law for composition of velocities, so that the the flux leaving the shock is only that for  $-u \leq \mu \leq 1$  . We thus assume that, at  $z = 0$  ,

$$f = f_{\circ}(\mu) \quad -u \leq \mu \leq 1,$$

is prescribed. The distribution function at  $z = 0$  with  $\mu < -u$  describes those particles which tried to leave the shock region, but were kicked back toward it in the course of the diffusion process; it is there clearly determined by  $f_{\circ}(\mu) \quad -u \leq \mu \leq 1$  , and cannot

be specified as an independent boundary condition - this yields a non-classical type of boundary value problems that will be treated in this note.

At downstream infinity ( $z \rightarrow +\infty$ ), we simply impose that the solution does not diverge:

$$f = O(1) \quad \text{for } z \rightarrow +\infty .$$

The natural way to deal with this problem is by means of separation of variables. We posit

$$f(\mu, z) = g(z)w(\mu)$$

to obtain

$$\frac{dg}{dz} = \lambda g(z)$$

$$\gamma(u + \mu)\lambda w(\mu) = \frac{d}{d\mu} \left( D(\mu)(1 - \mu^2) \frac{dw}{d\mu} \right) .$$

We see thus that the stated astrophysical problem can be reduced to a singular Sturm-Liouville problem with indefinite weight, and with boundary conditions  $w(\mu) = O(1)$  as  $\mu \rightarrow \pm 1$ ; this problem will be analyzed below in a slightly more abstract setting.

It can be shown that this boundary value problem has a countable set of eigenvalues, both positive and negative. Moreover, the system of eigenfunctions (with associated functions if necessary) form a basis on the whole interval  $-1 < \mu < 1$ . This is the so-called full-range expansion. However, the more important and difficult problem that is to be solved here is the so-called half-range expansion with respect to a part of eigenfunctions corresponding only to negative (or positive) part of the spectrum; notice that this problem appears here since the function  $f_\circ(\mu)$  is given only for  $-u \leq \mu \leq 1$ , and because the boundary conditions at infinity require the solution to be finite there, so that eigenfunctions belonging to positive eigenvalues (which diverge at  $z \rightarrow +\infty$ ) do not represent physically realizable configurations.

In order to solve the above-mentioned non-classical boundary eigenvalue and expansion problem we consider subsequently singular Sturm-Liouville boundary value problems  $\mathcal{L}$  of the form

$$-\left(p(x)Y'(x)\right)' = \lambda s(x)Y(x), \quad -1 < x < 1, \quad (1)$$

$$Y(x) = O(1), \quad x \rightarrow \pm 1, \quad (2)$$

where

$$p(x) = (1 - x^2)p_0(x), \quad s(x) = (x - x_0)s_0(x), \quad x_0 \in (-1, 1),$$

$p_0(x), s_0(x) \in C^2[-1, 1]$ ,  $p_0(x) > 0$ ,  $s_0(x) > 0$  for all  $x \in [-1, 1]$ , and  $\lambda$  is the spectral parameter. Hence (1) is singular at both endpoints and  $x = x_0$  is a turning point.

We prefer to use the notation introduced above since in a preceding paper [5] we studied already direct and inverse problems of spectral analysis for the boundary value problem  $\mathcal{L}$ . In particular we established properties of the spectrum, which consists of a countable sequence  $\{\lambda_n\}$  of positive and negative eigenvalues, proved completeness and expansion theorems and specified spectral characteristics which uniquely determine the differential equation.

The main goal of this note is to combine the results derived in [5] with the methods and results of Beals [1], Kaper et al. [7] and Langer and Curgus [4] in order to prove that

- (i) the system  $\{Y_n(x)\}$  of all eigen- and associated functions of (1), (2) is a Riesz basis in  $L_2((-1, 1), |s|)$ ,
- (ii) the system  $\{Y_n(x)\}_{\lambda_n \geq 0}$  forms a Riesz basis in  $L_2((x_0, 1), |s|)$ ,
- (iii) the system  $\{Y_n(x)\}_{\lambda_n < 0}$  forms a Riesz basis in  $L_2((-1, x_0), |s|)$ .

If  $f_0 \in L_2((x_0, 1), |s|)$  then, according to (ii), there exist unique coefficients  $a_n$ ,  $n \geq 0$ , such that we have the half-range expansion

$$f_0(x) = \sum_{\lambda_n \geq 0} a_n Y_n(x) \text{ in } L_2((x_0, 1), |s|);$$

if in addition

$$f_1(x) := \sum_{\lambda_n \geq 0} a_n Y_n(x) \text{ in } L_2(-1, (x_0), |s|)$$

then, according to (iii), there exist unique coefficients  $b_n$ ,  $n < 0$ , such that we have the alternative half-range expansion

$$f_1(x) = \sum_{\lambda_n < 0} a_n Y_n(x) \text{ in } L_2(-1, (x_0), |s|).$$

Therefore the theorems formulated in this paper represent the basis for a rigorous application of the method of separation of variables to the above-mentioned initial boundary value problems appearing in astrophysics, scattering theory, etc.. For convenience of readers mainly interested in these applications we summarize, adapt and extend below the corresponding results that are needed for this purpose; they are essentially contained already in [1], [7], [4] and [5].

Finally it is worth while to mention that problems similar to  $\mathcal{L}$  also appear in other fields of natural sciences. For example, in the particular case  $p_0(x) = s_0(x) \equiv 1$ ,  $q(x) \equiv 0$ ,  $x_0 = 0$ , the problem  $\mathcal{L}$  describes electron scattering in a one-dimensional slab configuration ([3], [11]).

## 2. Basic spectral properties, completeness and expansion theorems

We consider the singular Sturm-Liouville boundary value problem  $\mathcal{L}$  of the form (1), (2) and denote

$$r(x) = \frac{s(x)}{p(x)} = \frac{(x - x_0)s_0(x)}{(1 - x^2)p_0(x)} =: R^2(x), \quad -1 < x < 1,$$

where  $R(x) > 0$  for  $x > x_0$ , and  $-iR(x) > 0$  for  $x < x_0$ . Denote

$$R_- = \int_{-1}^{x_0} |R(\xi)| d\xi, \quad R_+ = \int_{x_0}^1 |R(\xi)| d\xi.$$

The following theorem concerns the existence and the asymptotic behavior of the eigenvalues of  $\mathcal{L}$  (see [5]).

**Theorem 1.** *The boundary value problem (1)-(2) has a countable set of eigenvalues  $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$  (counting with multiplicities). All eigenvalues are real, and*

$$\lambda_n = \pm \left( \left( n + \frac{1}{2} \right) \frac{\pi}{R_{\pm}} \right)^2 + O(1) \quad \text{as } n \rightarrow \pm\infty. \quad (3)$$

We note that  $\lambda = 0$  is an eigenvalue of  $\mathcal{L}$  (i.e.  $0 \in \Lambda$ ) with the eigenfunction  $Y_0(x) \equiv 1$ . All non-zero eigenvalues are simple but the eigenvalue  $\lambda = 0$  may have multiplicity 2.

*Example.* Let  $p(x) = 1 - x^2$ ,  $s(x) = x$ ,  $x_0 = 0$ . Then  $\lambda_{-n} = -\lambda_n$ ,  $n > 0$ , and the eigenvalue  $\lambda = 0$  has multiplicity 2 with the eigenfunction  $Y_0(x) = 1$  and with the associated function  $Y_0^0(x) = -x/2$ . We note that the associated function  $Y_0^0(x)$  is here a solution of the equation

$$((1 - x^2)Y'(x))' = x$$

satisfying (2).

Let  $B$  be a separable Banach space. Then a system  $E = \{e_n : n \in \mathbf{Z}\} \subset B$  is called complete in  $B$  if  $\overline{\text{span}\{E\}} = B$ . Notice that  $E$  is complete if and only if there does not exist any nontrivial linear functional  $F \in B^*$  such that  $F(e_n) = 0$  for all  $n \in \mathbf{Z}$  (here  $B^*$  is the dual space for  $B$ ).

Let  $\alpha$  be a real number and let  $1 \leq p < \infty$ . We consider the Banach spaces  $B_{\alpha,p} := \{f(x) : f(x)(1 - x^2)^{-\alpha} \in L_p(-1, 1)\}$  with the norm  $\|f\|_{\alpha,p} = \|f(x)(1 - x^2)^{-\alpha}\|_p$ , where  $\|\cdot\|_p$  is the norm in the space  $L_p(-1, 1)$ . One can prove (see, for example [5]) that

$$B_{\alpha,p} \subseteq B_{\beta,s}, \quad 1 \leq s \leq p < \infty, \quad \beta - \alpha < s^{-1} - p^{-1},$$

(here the symbol  $\subseteq$  denotes dense embedding [10], p.9). The following completeness theorem was proved in [5].

**Theorem 2.** *The system of eigen- and associated functions (e.a.f.)  $\{Y_n(x)\}_{n=-\infty}^{\infty}$  of the boundary-value problem  $\mathcal{L}$  is complete in the space  $B_{\alpha,p}$  for  $p \geq 1$ ,  $\alpha < 1/p$ . Moreover, for  $n \neq k$ ,*

$$\int_{-1}^1 s(x)Y_n(x)Y_k(x) dx = 0.$$

*In particular, the system of e.a.f.  $\{Y_n(x)\}_{n=-\infty}^{\infty}$  of the boundary-value problem  $\mathcal{L}$  is complete in  $L_p(-1, 1)$  for  $p \geq 1$ .*

Denote

$$\alpha_n := \int_{-1}^1 s(x)Y_n^2(x) dx.$$

It follows from Theorem 2 and the preceding remark that  $\alpha_n \neq 0$  for  $n \in \mathbf{Z}$ .

The following expansion theorem was also proved in [5].

**Theorem 3.** *Let  $g(x)$ ,  $x \in [-1, 1]$ , be an absolutely continuous function having an absolutely continuous derivative. Then*

$$g(x) = \sum_{n=-\infty}^{\infty} a_n Y_n(x), \quad a_n := \frac{1}{\alpha_n} \int_{-1}^1 s(x)g(x)Y_n(x) dx, \quad (4)$$

and the series converges uniformly on  $[-1, 1]$ .

### 3. Riesz basis properties, full- and half-range expansions

Let  $H$  be a complex (or real) Hilbert space with a scalar product  $(\cdot, \cdot)$ . We recall that a set  $\{v_j\}_{j \geq 1}$ ,  $v_j \in H$ , is called a basis in  $H$  if each element  $v \in H$  can be uniquely represented by a series

$$v = \sum_{j=1}^{\infty} c_j v_j, \quad c_j \in \mathbf{C} \quad (\text{or } c_j \in \mathbf{R}), \quad (5)$$

which converges in the norm of  $H$ . A basis  $\{v_j\}_{j \geq 1}$  is called a *Riesz basis* if there exists a linear bounded and boundedly invertible operator  $T$  in  $H$  such that  $\{e_j\}_{j \geq 1} := \{Tv_j\}_{j \geq 1}$  is an orthogonal basis in  $H$ .

If  $\{v_j\}_{j \geq 1}$  is a Riesz basis in  $H$ , then there exists a biorthonormal system  $\{v_j^*\}_{j \geq 1}$  in  $H$  such that  $(v_j, v_k^*) = \delta_{j,k}$  ( $\delta_{j,k}$  is the Kronecker delta). The system  $\{v_j^*\}_{j \geq 1}$  is also a Riesz basis in  $H$  and (with  $T$  as in the definition of a Riesz basis)  $v_j^* = T^* e_j$ , moreover, the coefficients  $c_j$  in (5) can be calculated by the formula

$$c_j = (v, v_j^*).$$

The basic properties of Riesz bases can be found, for example, in [13] and [6], Sec.1.8.5. We denote by  $H_0 = L_2((-1, 1), |s|)$  the real Hilbert space of functions  $f$  such that

$$\int_{-1}^1 |s(x)| |f(x)|^2 dx < \infty$$

with the scalar product

$$(f, g)_0 = \int_{-1}^1 |s(x)| f(x) g(x) dx$$

and with the norm

$$\|f\|_0 = \left( \int_{-1}^1 |s(x)| |f(x)|^2 dx \right)^{1/2}.$$

By  $H_+ = L_2((x_0, 1), |s|)$  we denote the real Hilbert space of functions  $f$  such that

$$\int_{x_0}^1 |s(x)| |f(x)|^2 dx < \infty$$

with the scalar product

$$(f, g)_+ = \int_{x_0}^1 |s(x)| f(x) g(x) dx$$

and with the norm

$$\|f\|_+ = \left( \int_{x_0}^1 |s(x)| |f(x)|^2 dx \right)^{1/2},$$

and by  $H_- = L_2((-1, x_0), |s|)$  we denote the real Hilbert space of functions  $f$  such that

$$\int_{-1}^{x_0} |s(x)| |f(x)|^2 dx < \infty$$

with the scalar product

$$(f, g)_- = \int_{-1}^{x_0} |s(x)| f(x) g(x) dx$$

and with the norm

$$\|f\|_- = \left( \int_{-1}^{x_0} |s(x)| |f(x)|^2 dx \right)^{1/2}.$$

**Theorem 4.** *The system of eigen- and associated functions  $\{Y_n(x)\}_{n=-\infty}^{\infty}$  of the boundary value problem (1)-(2) forms a Riesz basis in  $H_0$ . For any  $g \in H_0$ , (4) holds, and the series in (4) converges in the norm of  $H_0$ .*

Theorem 3 and Theorem 4 are called a *full-range expansion theorems*, since we use all e.a.f. and the convergence takes place in the whole interval  $[-1, 1]$ . Next we provide a so-called *half-range expansion theorem*, which is needed for the solution of the above-mentioned boundary value problems from various fields of physics. Denote

$$\gamma := \int_{-1}^1 s(x) dx.$$

**Theorem 5.** (i) *Let  $\gamma > 0$ . Then the system  $\{Y_n(x)\}_{\lambda_n \geq 0}$  forms a Riesz basis in  $H_+$ , and the system  $\{Y_n(x)\}_{\lambda_n < 0}$  forms a Riesz basis in  $H_-$ .*

(ii) *Let  $\gamma = 0$ . Then the system  $\{Y_n(x)\}_{\lambda_n \geq 0}$  forms a Riesz basis in  $H_+$ , and the system  $\{Y_n(x)\}_{\lambda_n \leq 0}$  forms a Riesz basis in  $H_-$ .*

(iii) *Let  $\gamma < 0$ . Then the system  $\{Y_n(x)\}_{\lambda_n > 0}$  forms a Riesz basis in  $H_+$ , and the system  $\{Y_n(x)\}_{\lambda_n \leq 0}$  forms a Riesz basis in  $H_-$ .*

For more details on the half-range expansion theorems see [1], [7], [4].

*Examples.* (i) Let  $p(x) = 1 - x^2$ ,  $s(x) = x$ ,  $x_0 = 0$ . Then  $\gamma = 0$ . In this case in Theorem 5 we do not use the associated function  $Y_0^0(x) = -x/2$ , but we use the eigenfunction  $Y_0(x) = 1$  twice, namely in  $H_+$  and in  $H_-$ .

(ii) Let us consider, as another example, the particular case when  $s_0(x) \equiv 1$ ,  $x_0 < 0$ . Then  $s(x) = x - x_0$ ,  $\gamma = ((1 - x_0)^2 - (1 + x_0)^2)/2 > 0$ , and the following theorem is a simple corollary of Theorem 5.

**Theorem 6.** *Let  $s_0(x) \equiv 1$ ,  $x_0 < 0$ . Then the system  $\{Y_n(x)\}_{\lambda_n \geq 0}$  forms a Riesz basis in  $H_+$ . For any  $g(x) \in H_+$ ,  $x \in (x_0, 1)$ , one has*

$$g(x) = \sum_{\lambda_n \geq 0} \gamma_n Y_n(x), \quad \gamma_n = \int_{x_0}^1 (x - x_0) g(x) Y_n^*(x) dx, \quad (6)$$

where  $Y_n^*(x)$  is the biorthogonal basis for  $\{Y_n(x)\}$  in  $H_+$ , i. e.

$$\int_{x_0}^1 (x - x_0) Y_n(x) Y_k^*(x) dx = \delta_{nk}.$$

The series in (6) converges in the norm of  $H_+$ .

#### 4. Transformation to the symmetrical case

The replacement

$$\xi = \frac{x - x_0}{1 - x_0 x} \quad (7)$$

reduces the boundary value problem  $\mathcal{L}$  to the symmetrical case when the turning point lies at the origin. Indeed it follows from (7) that

$$x = \frac{\xi + x_0}{1 + x_0 \xi} \quad (8)$$

and consequently

$$(1 - x^2) = \frac{(1 - \xi^2)(1 - x_0^2)}{(1 + x_0 \xi)^2}, \quad x - x_0 = \frac{\xi(1 - x_0^2)}{1 + x_0 \xi}, \quad \frac{d}{dx} = \frac{(1 + x_0 \xi)^2}{1 - x_0^2} \frac{d}{d\xi}.$$

Denote

$$\tilde{Y}(\xi) = Y\left(\frac{\xi + x_0}{1 + x_0 \xi}\right) = Y(x).$$

The equation

$$-\frac{d}{dx}((1 - x^2)p_0(x)Y'(x)) = \lambda(x - x_0)s_0(x)Y(x)$$

can be written as

$$-\frac{d}{d\xi}\left((1 - \xi^2)p_0\left(\frac{\xi + x_0}{1 + x_0 \xi}\right)\frac{d\tilde{Y}}{d\xi}\right) = \lambda\xi\frac{1 - x_0^2}{1 + x_0 \xi}s_0\left(\frac{\xi + x_0}{1 + x_0 \xi}\right)\tilde{Y}(\xi)$$

or

$$-\frac{d}{d\xi}\left((1 - \xi^2)\tilde{p}_0(\xi)\frac{d\tilde{Y}}{d\xi}\right) = \lambda\xi\tilde{s}_0(\xi)\tilde{Y}(\xi), \quad (9)$$

where

$$\begin{aligned} \tilde{p}_0(\xi) &:= p_0\left(\frac{\xi + x_0}{1 + x_0 \xi}\right) = p_0(x), \\ \tilde{s}_0(\xi) &:= s_0\left(\frac{\xi + x_0}{1 + x_0 \xi}\right)\frac{(1 - x_0^2)^2}{(1 + x_0 \xi)^3}. \end{aligned}$$

Clearly,

$$\int_{-1}^1 s(x) dx = \int_{-1}^1 \tilde{s}(\xi) d\xi,$$

where  $\tilde{s}(\xi) = \xi\tilde{s}_0(\xi)$ .

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